Gradient estimates for solutions of the Lamé system with infinity coefficients

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Abstract

We establish upper bounds on the blow up rate of the gradients of solutions of the Lamé system with infinity coefficients in dimension two as the distance between the surfaces of discontinuity of the coefficients of the system tends to zero.

1 Introduction

We consider the Lamé system in linear elasticity. Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded open set with C^2 boundary, and D_1 and D_2 be two disjoint strictly convex open sets in Ω with $C^{2,\gamma}$ boundaries, $0 < \gamma < 1$, which are ϵ-distance apart and far away from $\partial\Omega$. More precisely,

$$\begin{aligned} &\overline{D}_1, \overline{D}_2 \subset \Omega, \quad \text{the principle curvatures of } \partial D_1, \partial D_2 \geq \kappa_0 > 0, \\ &\epsilon := \mathrm{dist}(D_1, D_2) > 0, \quad \mathrm{dist}(D_1 \cup D_2, \partial\Omega) > \kappa_1 > 0, \end{aligned} \tag{1.1}$$

where κ_0, κ_1 are constants independent of ϵ.

Denote
$$\widetilde{\Omega} := \Omega \setminus \overline{D_1 \cup D_2}.$$

We assume that $\widetilde{\Omega}$ and $D_1 \cup D_2$ are occupied by two different homogeneous and isotropic materials with different Lamé constants (λ, μ) and (λ_1, μ_1). Then the elasticity tensors for the inclusions and the background can be written, respectively, as $\mathbb{C}^1$ and $\mathbb{C}^0$, with

$$C^1_{ijkl} = \lambda_1 \delta_{ij}\delta_{kl} + \mu_1(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

and

$$C^0_{ijkl} = \lambda \delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

where $i, j, k, l = 1, 2, \cdots, d$ and δ_{ij} is the Kronecker symbol: $\delta_{ij} = 0$ for $i \neq j$, $\delta_{ij} = 1$ for $i = j$.

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Let $u = \left(u^1, u^2, \cdots, u^d\right)^T : \Omega \to \mathbb{R}^d$ denote the displacement field. For a given vector-valued function φ, we consider the following Dirichlet problem

$$\begin{cases} \nabla \cdot \left(\left(\chi_{\widetilde{\Omega}} \mathbb{C}^0 + \chi_{D_1 \cup D_2} \mathbb{C}^1 \right) e(u) \right) = 0, & \text{in } \Omega, \\ u = \varphi, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where χ_D is the characteristic function of D,

$$e(u) := \frac{1}{2} \left(\nabla u + (\nabla u)^T \right)$$

is the strain tensor.

We assume that the standard ellipticity condition holds for (1.2), that is,

$$\mu > 0, \quad d\lambda + 2\mu > 0; \qquad \mu_1 > 0, \quad d\lambda_1 + 2\mu_1 > 0.$$

For $\varphi \in H^1(\Omega; \mathbb{R}^d)$, it is well known that there exists a unique solution $u \in H^1(\Omega; \mathbb{R}^d)$ of the Dirichlet problem (1.2), which is also the minimizer of the energy functional

$$J[u] = \frac{1}{2} \int_\Omega \left(\left(\chi_{\widetilde{\Omega}} \mathbb{C}^0 + \chi_{D_1 \cup D_2} \mathbb{C}^1 \right) e(u), e(u) \right) dx$$

on

$$H^1_\varphi(\Omega; \mathbb{R}^d) := \left\{ u \in H^1(\Omega; \mathbb{R}^d) \;\middle|\; u - \varphi \in H^1_0(\Omega; \mathbb{R}^d) \right\}.$$

Babuška, Andersson, Smith, and Levin [10] computationally analyzed the damage and fracture in fiber composite materials where the Lamé system is used. They observed numerically that the size of the strain tensor $e(u)$ remains bounded when the distance ϵ tends to zero. Stimulated by this, there have been many works on the analogous question for the scalar equation

$$\begin{cases} \nabla \cdot \left(a_k(x) \nabla u_k \right) = 0 & \text{in } \Omega, \\ u_k = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where φ is given, and

$$a_k(x) = \begin{cases} k \in (0, \infty) & \text{in } D_1 \cup D_2, \\ 1 & \text{in } \widetilde{\Omega}. \end{cases}$$

For touching disks D_1 and D_2 in dimension $d = 2$, Bonnetier and Vogelius [15] proved that $|\nabla u_k|$ remains bounded. The bound depends on the value of k. Li and Vogelius [28] extended the result to general divergence form second order elliptic equations with piecewise smooth coefficients in all dimensions, and they proved that $|\nabla u|$ remains bounded as $\epsilon \to 0$. They also established stronger, ϵ-independent, $C^{1,\alpha}$ estimates for solutions in the closure of each of the regions D_1, D_2 and $\widetilde{\Omega}$. This extension covers domains D_1 and D_2 of arbitrary smooth shapes. Li and Nirenberg extended in [27] the results in [28] to general divergence form second order elliptic systems including systems of elasticity. This in particular answered in the affirmative the question naturally led to by the above mentioned numerical indication in [10] for the boundedness of the strain tensor as ϵ tends to 0. For higher derivative estimates, we draw attention of readers to the open problem on page 894 of [27].

The estimates in [27] and [28] depend on the ellipticity of the coefficients. If ellipticity constants are allowed to deteriorate, the situation is very different. It was shown in various papers, see for example Budiansky and Carrier [17] and Markenscoff [31], that when $k = \infty$ in (1.3) the L^∞-norm of $|\nabla u_\infty|$ generally becomes unbounded as ϵ tends to 0. The rate at which the L^∞-norm of the gradient of a special solution blows up was shown in [17] to be $\epsilon^{-1/2}$ in dimension $d = 2$. Ammari, Kang and Lim [9] and Ammari, Kang, Lee, Lee and Lim [7] proved that when D_1 and D_2 are disks in $\mathbb{R}^2$, and when $k = \infty$ in (1.3), the blow up rate of $|\nabla u_\infty|$ is $\epsilon^{-1/2}$. This result was extended by Yun [36, 37] and Bao, Li and Yin [11] to strictly convex D_1 and D_2 in $\mathbb{R}^2$. In dimension $d = 3$ and $d \geq 4$, the blow up rate of $|\nabla u_\infty|$ turns out to be $(\epsilon|\ln\epsilon|)^{-1}$ and ϵ^{-1} respectively; see [11]. The results were extended to multi-inclusions in [12]. Further, more detailed, characterizations of the singular behavior of ∇u_∞ have been obtained by Ammari, Ciraolo, Kang, Lee and Yun [3], Ammari, Kang, Lee, Lim and Zribi [8], Bonnetier and Triki [13, 14], Kang, Lim and Yun [21, 22]. For related works, see [4, 5, 14, 16, 18, 19, 23, 24, 25, 26, 29, 30, 32, 34, 35] and the references therein.

In this paper we obtain gradient estimates for the Lamé system with infinity coefficients in dimension $d = 2$. In a subsequent paper we treat higher dimensional cases $d \geq 3$.

The linear space of rigid displacements in $\mathbb{R}^2$ is

$$\Psi := \left\{ \psi \in C^1(\mathbb{R}^2;\mathbb{R}^2) \,\middle|\, \nabla\psi + (\nabla\psi)^T = 0 \right\},$$

or equivalently [33],

$$\Psi = \mathrm{span}\left\{ \psi^1 = \begin{pmatrix}1\\0\end{pmatrix},\ \psi^2 = \begin{pmatrix}0\\1\end{pmatrix},\ \psi^3 = \begin{pmatrix}x_2\\-x_1\end{pmatrix} \right\}.$$

If $\xi \in H^1(D;\mathbb{R}^2)$, $e(\xi) = 0$ in D, and $D \subset \mathbb{R}^2$ is a connected open set, then ξ is a linear combination of $\{\psi^\alpha\}$ in D.

For fixed λ and μ satisfying $\mu > 0$ and $\lambda + \mu > 0$, denote u_{λ_1,μ_1} the solution of (1.2). Then, as proved in the Appendix,

$$u_{\lambda_1,\mu_1} \to u \text{ in } H^1(\Omega;\mathbb{R}^2) \ \text{ as } \ \min\{\mu_1, \lambda_1 + \mu_1\} \to \infty. \tag{1.4}$$

where u is a $H^1(\Omega;\mathbb{R}^2)$ solution of

$$\begin{cases} \mathcal{L}_{\lambda,\mu} u := \nabla\cdot\left(\mathbb{C}^0 e(u)\right) = 0, & \text{in } \widetilde{\Omega},\\ u\big|_+ = u\big|_-, & \text{on } \partial D_1 \cup \partial D_2,\\ e(u) = 0, & \text{in } D_1 \cup D_2,\\ \int_{\partial D_i} \frac{\partial u}{\partial \nu_0}\Big|_+ \cdot \psi^\alpha = 0, & \alpha = 1,2,3,\ i = 1,2,\\ u = \varphi, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

where

$$\frac{\partial u}{\partial \nu_0}\Big|_+ := \left(\mathbb{C}^0 e(u)\right)\vec{n} = \lambda\,(\nabla\cdot u)\,\vec{n} + \mu\left(\nabla u + (\nabla u)^T\right)\vec{n},$$

and $\vec{n}$ is the unit outer normal of D_i, $i = 1, 2$.

Here and throughout this paper the subscript $\pm$ indicates the limit from outside and inside the domain, respectively. The existence, uniqueness and regularity of weak solutions to (1.5) are proved in the Appendix. In particular, the H^1 weak solution to (1.5) is in $C^1(\overline{\widetilde{\Omega}}) \cap C^1(\overline{D_1 \cup D_2})$.

The convergence (1.4) in the case $\mu_1 \to \infty$ while λ_1 remains bounded was established in [6]. Our proof of (1.4) in the Appendix is different and is an extension to systems of that in [11].

The solution of (1.5) is also the unique function which has the least energy in appropriate functional spaces, characterized by

$$I_\infty[u] = \min_{v \in \mathcal{A}} I_\infty[v],$$

where

$$I_\infty[v] := \frac{1}{2} \int_{\widetilde{\Omega}} \left(\mathbb{C}^{(0)} e(v), e(v) \right) dx,$$

and

$$\mathcal{A} := \left\{ u \in H^1_\varphi(\Omega; \mathbb{R}^2) \;\middle|\; e(u) = 0 \ \text{ in } D_1 \cup D_2 \right\}.$$

A calculation gives

$$\left(\mathcal{L}_{\lambda,\mu} u \right)^i = \mu \Delta u^i + (\lambda + \mu) \left[\partial_{x_i x_1} u^1 + \partial_{x_i x_2} u^2 \right], \qquad i = 1, 2. \tag{1.6}$$

Since D_1 and D_2 are two strictly convex subdomains of Ω, there exist two points $P_1 \in \partial D_1$ and $P_2 \in \partial D_2$ such that

$$\mathrm{dist}(P_1, P_2) = \mathrm{dist}(\partial D_1, \partial D_2) = \epsilon. \tag{1.7}$$

We use $\overline{P_1 P_2}$ to denote the line segment connecting P_1 and P_2.

The main result in this paper is as follows. Assume that for some $\delta_0 > 0$,

$$\delta_0 \le \mu, \lambda + \mu \le \frac{1}{\delta_0}. \tag{1.8}$$

Theorem 1.1. *Assume that $\Omega, D_1, D_2, \epsilon$ are defined in* (1.1) *with $d = 2$, λ and μ satisfy (1.8), and $\varphi \in C^{1,\gamma}(\partial\Omega; \mathbb{R}^2)$ for some $0 < \gamma < 1$. Let $u \in H^1(\Omega; \mathbb{R}^2) \cap C^1(\overline{\widetilde{\Omega}}; \mathbb{R}^2)$ be a solution to* (1.5)*. Then for $0 < \epsilon < 1$, we have*

$$|\nabla u(x)| \le \begin{cases} \dfrac{C}{\sqrt{\epsilon} + \mathrm{dist}(x, \overline{P_1 P_2})} \|\varphi\|_{C^{1,\gamma}(\partial\Omega; \mathbb{R}^2)}, & x \in \widetilde{\Omega}, \\ C \|\varphi\|_{C^{1,\gamma}(\partial\Omega; \mathbb{R}^2)}, & x \in D_1 \cup D_2. \end{cases} \tag{1.9}$$

where C is a universal constant. In particular,

$$\|\nabla u\|_{L^\infty(\Omega)} \le C \epsilon^{-1/2} \|\varphi\|_{C^{1,\gamma}(\partial\Omega; \mathbb{R}^2)}. \tag{1.10}$$

Note that throughout the paper, unless otherwise stated, C denotes some constant, whose value may vary from line to line, depending only on κ_0, κ_1, γ, δ_0, $\|\partial D_1\|_{C^{2,\gamma}}$, $\|\partial D_2\|_{C^{2,\gamma}}$, $\|\partial\Omega\|_{C^2}$ and the Lebesgue measure of Ω, and is in particular independent of ϵ. Also, we call a constant having such dependence a universal constant.

Since the blow up rate of $|\nabla u_\infty|$ for solutions of (1.3) when $k = \infty$ is known to reach the magnitude $\epsilon^{-1/2}$, estimate (1.10) is expected to be optimal. This is also supported by the numerical indication in [20].

The paper is organized as follows. In Section 2, we first introduce the setup of the proof of Theorem 1.1. Then we state a proposition, Proposition 2.1, containing key estimates, and deduce Theorem 1.1 from the proposition. In Sections 3 and 4, we prove Proposition 2.1. In Section 5, we prove Theorem 5.1 which extends Theorem 1.1 in two aspects. One is that the strict convexity assumption on ∂D_1 and ∂D_2 can be replaced by a weaker relative strict convexity assumption. The other is an upper bound of the gradient when the flatness order near the closest points between ∂D_1 and ∂D_2 is $m \geq 2$ instead of $m = 2$ for the strictly convex ∂D_1 and ∂D_2. In the Appendix, we give a variational characterization of solutions of the Lamé system with infinity coefficients and prove the previously mentioned convergence result (1.4).

2 Outline of the proof of Theorem 1.1

The proof of Theorem 1.1 makes use of the following decomposition. By the third line of (1.5), u is a linear combination of $\{\psi^\alpha\}$ in D_1 and D_2, respectively. Since $\mathcal{L}_{\lambda,\mu}\xi = 0$ in $\widetilde{\Omega}$ and $\xi = 0$ on $\partial\widetilde{\Omega}$ imply that $\xi = 0$ in $\widetilde{\Omega}$, we decompose the solution of (1.5), in the spire of [11], as follows:

$$u = \begin{cases} \sum_{\alpha=1}^{3} C_1^\alpha \psi^\alpha, & \text{in } \overline{D}_1, \\ \sum_{\alpha=1}^{3} C_2^\alpha \psi^\alpha, & \text{in } \overline{D}_2, \\ \sum_{\alpha=1}^{3} C_1^\alpha v_1^\alpha + \sum_{\alpha=1}^{3} C_2^\alpha v_2^\alpha + v_3, & \text{in } \widetilde{\Omega}, \end{cases} \tag{2.1}$$

where $v_i^\alpha \in C^1(\overline{\widetilde{\Omega}}; \mathbb{R}^2) \cap C^2(\widetilde{\Omega}; \mathbb{R}^2)$, $\alpha = 1, 2, 3$, $i = 1, 2$, satisfy

$$\begin{cases} \mathcal{L}_{\lambda,\mu} v_i^\alpha = 0, & \text{in } \widetilde{\Omega}, \\ v_i^\alpha = \psi^\alpha, & \text{on } \partial D_i, \\ v_i^\alpha = 0, & \text{on } \partial D_j \cup \partial\Omega, \ j \neq i; \end{cases} \tag{2.2}$$

$v_3 \in C^1(\overline{\widetilde{\Omega}}; \mathbb{R}^2) \cap C^2(\widetilde{\Omega}; \mathbb{R}^2)$ satisfies

$$\begin{cases} \mathcal{L}_{\lambda,\mu} v_3 = 0, & \text{in } \widetilde{\Omega}, \\ v_3 = 0, & \text{on } \partial D_1 \cup \partial D_2, \\ v_3 = \varphi, & \text{on } \partial\Omega; \end{cases} \tag{2.3}$$

and the constants $\{C_i^\alpha\}$ are uniquely determined by u.

By the decomposition (2.1), we write

$$\nabla u=\sum_{\alpha=1}^{2}(C_1^\alpha-C_2^\alpha)\,\nabla v_1^\alpha+\sum_{\alpha=1}^{2}C_2^\alpha(\nabla v_1^\alpha+\nabla v_2^\alpha)+\sum_{i=1}^{2}C_i^3\nabla v_i^3+\nabla v_3,\quad \text{in } \widetilde{\Omega}. \tag{2.4}$$

Theorem 1.1 can be deduced from the following proposition.

Proposition 2.1. *Under the hypotheses of Theorem 1.1 and a normalization* $\|\varphi\|_{C^{1,\gamma}(\partial\Omega)}=1$*, we have, for* $0<\epsilon<1$*,*

$$\|\nabla v_3\|_{L^\infty(\widetilde{\Omega})}\leq C; \tag{2.5}$$

$$\|\nabla v_1^\alpha+\nabla v_2^\alpha\|_{L^\infty(\widetilde{\Omega})}\leq C,\quad \alpha=1,2,3; \tag{2.6}$$

$$|\nabla v_i^\alpha(x)|\leq\frac{C}{\epsilon+\mathrm{dist}^2(x,\overline{P_1P_2})},\quad i,\alpha=1,2,\quad x\in\widetilde{\Omega}; \tag{2.7}$$

$$|\nabla v_i^3(x)|\leq C\frac{\epsilon+\mathrm{dist}(x,\overline{P_1P_2})}{\epsilon+\mathrm{dist}^2(x,\overline{P_1P_2})},\quad i=1,2,\quad x\in\widetilde{\Omega}; \tag{2.8}$$

and

$$|C_i^\alpha|\leq C,\quad i=1,2,\ \alpha=1,2,3; \tag{2.9}$$

$$|C_1^\alpha-C_2^\alpha|\leq C\sqrt{\epsilon},\quad \alpha=1,2. \tag{2.10}$$

Proof of Theorem 1.1 by using Proposition 2.1. Clearly, we only need to prove the theorem under the normalization $\|\varphi\|_{C^{1,\gamma}(\partial\Omega)}=1$.

Since

$$\nabla u=C_i^3\begin{pmatrix}0&1\\-1&0\end{pmatrix}\quad \text{in } D_i,\quad i=1,2,$$

the second estimate in (1.9) follows easily from (2.9).

By (2.4) and Proposition 2.1, we have, for x in $\widetilde{\Omega}$,

$$|\nabla u(x)|\leq\sum_{\alpha=1}^{2}\left|C_1^\alpha-C_2^\alpha\right|\left|\nabla v_1^\alpha(x)\right|+C\sum_{i=1}^{2}\left|\nabla v_i^3(x)\right|+C\leq\frac{C}{\sqrt{\epsilon}+\mathrm{dist}(x,\overline{P_1P_2})}.$$

Theorem 1.1 follows. □

To complete this section, we recall some properties of the tensor $\mathbb{C}$. For the isotropic elastic material, let

$$\mathbb{C}:=(C_{ijkl})=\left(\lambda\delta_{ij}\delta_{kl}+\mu\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right)\right),\quad \mu>0,\quad d\lambda+2\mu>0.$$

The components C_{ijkl} satisfy the following symmetric condition:

$$C_{ijkl}=C_{klij}=C_{klji},\quad i,j,k,l=1,2,\cdots,d. \tag{2.11}$$

We will use the following notations:

$$(\mathbb{C}A)_{ij}=\sum_{k,l=1}^{d}C_{ijkl}A_{kl},\quad\text{and}\quad(A,B)\equiv A:B=\sum_{i,j=1}^{d}A_{ij}B_{ij},$$

for every pair of $d \times d$ matrices $A = (A_{ij})$, $B = (B_{ij})$. Clearly

$$(\mathbb{C}A, B) = (A, \mathbb{C}B).$$

If A is symmetric, then, by the symmetry condition (2.11), we have that

$$(\mathbb{C}A, A) = C_{ijkl} A_{kl} A_{ij} = \lambda A_{ii} A_{kk} + 2\mu A_{kj} A_{kj}.$$

Thus $\mathbb{C}$ satisfies the following ellipticity condition: For every $d \times d$ real symmetric matrix $A = (A_{ij})$,

$$\min\{2\mu, d\lambda + 2\mu\}|A|^2 \leq (\mathbb{C}A, A) \leq \max\{2\mu, d\lambda + 2\mu\}|A|^2, \tag{2.12}$$

where $|A|^2 = \sum\limits_{i,j} A_{ij}^2$.

3 Estimates of ∇v_1^α, ∇v_2^α and ∇v_3

Before proceeding to prove Proposition 2.1, we first fix notations. By a translation and rotation of the coordinates if necessary, we may assume without loss of generality that the points P_1 and P_2 in (1.7) satisfy

$$P_1 = \left(0, \frac{\epsilon}{2}\right) \in \partial D_1, \quad \text{and} \quad P_2 = \left(0, -\frac{\epsilon}{2}\right) \in \partial D_2.$$

Fix a small universal constant R, such that the portions of ∂D_i near P_i can be represented respectively by

$$x_2 = \frac{\epsilon}{2} + h_1(x_1), \quad \text{and} \quad x_2 = -\frac{\epsilon}{2} + h_2(x_1), \quad \text{for } |x_1| < 2R.$$

Moreover, by the assumptions on ∂D_i, h_i satisfies

$$\frac{\epsilon}{2} + h_1(x_1) > -\frac{\epsilon}{2} + h_2(x_1), \quad \text{for } |x_1| < 2R,$$

$$h_1(0) = h_2(0) = h_1'(0) = h_2'(0) = 0, \tag{3.1}$$

$$h_1''(0) \geq \kappa_0 > 0, \quad h_2''(0) \leq -\kappa_0 < 0, \tag{3.2}$$

and

$$\|h_1\|_{C^{2,\gamma}([-2R,2R])} + \|h_2\|_{C^{2,\gamma}([-2R,2R])} \leq C. \tag{3.3}$$

For $0 < r \leq 2R$, denote

$$\Omega_r := \left\{x \in \mathbb{R}^2 \;\middle|\; -\frac{\epsilon}{2} + h_2(x_1) < x_2 < \frac{\epsilon}{2} + h_1(x_1),\ |x_1| < r\right\}.$$

The top and bottom boundaries of Ω_r are

$$\Gamma_r^+ = \left\{x \in \mathbb{R}^2 \;\middle|\; x_2 = \frac{\epsilon}{2} + h_1(x_1),\ |x_1| < r\right\},$$

and

$$\Gamma_r^- = \left\{x \in \mathbb{R}^2 \;\middle|\; x_2 = -\frac{\epsilon}{2} + h_2(x_1),\ |x_1| < r\right\}.$$

Here $x = (x_1, x_2)$.

3.1 Estimates of v_3 and $v_1^\alpha + v_2^\alpha$, $\alpha = 1, 2, 3$

Lemma 3.1.

$$\|v_3\|_{L^\infty(\widetilde{\Omega})} + \|\nabla v_3\|_{L^\infty(\widetilde{\Omega})} \leq C.$$
$$\|v_1^\alpha + v_2^\alpha\|_{L^\infty(\widetilde{\Omega})} + \|\nabla v_1^\alpha + \nabla v_2^\alpha\|_{L^\infty(\widetilde{\Omega})} \leq C, \quad \alpha = 1, 2, 3.$$

Proof. As mentioned before, we may assume without loss of generality that $\|\varphi\|_{C^{1,\gamma}(\partial\Omega)} = 1$. Extending φ to $\Phi \in C^{1,\gamma}(\overline{\Omega})$ satisfying $\Phi(x) = 0$ for all $dist(x, \partial\Omega) > \kappa_1/2$. In particular, $\Phi = 0$ near $\overline{D_1 \cup D_2}$, and

$$\int_{\widetilde{\Omega}} |\nabla\Phi|^2 dx \leq C\|\varphi\|_{C^{1,\gamma}(\partial\Omega)} = C.$$

Then, in view of (2.3),

$$I_\infty[v_3] := \frac{1}{2}\int_{\widetilde{\Omega}} \left(\mathbb{C}^0 e(v_3), e(v_3)\right) dx \leq I_\infty[\Phi] \leq C.$$

By the first Korn's inequality (see, e.g. theorem 2.1 in [33]) and (2.12),

$$\begin{aligned}
\|\nabla(v_3 - \Phi)\|^2_{L^2(\widetilde{\Omega})} &\leq 2\|e(v_3 - \Phi)\|^2_{L^2(\widetilde{\Omega})} \\
&\leq C\left(\|e(v_3)\|^2_{L^2(\widetilde{\Omega})} + \|e(\Phi)\|^2_{L^2(\widetilde{\Omega})}\right) \\
&\leq C\left(I_\infty[v_3] + I_\infty[\Phi]\right) \\
&\leq C.
\end{aligned}$$

It follows that

$$\|\nabla v_3\|_{L^2(\widetilde{\Omega})} \leq C.$$

Consequently,

$$\|v_3\|_{L^2(\widetilde{\Omega})} \leq C\|\nabla v_3\|_{L^2(\widetilde{\Omega})} \leq C.$$

Note that the constant C above is independent of ϵ. By the interior estimates and the boundary estimates for elliptic systems (see Agmon, Douglis and Nirenberg [1] and [2]), we have

$$\|\nabla v_3\|_{L^\infty(\widetilde{\Omega}\setminus\Omega_{R/2})} \leq C.$$

We apply theorem 1.1 in [26] to v_3 and obtain

$$\|\nabla v_3\|_{L^\infty(\Omega_{R/2})} \leq C.$$

Since

$$\begin{cases}
\mathcal{L}_{\lambda,\mu}(v_1^\alpha + v_2^\alpha - \psi^\alpha) = 0, & \text{in } \widetilde{\Omega}, \\
v_1^\alpha + v_2^\alpha - \psi^\alpha = 0, & \text{on } \partial D_1 \cup \partial D_2, \\
v_1^\alpha + v_2^\alpha - \psi^\alpha = -\psi^\alpha, & \text{on } \partial\Omega,
\end{cases}$$

the above arguments yield, with $\varphi = -\psi^\alpha$,

$$\left\|\nabla v_1^\alpha + \nabla v_2^\alpha\right\|_{L^\infty(\widetilde{\Omega})} \leq C, \quad \alpha = 1, 2, 3. \tag{3.4}$$

Lemma 3.1 follows from the above. □

3.2 Estimates of v_i^α, $i, \alpha = 1, 2$

To estimate v_i^α, $i, \alpha = 1, 2$, we introduce a scalar function $\bar{u} \in C^2(\mathbb{R}^2)$, such that $\bar{u} = 1$ on ∂D_1, $\bar{u} = 0$ on $\partial D_2 \cup \partial\Omega$,

$$\bar{u}(x) = \frac{x_2 - h_2(x_1) + \frac{\epsilon}{2}}{\epsilon + h_1(x_1) - h_2(x_1)}, \quad \text{in } \Omega_R, \tag{3.5}$$

and

$$\|\bar{u}\|_{C^2(\mathbb{R}^2\setminus\Omega_R)} \leq C. \tag{3.6}$$

A calculation gives

$$|\partial_{x_1}\bar{u}(x)| \leq \frac{C|x_1|}{\epsilon + |x_1|^2}, \quad |\partial_{x_2}\bar{u}(x)| \leq \frac{C}{\epsilon + |x_1|^2}, \quad x \in \Omega_R, \tag{3.7}$$

$$|\partial_{x_1x_1}\bar{u}(x)| \leq \frac{C}{\epsilon + |x_1|^2}, \quad |\partial_{x_1x_2}\bar{u}(x)| \leq \frac{C|x_1|}{(\epsilon + |x_1|^2)^2}, \quad \partial_{x_2x_2}\bar{u}(x) = 0, \quad x \in \Omega_R. \tag{3.8}$$

Define

$$\bar{u}_1^1 = (\bar{u}, 0)^T, \quad \bar{u}_1^2 = (0, \bar{u})^T, \quad \text{in } \widetilde{\Omega}, \tag{3.9}$$

then $v_1^\alpha = \bar{u}_1^\alpha$ on $\partial\widetilde{\Omega}$. Similarly, we can define

$$\bar{u}_2^1 = (\underline{u}, 0)^T, \quad \bar{u}_2^2 = (0, \underline{u})^T, \quad \text{in } \widetilde{\Omega}, \tag{3.10}$$

where $\underline{u}$ is a scalar function in $C^2(\mathbb{R}^2)$ satisfying $\underline{u} = 1$ on ∂D_2, $\underline{u} = 0$ on $\partial D_1 \cup \partial\Omega$,

$$\underline{u}(x) = \frac{-x_2 + h_1(x_1) + \frac{\epsilon}{2}}{\epsilon + h_1(x_1) - h_2(x_1)}, \quad x \in \Omega_R, \tag{3.11}$$

and

$$\|\underline{u}\|_{C^2(\mathbb{R}^2\setminus\Omega_R)} \leq C. \tag{3.12}$$

By (1.6), (3.7) and (3.8),

$$\left|\mathcal{L}_{\lambda,\mu}\bar{u}_i^\alpha(x)\right| \leq \frac{C}{\epsilon + |x_1|^2} + \frac{C|x_1|}{(\epsilon + |x_1|^2)^2}, \quad i, \alpha = 1, 2, \quad x \in \Omega_R. \tag{3.13}$$

For $|z_1| \leq R$, we always use δ to denote

$$\delta := \delta(z_1) = \frac{\epsilon + h_1(z_1) - h_2(z_1)}{2}. \tag{3.14}$$

Clearly,

$$\frac{1}{C}(\epsilon + |z_1|^2) \leq \delta(z_1) \leq C(\epsilon + |z_1|^2). \tag{3.15}$$

For $|z_1| \leq R/2$, $s < R/2$, let

$$\widehat{\Omega}_s(z_1) := \left\{(x_1, x_2) \,\Big|\, -\frac{\epsilon}{2} + h_2(x_1) < x_2 < \frac{\epsilon}{2} + h_1(x_1),\ |x_1 - z_1| < s\right\}. \tag{3.16}$$

We denote

$$w_i^\alpha := v_i^\alpha - \bar{u}_i^\alpha, \quad i, \alpha = 1, 2. \tag{3.17}$$

In order to prove (2.7), it suffices to prove the following proposition.

Proposition 3.2. *Assume the above, let* $v_i^\alpha \in C^2(\widetilde{\Omega};\mathbb{R}^2)\cap C^1(\overline{\widetilde{\Omega}};\mathbb{R}^2)$ *be the weak solution of* (2.2)*. Then, for* $i,\alpha=1,2$*,*

$$\int_{\widetilde{\Omega}}\left|\nabla w_i^\alpha\right|^2 dx \leq C, \tag{3.18}$$

$$\int_{\widehat{\Omega}_\delta(z_1)}\left|\nabla w_i^\alpha\right|^2 dx \leq \begin{cases} C(\epsilon^2+|z_1|^2), & |z_1|\leq\sqrt{\epsilon},\\ C|z_1|^2, & \sqrt{\epsilon}<|z_1|\leq R,\end{cases} \tag{3.19}$$

and

$$|\nabla w_i^\alpha(x)|\leq \begin{cases} C\frac{\epsilon+|x_1|}{\epsilon}, & |x_1|\leq\sqrt{\epsilon},\\ \frac{C}{|x_1|}, & \sqrt{\epsilon}<|x_1|\leq R.\end{cases} \tag{3.20}$$

Corollary 3.3. *For* $i,\alpha=1,2$*,*

$$\left|\nabla v_i^\alpha(x)\right|\leq\frac{C}{\epsilon+\mathrm{dist}^2(x,\overline{P_1P_2})}, \qquad x\in\widetilde{\Omega}. \tag{3.21}$$

Proof of Corollary 3.3. A consequence of (3.18) is

$$\int_{\widetilde{\Omega}\setminus\Omega_{R/2}}\left|\nabla v_i^\alpha\right|^2 dx\leq 2\int_{\widetilde{\Omega}\setminus\Omega_{R/2}}\left(\left|\nabla\overline{u}_i^\alpha\right|^2+\left|\nabla w_i^\alpha\right|^2\right)dx\leq C,$$

With this we can apply classical elliptic estimates to obtain

$$\left\|\nabla v_i^\alpha\right\|_{L^\infty(\widetilde{\Omega}\setminus\Omega_R)}\leq C, \quad i,\alpha=1,2. \tag{3.22}$$

Under assumption (1.1),

$$\frac{1}{C}(\epsilon+|x_1|^2)\leq\mathrm{dist}(x,\overline{P_1P_2})\leq C(\epsilon+|x_1|^2).$$

Estimate (3.21) in Ω_R follows from (3.20) and the fact that

$$\left|\nabla\overline{u}_i^\alpha(x)\right|\leq\frac{C}{\epsilon+|x_1|^2}, \quad \text{in } \Omega_R.$$

□

Proof of Proposition 3.2. We only prove it for $i=\alpha=1$, since the same proof applies to the other cases. For simplicity, denote $w:=w_1^1$. We divide into three steps.
STEP 1. Proof of (3.18).

By (3.17),

$$\begin{cases}\mathcal{L}_{\lambda,\mu}w=-\mathcal{L}_{\lambda,\mu}\overline{u}_1^1, & \text{in } \widetilde{\Omega},\\ w=0, & \text{on } \partial\widetilde{\Omega}.\end{cases} \tag{3.23}$$

Multiplying the equation in (3.23) by w and integrating by parts, we have

$$\int_{\widetilde{\Omega}}\left(\mathbb{C}^0e(w),e(w)\right)dx=\int_{\widetilde{\Omega}}w\left(\mathcal{L}_{\lambda,\mu}\overline{u}_1^1\right)dx. \tag{3.24}$$

By the mean value theorem, there exists $r_0 \in (R/2, 2R/3)$ such that

$$\begin{aligned}
\int_{\substack{|x_1|=r_0,\\ -\epsilon/2+h_2(x_1)<x_2<\epsilon/2+h_1(x_1)}} |w| dx_2 &= \frac{6}{R} \int_{\substack{R/2<|x_1|<2R/3,\\ -\epsilon/2+h_2(x_1)<x_2<\epsilon/2+h_1(x_1)}} |w| dx \\
&\leq C \int_{\Omega_{2R/3}\setminus\Omega_{R/2}} |\nabla w| dx \\
&\leq C \left(\int_{\widetilde{\Omega}} |\nabla w|^2 \, dx \right)^{1/2}. \qquad (3.25)
\end{aligned}$$

It follows from (2.12), (3.24) and the first Korn's inequality that

$$\begin{aligned}
&\int_{\widetilde{\Omega}} |\nabla w|^2 \, dx \\
&\leq 2 \int_{\widetilde{\Omega}} |e(w)|^2 dx \\
&\leq C \left| \int_{\Omega_{r_0}} w \left(\mathcal{L}_{\lambda,\mu} \overline{u}_1^1 \right) dx \right| + C \left| \int_{\widetilde{\Omega}\setminus\Omega_{r_0}} w \left(\mathcal{L}_{\lambda,\mu} \overline{u}_1^1 \right) dx \right| \\
&\leq C \left| \int_{\Omega_{r_0}} w \left(\mathcal{L}_{\lambda,\mu} \overline{u}_1^1 \right) dx \right| + C \int_{\widetilde{\Omega}\setminus\Omega_{r_0}} |w| dx \\
&\leq C \left(\left| \int_{\Omega_{r_0}} w^{(1)} \partial_{x_1x_1} \overline{u} \, dx \right| + \left| \int_{\Omega_{r_0}} w^{(2)} \partial_{x_1x_2} \overline{u} \, dx \right| \right) + C \left(\int_{\widetilde{\Omega}\setminus\Omega_{r_0}} |\nabla w|^2 dx \right)^{1/2}. \qquad (3.26)
\end{aligned}$$

First,

$$\begin{aligned}
\int_{\Omega_{r_0}} w^{(1)} \partial_{x_1x_1} \overline{u} \, dx &= -\int_{\Omega_{r_0}} \partial_{x_1} w^{(1)} \partial_{x_1} \overline{u} \, dx + \int_{\substack{|x_1|=r_0,\\ -\epsilon/2+h_2(x_1)<x_2<\epsilon/2+h_1(x_1)}} (\partial_{x_1} \overline{u}) \, w^{(1)} \, dx_2 \\
&:= I + II.
\end{aligned}$$

Then, by (3.7),

$$|I| \leq C \left(\int_{\Omega_{r_0}} |\partial_{x_1} \overline{u}|^2 dx \right)^{1/2} \left(\int_{\widetilde{\Omega}} |\nabla w|^2 dx \right)^{1/2} \leq C \left(\int_{\widetilde{\Omega}} |\nabla w|^2 dx \right)^{1/2}.$$

By (3.25), we have

$$|II| \leq C \int_{\substack{|x_1|=r_0,\\ -\epsilon/2+h_2(x_1)<x_2<\epsilon/2+h_1(x_1)}} |w| \, dx_2 \leq C \left(\int_{\widetilde{\Omega}} |\nabla w|^2 dx \right)^{1/2}.$$

Hence

$$\left| \int_{\Omega_{r_0}} w^{(1)} \partial_{x_1x_1} \overline{u} \, dx \right| \leq C \left(\int_{\widetilde{\Omega}} |\nabla w|^2 dx \right)^{1/2}. \qquad (3.27)$$

Similarly, using $w=0$ on $\partial D_1\cup\partial D_2$,

$$\begin{aligned}\left|\int_{\Omega_{r_0}} w^{(2)}\partial_{x_1x_2}\bar{u}\,dx\right| = \left|\int_{\Omega_{r_0}} \partial_{x_2}w^{(2)}\partial_{x_1}\bar{u}\,dx\right| &\\ \leq\; C\left(\int_{\Omega_{r_0}}|\partial_{x_1}\bar{u}|^2\,dx\right)^{1/2}\left(\int_{\widetilde{\Omega}}|\nabla w|^2\,dx\right)^{1/2} &\\ \leq\; C\left(\int_{\widetilde{\Omega}}|\nabla w|^2\,dx\right)^{1/2}. &\end{aligned}$$

Therefore, combining this estimate with (3.27) and (3.26),

$$\int_{\widetilde{\Omega}}|\nabla w|^2\,dx\leq\; C\left(\int_{\widetilde{\Omega}}|\nabla w|^2\,dx\right)^{1/2},$$

which implies (3.18).

STEP 2. Proof of (3.19).

For $0<t<s<R$, let η be a smooth function satisfying $\eta(x_1)=1$ if $|x_1-z_1|<t$, $\eta(x_1)=0$ if $|x_1-z_1|>s$, $0\leq\eta(x_1)\leq 1$ if $t\leq|x_1-z_1|\leq\; s$, and $|\eta'(x_1)|\leq\frac{2}{s-t}$. Multiplying the equation in (3.23) by $w\eta^2$ and integrating by parts lead to

$$\int_{\widehat{\Omega}_s(z_1)}(\mathbb{C}^0e(w),e(w\eta^2))dx=-\int_{\widehat{\Omega}_s(z_1)}(w\eta^2)\mathcal{L}_{\lambda,\mu}\bar{u}_1^1\,dx. \tag{3.28}$$

Using the first Korn's inequality and some standard arguments, we have

$$\int_{\widehat{\Omega}_s(z_1)}(\mathbb{C}^0e(w),e(w\eta^2))dx\geq\frac{1}{C}\int_{\widehat{\Omega}_s(z_1)}|\nabla(w\eta)|^2dx-C\int_{\widehat{\Omega}_s(z_1)}|w|^2|\nabla\eta|^2dx, \tag{3.29}$$

and

$$\left|\int_{\widehat{\Omega}_s(z_1)}(w\eta^2)\mathcal{L}_{\lambda,\mu}\bar{u}_1^1\,dx\right|\leq\;\frac{C}{(s-t)^2}\int_{\widehat{\Omega}_s(z_1)}|w|^2dx+(s-t)^2\int_{\widehat{\Omega}_s(z_1)}\left|\mathcal{L}_{\lambda,\mu}\bar{u}_1^1\right|^2dx.$$

It follows that

$$\int_{\widehat{\Omega}_t(z_1)}|\nabla w|^2dx\leq\;\frac{C}{(s-t)^2}\int_{\widehat{\Omega}_s(z_1)}|w|^2dx+(s-t)^2\int_{\widehat{\Omega}_s(z_1)}\left|\mathcal{L}_{\lambda,\mu}\bar{u}_1^1\right|^2dx. \tag{3.30}$$

Case 1. For $\sqrt{\epsilon}\leq|z_1|\leq\; R$.

Note that for $0<s<\frac{2|z_1|}{3}$, we have

$$\begin{aligned}\int_{\widehat{\Omega}_s(z_1)}|w|^2dx&=\int_{|x_1-z_1|\leq s}\int_{-\frac{\epsilon}{2}+h_2(x_1)}^{\frac{\epsilon}{2}+h_1(x_1)}|w(x_1,x_2)|^2dx_2dx_1\\&\leq\int_{|x_1-z_1|\leq s}(\epsilon+h_1(x_1)-h_2(x_1))^2\int_{-\frac{\epsilon}{2}+h_2(x_1)}^{\frac{\epsilon}{2}+h_1(x_1)}\left|\partial_{x_2}w(x_1,x_2)\right|^2dx_2dx_1\\&\leq\; C|z_1|^4\int_{\widehat{\Omega}_s(z_1)}|\nabla w|^2dx,\end{aligned} \tag{3.31}$$

By (3.13), we have

$$\int_{\widehat{\Omega}_s(z_1)} \left|\mathcal{L}_{\lambda,\mu}\overline{u}_1^1\right|^2 dx \leq \frac{Cs}{|z_1|^4}, \quad 0 < s < \frac{2|z_1|}{3}. \tag{3.32}$$

Denote

$$\widehat{F}(t) := \int_{\widehat{\Omega}_t(z_1)} |\nabla w|^2 dx.$$

It follows from the above that

$$\widehat{F}(t) \leq \left(\frac{C_0|z_1|^2}{s-t}\right)^2 \widehat{F}(s) + C(s-t)^2 \frac{s}{|z_1|^4}, \quad \forall\, 0 < t < s < \frac{2|z_1|}{3}, \tag{3.33}$$

where C_0 is also a universal constant.

Let $t_i = 2C_0 i\,|z_1|^2$, $i = 1, 2, \cdots$. Then

$$\frac{C_0|z_1|^2}{t_{i+1} - t_i} = \frac{1}{2}.$$

Let $k = \left[\frac{1}{4C_0|z_1|}\right]$. Then by (3.33) with $s = t_{i+1}$ and $t = t_i$, we have

$$\widehat{F}(t_i) \leq \frac{1}{4}\widehat{F}(t_{i+1}) + \frac{C(t_{i+1} - t_i)^2 t_{i+1}}{|z_1|^4} \leq \frac{1}{4}\widehat{F}(t_{i+1}) + C(i+1)|z_1|^2,$$

After k iterations, we have, using (3.18),

$$\begin{aligned}\widehat{F}(t_1) &\leq \left(\frac{1}{4}\right)^k \widehat{F}(t_{k+1}) + C|z_1|^2 \sum_{l=1}^{k} \left(\frac{1}{4}\right)^{l-1} (l+1) \leq C\left(\frac{1}{4}\right)^k + C|z_1|^2 \sum_{l=1}^{k} \left(\frac{1}{4}\right)^{l-1} (l+1)\\ &\leq C|z_1|^2.\end{aligned}$$

This implies that

$$\int_{\widehat{\Omega}_\delta(z_1)} |\nabla w|^2 dx \leq C|z_1|^2.$$

Case 2. For $|z_1| \leq \sqrt{\epsilon}$.

For $0 < t < s < \sqrt{\epsilon}$, we still have (3.30). Estimate (3.31) becomes

$$\int_{\widehat{\Omega}_s(z_1)} |w|^2 dx \leq C\epsilon^2 \int_{\widehat{\Omega}_s(z_1)} |\nabla w|^2 dx, \quad 0 < s < \sqrt{\epsilon}. \tag{3.34}$$

Estimate (3.32) becomes

$$\int_{\widehat{\Omega}_s(z_1)} \left|\mathcal{L}_{\lambda,\mu}\overline{u}_1^1\right|^2 dx \leq \frac{Cs}{\epsilon} + \frac{C|z_1|^2 s}{\epsilon^3} + \frac{Cs^3}{\epsilon^3}, \quad 0 < s < \sqrt{\epsilon}. \tag{3.35}$$

Estimate (3.33) becomes, in view of (3.30),

$$\widehat{F}(t) \leq \left(\frac{C_0\epsilon}{s-t}\right)^2 \widehat{F}(s) + C(s-t)^2 s\left(\frac{1}{\epsilon} + \frac{|z_1|^2}{\epsilon^3} + \frac{s^2}{\epsilon^3}\right), \quad \forall\, 0 < t < s < \sqrt{\epsilon}. \tag{3.36}$$

Let $t_i = 2C_0 i\epsilon$, $i = 1, 2, \cdots$. Then

$$\frac{C_0\epsilon}{t_{i+1} - t_i} = \frac{1}{2}.$$

Let $k = \left[\frac{1}{4C_0\sqrt{\epsilon}}\right]$. Then by (3.36) with $s = t_{i+1}$ and $t = t_i$, we have

$$\widehat{F}(t_i) \leq \frac{1}{4}\widehat{F}(t_{i+1}) + Ci^3(\epsilon^2 + |z_1|^2).$$

After k iterations, we have, using (3.18),

$$\begin{aligned}\widehat{F}(t_1) &\leq \left(\frac{1}{4}\right)^k \widehat{F}(t_{k+1}) + C\sum_{l=1}^{k}\left(\frac{1}{4}\right)^{l-1} l^3(\epsilon^2 + |z_1|^2) \leq C\left(\frac{1}{4}\right)^{\frac{1}{C\sqrt{\epsilon}}} + C(\epsilon^2 + |z_1|^2)\\ &\leq C(\epsilon^2 + |z_1|^2).\end{aligned}$$

This implies that

$$\int_{\widehat{\Omega}_\delta(z_1)} |\nabla w|^2 dx \leq C(\epsilon^2 + |z_1|^2).$$

STEP 3. Proof of (3.20).

Making a change of variables

$$\begin{cases} x_1 - z_1 = \delta y_1, \\ x_2 = \delta y_2, \end{cases} \tag{3.37}$$

then $\widehat{\Omega}_\delta(z_1)$ becomes Q_1', where

$$Q_r' = \left\{ y \in \mathbb{R}^2 \middle| -\frac{\epsilon}{2\delta} + \frac{1}{\delta}h_2(\delta y_1 + z_1) < y_2 < \frac{\epsilon}{2\delta} + \frac{1}{\delta}h_1(\delta y_1 + z_1), |y_1| < r \right\}, \quad \text{for } r \leq 1,$$

and the boundaries $\Gamma_1^{\pm}$ become

$$y_2 = \hat{h}_1(y_1) := \frac{1}{\delta}\left(\frac{\epsilon}{2} + h_1(\delta y_1 + z_1)\right), \quad |y_1| < 1,$$

and

$$y_2 = \hat{h}_2(y_1) := \frac{1}{\delta}\left(-\frac{\epsilon}{2} + h_2(\delta y_1 + z_1)\right), \quad |y_1| < 1.$$

Then

$$\hat{h}_1(0) - \hat{h}_2(0) := \frac{1}{\delta}\left(\epsilon + h_1(z_1) - h_2(z_1)\right) = 2,$$

and by (3.1) and (3.2),

$$|\hat{h}_1'(0)| + |\hat{h}_2'(0)| \leq C|z_1|, \quad |\hat{h}_1''(0)| + |\hat{h}_2''(0)| \leq C\delta.$$

Since R is small, $\|\hat{h}_1\|_{C^{1,1}((-1,1))}$ and $\|\hat{h}_2\|_{C^{1,1}((-1,1))}$ are small and $\frac{1}{2}Q_1'$ is essentially a unit square as far as applications of Sobolev embedding theorems and classical L^p estimates for elliptic systems are concerned. Let

$$U_1^1(y_1, y_2) := \bar{u}_1^1(x_1, x_2), \quad W_1^1(y_1, y_2) := w_1^1(x_1, x_2), \quad y \in Q_1', \tag{3.38}$$

then by (3.23),

$$\mathcal{L}_{\lambda,\mu} W_1^1 = -\mathcal{L}_{\lambda,\mu} U_1^1, \qquad y \in Q_1'. \tag{3.39}$$

where

$$\left|\mathcal{L}_{\lambda,\mu} U_1^1\right| = \delta^2 \left|\mathcal{L}_{\lambda,\mu} \overline{u}_1^1\right|.$$

Since $W_1^1 = 0$ on the top and bottom boundaries of Q_1', we have, using Poincaré inequality, that

$$\left\|W_1^1\right\|_{H^1(Q_1')} \leq C \left\|\nabla W_1^1\right\|_{L^2(Q_1')}.$$

By $W^{2,p}$ estimates for elliptic systems (see [2]) and Sobolev embedding theorems, we have, with $p = 3$,

$$\left\|\nabla W_1^1\right\|_{L^\infty(Q_{1/2}')} \leq C \left\|W_1^1\right\|_{W^{2,p}(Q_{1/2}')} \leq C \left(\left\|\nabla W_1^1\right\|_{L^2(Q_1')} + \left\|\mathcal{L}_{\lambda,\mu} U_1^1\right\|_{L^\infty(Q_1')}\right).$$

It follows that

$$\left\|\nabla w_1^1\right\|_{L^\infty(\widehat{\Omega}_{\frac{\delta}{2}}(z_1))} \leq \frac{C}{\delta} \left(\left\|\nabla w_1^1\right\|_{L^2(\widehat{\Omega}_\delta(z_1))} + \delta^2 \left\|\mathcal{L}_{\lambda,\mu} \overline{u}_1^1\right\|_{L^\infty(\widehat{\Omega}_\delta(z_1))}\right). \tag{3.40}$$

Case 1. For $\sqrt{\epsilon} \leq |z_1| \leq R$.

By (3.19),

$$\int_{\widehat{\Omega}_\delta(z_1)} \left|\nabla w_1^1\right|^2 dx \leq C|z_1|^2.$$

By (3.13),

$$\delta^2 \left|\mathcal{L}_{\lambda,\mu} \overline{u}_1^1\right| \leq \delta^2 \left(\frac{C}{|z_1|^2} + \frac{C}{|z_1|^3}\right) \leq C|z_1|, \qquad \text{in } \widehat{\Omega}_\delta(z_1).$$

We deduce from (3.40) that

$$\left|\nabla w_1^1(z_1, x_2)\right| = \frac{C|z_1|}{\delta} \leq \frac{C}{|z_1|}, \qquad \forall\ -\frac{\epsilon}{2} + h_2(z_1) < x_2 < \frac{\epsilon}{2} + h_1(z_1).$$

Case 2. For $|z_1| \leq \sqrt{\epsilon}$.

By (3.19),

$$\int_{\widehat{\Omega}_\delta(z_1)} \left|\nabla w_1^1\right|^2 dx \leq C(\epsilon^2 + |z_1|^2).$$

By (3.13),

$$\delta^2 \left|\mathcal{L}_{\lambda,\mu} \overline{u}_1^1\right| \leq C\delta^2 \left(\frac{1}{\epsilon} + \frac{\epsilon + |z_1|}{\epsilon^2}\right) \leq C(\epsilon + |z_1|), \qquad \text{in } \widehat{\Omega}_\delta(z_1).$$

We deduce from (3.40) that

$$\left|\nabla w_1^1(z_1, x_2)\right| = \frac{C}{\delta}(\epsilon + |z_1|) \leq C\frac{\epsilon + |z_1|}{\epsilon}, \qquad \forall\ -\frac{\epsilon}{2} + h_2(z_1) < x_2 < \frac{\epsilon}{2} + h_1(z_1).$$

Proposition 3.2 is established. □

3.3 Estimates of v_i^3, $i = 1, 2$

Define

$$\vec{u}_1^3 = (x_2\bar{u}, -x_1\bar{u})^T, \quad \text{and} \quad \vec{u}_2^3 = \left(x_2\underline{u}, -x_1\underline{u}\right)^T \tag{3.41}$$

then $v_i^3 = \vec{u}_i^3$ on $\partial\widetilde{\Omega}$, $i = 1, 2$. Using (3.7), (3.1) and (3.3), we obtain

$$\left|\nabla\vec{u}_i^3(x)\right| \leq \frac{C(\epsilon + |x_1|)}{\epsilon + |x_1|^2}, \quad i = 1, 2, \quad x \in \Omega_R, \tag{3.42}$$

and

$$\left|\nabla\vec{u}_i^3(x)\right| \leq C, \quad i = 1, 2, \quad x \in \widetilde{\Omega} \setminus \Omega_R. \tag{3.43}$$

It follows from (3.41), (1.6), (3.7) and (3.8) that

$$\left|\mathcal{L}_{\lambda,\mu}\vec{u}_i^3\right| \leq \frac{C}{\epsilon + |x_1|^2}, \quad i = 1, 2, \quad x \in \Omega_R. \tag{3.44}$$

We estimate the energy of v_i^3, $i = 1, 2$.

Lemma 3.4.

$$\int_{\widetilde{\Omega}} \left|v_i^3\right|^2 dx + \int_{\widetilde{\Omega}} \left|\nabla v_i^3\right|^2 dx \leq C, \quad i = 1, 2, \tag{3.45}$$

and

$$\left\|\nabla v_i^3\right\|_{L^\infty(\widetilde{\Omega}\setminus\Omega_R)} \leq C, \quad i = 1, 2. \tag{3.46}$$

Proof. By (3.42) and (3.43), we have

$$I_\infty[v_i^3] \leq I_\infty[\vec{u}_i^3] \leq C\left\|\nabla\vec{u}_i^3\right\|^2_{L^2(\widetilde{\Omega})} \leq C,$$

and, by (1.8) and (2.12) and the first Korn's inequality,

$$\begin{aligned} \left\|\nabla v_i^3\right\|_{L^2(\widetilde{\Omega})} &\leq \left\|\nabla(v_i^3 - \vec{u}_i^3)\right\|_{L^2(\widetilde{\Omega})} + \left\|\nabla\vec{u}_i^3\right\|_{L^2(\widetilde{\Omega})} \leq \sqrt{2}\left\|e(v_i^3 - \vec{u}_i^3)\right\|_{L^2(\widetilde{\Omega})} + C \\ &\leq C\left\|e(v_i^3)\right\|_{L^2(\widetilde{\Omega})} + C \leq CI_\infty[v_i^3] + C \leq C. \end{aligned}$$

We know from the Poincaré inequality that

$$\int_{\widetilde{\Omega}} \left|v_i^3\right|^2 dx \leq C\int_{\widetilde{\Omega}} \left|\nabla v_i^3\right|^2 dx \leq C.$$

Note that the above constant C is independent of ϵ.

With (3.45), we can apply classical elliptic estimates, see [1] and [2], to obtain (3.46). □

Denote

$$w_i^3 := v_i^3 - \vec{u}_i^3, \quad i = 1, 2.$$

It is easy to see from (3.42), (3.43) and (3.45) that

$$\int_{\widetilde{\Omega}} \left|\nabla w_i^3\right|^2 \leq C. \tag{3.47}$$

Lemma 3.5. *With $\delta = \delta(z_1)$ in* (3.14)*, we have, for $i = 1, 2$,*

$$\int_{\widehat{\Omega}_\delta(z_1)} \left|\nabla w_i^3\right|^2 dx \le \begin{cases} C\epsilon^2, & |z_1| < \sqrt{\epsilon}, \\ C|z_1|^4, & \sqrt{\epsilon} \le |z_1| < R. \end{cases} \tag{3.48}$$

Proof. The proof is similar to that of (3.19). We will only prove it for $i = 1$, since the proof for $i = 2$ is the same. For simplicity, denote $w := w_1^3$, then

$$\begin{cases} \mathcal{L}_{\lambda,\mu} w = -\mathcal{L}_{\lambda,\mu} \bar{u}_1^3, & \text{in } \widetilde{\Omega}, \\ w = 0, & \text{on } \partial\widetilde{\Omega}. \end{cases} \tag{3.49}$$

As in the proof of (3.19), we have, instead of (3.30),

$$\int_{\widehat{\Omega}_t(z_1)} |\nabla w|^2 dx \le \frac{C}{(s-t)^2} \int_{\widehat{\Omega}_s(z_1)} |w|^2 dx + (s-t)^2 \int_{\widehat{\Omega}_s(z_1)} \left|\mathcal{L}_{\lambda,\mu} \bar{u}_1^3\right|^2 dx. \tag{3.50}$$

Case 1. $\sqrt{\epsilon} < |z_1| < R$**.**

We still have (3.31) for $0 < s < \frac{2|z_1|}{3}$. Instead of (3.32), we have, using (3.44),

$$\int_{\widehat{\Omega}_s(z_1)} \left|\mathcal{L}_{\lambda,\mu} \bar{u}_1^3\right|^2 dx \le \frac{Cs}{|z_1|^2}. \tag{3.51}$$

Instead of (3.33), we have

$$\widehat{F}(t) \le \left(\frac{C_0 |z_1|^2}{s-t}\right)^2 \widehat{F}(s) + C(s-t)^2 \frac{s}{|z_1|^2}, \quad \forall\, 0 < t < s < \frac{2|z_1|}{3}. \tag{3.52}$$

We define $\{t_i\}$, k and iterate as in the proof of (3.19), right below formula (3.33), to obtain, using (3.47),

$$\widehat{F}(t_1) \le \left(\frac{1}{4}\right)^k \widehat{F}(\frac{2|z_1|}{3}) + C|z_1|^4 \sum_{l=1}^{k} \left(\frac{1}{4}\right)^l l \le C|z_1|^4.$$

This implies that

$$\int_{\widehat{\Omega}_\delta(z_1)} |\nabla w|^2 dx \le C|z_1|^4.$$

Case 2. $|z_1| < \sqrt{\epsilon}$**.**

Estimate (3.34) remains the same. Estimate (3.35) becomes

$$\int_{\widehat{\Omega}_s(z_1)} \left|\mathcal{L}_{\lambda,\mu} \bar{u}_1^3\right|^2 dx \le \frac{Cs}{\epsilon}, \quad 0 < s < \sqrt{\epsilon}. \tag{3.53}$$

Estimate (3.36) becomes

$$\widehat{F}(t) \le \left(\frac{C_0 \epsilon}{s-t}\right)^2 \widehat{F}(s) + \frac{C(s-t)^2 s}{\epsilon}, \quad \forall\, 0 < t < s < \sqrt{\epsilon}. \tag{3.54}$$

Define $\{t_i\}$, k and iterate as in the proof of (3.19), right below formula (3.36), to obtain

$$\widehat{F}(t_1) \le \left(\frac{1}{4}\right)^k \widehat{F}(t_{k+1}) + C \sum_{l=1}^{k} \left(\frac{1}{4}\right)^{l-1} l\epsilon^2 \le C\epsilon^2.$$

This implies that

$$\int_{\widehat{\Omega}_\delta(z_1)} |\nabla w|^2 dx \leq C\epsilon^2.$$

□

Lemma 3.6.

$$\left\|\nabla w_i^3\right\|_{L^\infty(\widetilde{\Omega})} \leq C, \quad i = 1, 2. \tag{3.55}$$

Consequently,

$$\left|\nabla v_i^3(x)\right| \leq \frac{C(\epsilon + |x_1|)}{\epsilon + |x_1|^2}, \quad i = 1, 2, \quad x \in \Omega_R. \tag{3.56}$$

Proof. The proof is the same as that of (3.20). In Case 1, $\sqrt{\epsilon} \leq |z_1| \leq R$, we use estimates

$$\int_{\widehat{\Omega}_\delta(z_1)} \left|\nabla w_1^3\right|^2 dx \leq C|z_1|^4,$$

and

$$\delta^2 \left|\mathcal{L}_{\lambda,\mu}\bar{u}_1^3\right| \leq C|z_1|^2.$$

In Case 2, $|z_1| \leq \sqrt{\epsilon}$. we use

$$\int_{\widehat{\Omega}_\delta(z_1)} \left|\nabla w_1^3\right|^2 dx \leq C\epsilon^2,$$

and

$$\delta^2 \left|\mathcal{L}_{\lambda,\mu}\bar{u}_1^3\right| \leq C\epsilon.$$

□

4 Estimates of C_1^α and C_2^α

In this section, we first prove that C_1^α and C_2^α are uniformly bounded with respect to ϵ, and then estimate the difference $C_1^\alpha - C_2^\alpha$.

4.1 Boundedness of C_i^α, $i = 1, 2, \alpha = 1, 2, 3$

Lemma 4.1. *Let C_i^α be defined in* (2.1). *Then*

$$|C_i^\alpha| \leq C, \qquad i = 1, 2; \quad \alpha = 1, 2, 3.$$

Proof. We only need to prove it for $i = 1$, since the proof for $i = 2$ is the same. Let u_ϵ be the solution of (1.5). By Theorem 6.5 and Theorem 6.6 in the Appendix,

$$I_\infty[u_\epsilon] := \frac{1}{2}\int_{\widetilde{\Omega}} \left(\mathbb{C}^{(0)}e(u_\epsilon), e(u_\epsilon)\right) \leq I_\infty[\Phi] \leq C$$

where Φ is the one in the proof of Lemma 3.1.

It follows that

$$\|u_\epsilon\|_{H^1(\widetilde{\Omega})} \leq C\|e(u_\epsilon)\|_{L^2(\widetilde{\Omega})} \leq CI_\infty[u_\epsilon] \leq C.$$

By the trace embedding theorem,

$$\|u_\epsilon\|_{L^2(\partial D_1 \setminus B_R)} \leq C.$$

On ∂D_1,

$$u_\epsilon = \sum_{\alpha=1}^{3} C_1^\alpha \psi^\alpha.$$

If $C_1 := (C_1^1, C_1^2, C_1^3) = 0$, there is nothing to prove. Otherwise

$$C \geq |C_1| \left\| \sum_{\alpha=1}^{3} \widehat{C}_1^\alpha \psi^\alpha \right\|_{L^2(\partial D_1 \setminus B_R)}, \tag{4.1}$$

where $\widehat{C}_1^\alpha = \frac{C_1^\alpha}{|C_1|}$ and $|\widehat{C}_1| = 1$. It is easy to see that

$$\left\| \sum_{\alpha=1}^{3} \widehat{C}_1^\alpha \psi^\alpha \right\|_{L^2(\partial D_1 \setminus B_R)} \geq \frac{1}{C}. \tag{4.2}$$

Indeed, if not, along a subsequence $\epsilon \to 0$, $\widehat{C}_1^\alpha \to \bar{C}_1^\alpha$, and

$$\left\| \sum_{\alpha=1}^{3} \bar{C}_1^\alpha \psi^\alpha \right\|_{L^2(\partial D_1^* \setminus B_R)} = 0,$$

where ∂D_1^* is the limit of ∂D_1 as $\epsilon \to 0$ and $|\bar{C}_1| = 1$. This implies $\sum_{\alpha=1}^{3} \bar{C}_1^\alpha \psi^\alpha = 0$ on $\partial D_1^* \setminus B_R$. But $\left\{ \psi^\alpha \big|_{\partial D_1^* \setminus B_R} \right\}$ is easily seen to be linear independent, we must have $\bar{C}_1 = 0$. This is a contradiction. Lemma 4.1 for $i = 1$ follows from (4.1) and (4.2). □

4.2 Estimates of $|C_1^\alpha - C_2^\alpha|$, $\alpha = 1, 2$

In the rest of this section, we prove

Proposition 4.2. *Let C_i^α be defined in* (2.1). *Then*

$$|C_1^\alpha - C_2^\alpha| \leq C\sqrt{\epsilon}, \quad \alpha = 1, 2.$$

By the fourth line of (1.5),

$$\sum_{\alpha=1}^{3} C_1^\alpha \int_{\partial D_j} \frac{\partial v_1^\alpha}{\partial \nu_0}\Big|_+ \cdot \psi^\beta + \sum_{\alpha=1}^{3} C_2^\alpha \int_{\partial D_j} \frac{\partial v_2^\alpha}{\partial \nu_0}\Big|_+ \cdot \psi^\beta + \int_{\partial D_j} \frac{\partial v_3}{\partial \nu_0}\Big|_+ \cdot \psi^\beta = 0,$$
$$j = 1, 2; \;\; \beta = 1, 2, 3. \tag{4.3}$$

Denote

$$a_{ij}^{\alpha\beta} = -\int_{\partial D_j} \frac{\partial v_i^\alpha}{\partial \nu_0}\Big|_+ \cdot \psi^\beta, \quad b_j^\beta = \int_{\partial D_j} \frac{\partial v_3}{\partial \nu_0}\Big|_+ \cdot \psi^\beta, \quad i, j = 1, 2; \; \alpha, \beta = 1, 2, 3.$$

Integrating by parts over $\widetilde{\Omega}$ and using (2.2), we have

$$a_{ij}^{\alpha\beta} = \int_{\widetilde{\Omega}} \left(\mathbb{C}^0 e(v_i^\alpha), e(v_j^\beta)\right) dx, \quad b_j^\beta = -\int_{\widetilde{\Omega}} \left(\mathbb{C}^0 e(v_3), e(v_j^\beta)\right) dx.$$

Then (4.3) can be written as

$$\begin{cases} \sum_{\alpha=1}^{3} C_1^\alpha a_{11}^{\alpha\beta} + \sum_{\alpha=1}^{3} C_2^\alpha a_{21}^{\alpha\beta} - b_1^\beta = 0, \\ \sum_{\alpha=1}^{3} C_1^\alpha a_{12}^{\alpha\beta} + \sum_{\alpha=1}^{3} C_2^\alpha a_{22}^{\alpha\beta} - b_2^\beta = 0, \end{cases} \qquad \beta = 1, 2, 3. \tag{4.4}$$

For simplicity, we use a_{ij} to denote the 3×3 matrix $(a_{ij}^{\alpha\beta})$. To estimate $|C_1^\alpha - C_2^\alpha|$, $\alpha = 1, 2$, we only need to use the first three equations in (4.4):

$$a_{11} C_1 + a_{21} C_2 = b_1,$$

where

$$C_1 = (C_1^1, C_1^2, C_1^3)^T, \quad C_2 = (C_2^1, C_2^2, C_2^3)^T, \quad b_1 = (b_1^1, b_1^2, b_1^3)^T.$$

We write the equation as

$$a_{11}(C_1 - C_2) = p := b_1 - (a_{11} + a_{21}) C_2. \tag{4.5}$$

Namely,

$$a_{11}(C_1 - C_2) \equiv \begin{pmatrix} a_{11}^{11} & a_{11}^{12} & a_{11}^{13} \\ a_{11}^{21} & a_{11}^{22} & a_{11}^{23} \\ a_{11}^{31} & a_{11}^{32} & a_{11}^{33} \end{pmatrix} \begin{pmatrix} C_1^1 - C_2^1 \\ C_1^2 - C_2^2 \\ C_1^3 - C_2^3 \end{pmatrix} = \begin{pmatrix} p^1 \\ p^2 \\ p^3 \end{pmatrix}. \tag{4.6}$$

We will show that a_{11} is positive definite, which we assume for the time being. By Cramer's rule, we see from (4.6),

$$C_1^1 - C_2^1 = \frac{1}{\det a_{11}} \begin{vmatrix} p^1 & a_{11}^{12} & a_{11}^{13} \\ p^2 & a_{11}^{22} & a_{11}^{23} \\ p^3 & a_{11}^{32} & a_{11}^{33} \end{vmatrix}, \quad C_1^2 - C_2^2 = \frac{1}{\det a_{11}} \begin{vmatrix} a_{11}^{11} & p^1 & a_{11}^{13} \\ a_{11}^{21} & p^2 & a_{11}^{23} \\ a_{11}^{31} & p^3 & a_{11}^{33} \end{vmatrix}.$$

Therefore

$$C_1^1 - C_2^1 = \frac{1}{\det a_{11}} \left(p^1 \begin{vmatrix} a_{11}^{22} & a_{11}^{23} \\ a_{11}^{32} & a_{11}^{33} \end{vmatrix} - p^2 \begin{vmatrix} a_{11}^{12} & a_{11}^{13} \\ a_{11}^{32} & a_{11}^{33} \end{vmatrix} + p^3 \begin{vmatrix} a_{11}^{12} & a_{11}^{13} \\ a_{11}^{22} & a_{11}^{23} \end{vmatrix} \right), \tag{4.7}$$

and

$$C_1^2 - C_2^2 = \frac{1}{\det a_{11}} \left(-p^1 \begin{vmatrix} a_{11}^{21} & a_{11}^{23} \\ a_{11}^{31} & a_{11}^{33} \end{vmatrix} + p^2 \begin{vmatrix} a_{11}^{11} & a_{11}^{13} \\ a_{11}^{31} & a_{11}^{33} \end{vmatrix} - p^3 \begin{vmatrix} a_{11}^{11} & a_{11}^{13} \\ a_{11}^{21} & a_{11}^{23} \end{vmatrix} \right). \tag{4.8}$$

In order to prove Proposition 4.2, we first study the right hand side of (4.6) and have the following estimates.

Lemma 4.3.

$$\left|a_{11}^{\alpha\beta}+a_{21}^{\alpha\beta}\right|\leq C,\quad \alpha,\beta=1,2,3;$$

$$\left|b_1^{\beta}\right|\leq C,\quad \beta=1,2,3.$$

Consequently,

$$|p|\leq C. \tag{4.9}$$

Proof. For $\beta=1,2,3$, using (3.21) and (3.45),

$$\int_{\widetilde{\Omega}}\left|\nabla v_1^{\beta}\right|dx\leq\int_{\Omega_{R/2}}\left|\nabla v_1^{\beta}\right|dx+\int_{\widetilde{\Omega}\setminus\Omega_{R/2}}\left|\nabla v_1^{\beta}\right|dx\leq C. \tag{4.10}$$

For $\alpha,\beta=1,2,3$, by Lemma 3.1 and (4.10), we have

$$\begin{aligned}\left|a_{11}^{\alpha\beta}+a_{21}^{\alpha\beta}\right|&=\left|\int_{\widetilde{\Omega}}\left(\mathbb{C}^0e(v_1^{\alpha}+v_2^{\alpha}),e(v_1^{\beta})\right)dx\right|\\&\leq C\left\|\nabla(v_1^{\alpha}+v_2^{\alpha})\right\|_{L^{\infty}(\widetilde{\Omega})}\int_{\widetilde{\Omega}}\left|\nabla v_1^{\beta}\right|dx\\&\leq C.\end{aligned}$$

Similarly, it follows from Lemma 3.1 and (4.10) that

$$\left|b_1^{\beta}\right|=\left|\int_{\widetilde{\Omega}}\left(\mathbb{C}^0e(v_1^{\beta}),e(v_3)\right)dx\right|\leq C\|\nabla v_3\|_{L^{\infty}(\widetilde{\Omega})}\int_{\widetilde{\Omega}}\left|\nabla v_1^{\beta}\right|dx\leq C,\quad \beta=1,2,3.$$

Lemma 4.3 follows immediately, in view of Lemma 4.1. □

Lemma 4.4. *a_{11} is positive definite, and*

$$\frac{1}{C\sqrt{\epsilon}}\leq a_{11}^{\alpha\alpha}\leq\frac{C}{\sqrt{\epsilon}},\quad \alpha=1,2, \tag{4.11}$$

$$\frac{1}{C}\leq a_{11}^{33}\leq C,\quad \alpha=1,2; \tag{4.12}$$

$$\left|a_{11}^{12}\right|=\left|a_{11}^{21}\right|\leq\frac{C}{\epsilon^{1/4}}, \tag{4.13}$$

$$|a_{11}^{\alpha 3}|=|a_{11}^{3\alpha}|\leq C,\quad \alpha=1,2; \tag{4.14}$$

and

$$\frac{1}{C\epsilon}\leq\det a_{11}\leq\frac{C}{\epsilon}. \tag{4.15}$$

Proof. **STEP 1.** Proof of (4.11) and (4.12).

For any $\xi=(\xi^{(1)},\xi^{(2)},\xi^{(3)})^T\neq 0$,

$$\xi^Ta_{11}\xi=\int_{\widetilde{\Omega}}\left(\mathbb{C}^0e\left(\xi^{(\alpha)}v_1^{\alpha}\right),e\left(\xi^{(\beta)}v_1^{\beta}\right)\right)dx\geq\frac{1}{C}\int_{\widetilde{\Omega}}\left|e\left(\xi^{(\alpha)}v_1^{\alpha}\right)\right|^2dx>0.$$

In the last inequality we have used the fact that $e\left(\xi^{(\alpha)}v_1^\alpha\right)$ is not identically zero. Indeed if $e\left(\xi^{(\alpha)}v_1^\alpha\right) = 0$, then $\xi^{(\alpha)}v_1^\alpha = a\psi^1 + b\psi^2 + c\psi^3$ in $\widetilde{\Omega}$ for some constants a, b and c. On the other hand, $\xi^{(\alpha)}v_1^\alpha = 0$ on ∂D_2, and $\psi^1\big|_{\partial D_2}$, $\psi^2\big|_{\partial D_2}$ and $\psi^3\big|_{\partial D_2}$ are clearly independent. This implies that $a = b = c = 0$. Thus on ∂D_1, $\xi^{(\alpha)}v_1^\alpha = 0$, violating the linear independence of $\psi^1\big|_{\partial D_1}$, $\psi^2\big|_{\partial D_1}$ and $\psi^3\big|_{\partial D_1}$. We have proved that a_{11} is positive definite.

By (1.8), (2.12) and (2.7),

$$a_{11}^{\alpha\alpha} = \int_{\widetilde{\Omega}} \left(\mathbb{C}^0 e\left(v_1^\alpha\right), e\left(v_1^\alpha\right)\right) dx \leq C \int_{\widetilde{\Omega}} \left|\nabla v_1^\alpha\right|^2 dx \leq \frac{C}{\sqrt{\epsilon}}, \quad \alpha = 1, 2.$$

With (3.17), we have, by (3.18),

$$\begin{aligned} a_{11}^{11} &= \int_{\widetilde{\Omega}} \left(\mathbb{C}^0 e\left(v_1^1\right), e\left(v_1^1\right)\right) dx \geq \frac{1}{C} \int_{\widetilde{\Omega}} \left|e\left(v_1^1\right)\right|^2 dx \\ &\geq \frac{1}{2C} \int_{\widetilde{\Omega}} \left|e\left(\bar{u}_1^1\right)\right|^2 dx - C \int_{\widetilde{\Omega}} \left|e\left(w_1^1\right)\right|^2 dx \\ &\geq \frac{1}{2C} \int_{\widetilde{\Omega}} \left|e\left(\bar{u}_1^1\right)\right|^2 dx - C. \end{aligned}$$

Since

$$\left|e\left(\bar{u}_1^1\right)\right|^2 \geq \frac{1}{4}|\partial_{x_2}\bar{u}|^2, \tag{4.16}$$

we have

$$\begin{aligned} \int_{\widetilde{\Omega}} \left|e\left(\bar{u}_1^1\right)\right|^2 dx &\geq \frac{1}{4} \int_{\widetilde{\Omega}} |\partial_{x_2}\bar{u}|^2 dx \geq \frac{1}{4} \int_{\Omega_R} \frac{dx}{(\epsilon + h_1(x_1) - h_2(x_1))^2} \\ &\geq \frac{1}{C} \int_{\Omega_R} \frac{dx}{(\epsilon + |x_1|^2)^2} \geq \frac{1}{C\sqrt{\epsilon}}. \end{aligned}$$

Thus

$$a_{11}^{11} \geq \frac{1}{C\sqrt{\epsilon}}.$$

Similarly, we have

$$a_{11}^{22} \geq \frac{1}{C\sqrt{\epsilon}}.$$

Estimate (4.11) is proved.

By Lemma 3.4,

$$a_{11}^{33} = \int_{\widetilde{\Omega}} \left(\mathbb{C}^0 e\left(v_1^3\right), e\left(v_1^3\right)\right) dx \leq C.$$

By a version of the second Korn's inequality,

$$a_{11}^{33} \geq \frac{1}{C} \int_{\Omega_R \setminus \Omega_{R/2}} |e\left(v_1^3\right)|^2 dx \geq \frac{1}{C} \int_{\Omega_R \setminus \Omega_{R/2}} |\nabla v_1^3|^2 dx \geq \frac{1}{C}.$$

Estimate (4.12) is proved.

STEP 2. Proof of (4.13).

Notice that

$$a_{11}^{12} = a_{11}^{21} = \int_{\widetilde{\Omega}} \left(\mathbb{C}^0 e\left(v_1^1\right), e(v_1^2)\right) dx = \int_{\widetilde{\Omega}} \left(\mathbb{C}^0 \nabla v_1^1, \nabla v_1^2\right) dx.$$

With (3.17), we have

$$\begin{aligned}\int_{\Omega_{R/2}} \left(\mathbb{C}^0 \nabla v_1^1, \nabla v_1^2\right) dx &= \int_{\Omega_{R/2}} \left(\mathbb{C}^0 \nabla\left(\overline{u}_1^1 + w_1^1\right), \nabla\left(\overline{u}_1^2 + w_1^2\right)\right) dx \\ &= \int_{\Omega_{R/2}} \left(\mathbb{C}^0 \nabla \overline{u}_1^1, \nabla \overline{u}_1^2\right) dx + \int_{\Omega_{R/2}} \left(\mathbb{C}^0 \nabla \overline{u}_1^1, \nabla w_1^2\right) dx \\ &\quad + \int_{\Omega_{R/2}} \left(\mathbb{C}^0 \nabla \overline{u}_1^2, \nabla w_1^1\right) dx + \int_{\Omega_{R/2}} \left(\mathbb{C}^0 \nabla w_1^1, \nabla w_1^2\right) dx. \end{aligned} \tag{4.17}$$

By the definition $\overline{u}_1^1 = (\overline{u}, 0)^T$ and $\overline{u}_1^2 = (0, \overline{u})^T$, we have

$$\nabla \overline{u}_1^1 = \begin{pmatrix} \partial_{x_1}\overline{u} & \partial_{x_2}\overline{u} \\ 0 & 0 \end{pmatrix}, \quad \text{and} \quad \nabla \overline{u}_1^2 = \begin{pmatrix} 0 & 0 \\ \partial_{x_1}\overline{u} & \partial_{x_2}\overline{u} \end{pmatrix}. \tag{4.18}$$

By (3.18),

$$\left|\int_{\Omega_{R/2}} \left(\mathbb{C}^0 \nabla w_1^1, \nabla w_1^2\right) dx\right| \leq C \left(\int_{\Omega_{R/2}} \left|\nabla w_1^1\right|^2 dx\right)^{1/2} \left(\int_{\Omega_{R/2}} \left|\nabla w_1^2\right|^2 dx\right)^{1/2} \leq C,$$

and

$$\left|\int_{\Omega_{R/2}} \left(\mathbb{C}^0 \nabla \overline{u}_1^1, \nabla w_1^2\right) dx\right| \leq C \left(\int_{\Omega_{R/2}} \left|\nabla \overline{u}_1^1\right|^2 dx\right)^{1/2} \left(\int_{\Omega_{R/2}} \left|\nabla w_1^2\right|^2 dx\right)^{1/2} \leq \frac{C}{\epsilon^{1/4}}. \tag{4.19}$$

Similarly,

$$\left|\int_{\Omega_{R/2}} \left(\mathbb{C}^0 \nabla \overline{u}_1^2, \nabla w_1^1\right) dx\right| \leq \frac{C}{\epsilon^{1/4}}. \tag{4.20}$$

On the other hand,

$$\left(\mathbb{C}^0 \nabla \overline{u}_1^1, \nabla \overline{u}_1^2\right) = \begin{pmatrix} (\lambda + 2\mu)\partial_{x_1}\overline{u} & \mu\partial_{x_2}\overline{u} \\ \mu\partial_{x_2}\overline{u} & \lambda\partial_{x_1}\overline{u} \end{pmatrix} : \begin{pmatrix} 0 & 0 \\ \partial_{x_1}\overline{u} & \partial_{x_2}\overline{u} \end{pmatrix} = (\lambda + \mu)\partial_{x_1}\overline{u}\partial_{x_2}\overline{u}.$$

Thus,

$$\left|\int_{\Omega_{R/2}} \left(\mathbb{C}^0 \nabla \overline{u}_1^1, \nabla \overline{u}_1^2\right) dx\right| \leq C \int_{\Omega_{R/2}} |\partial_{x_1}\overline{u}||\partial_{x_2}\overline{u}| dx \leq C \int_{\Omega_{R/2}} \frac{|x_1| dx}{(\epsilon + |x_1|^2)^2} \leq C|\ln \epsilon|.$$

Substituting these estimates above into (4.17), and using (3.21), we have

$$\left|a_{11}^{12}\right| = \left|a_{11}^{21}\right| = \left|\int_{\widetilde{\Omega}} \left(\mathbb{C}^0 \nabla v_1^1, \nabla v_1^2\right) dx\right| \leq \left|\int_{\Omega_{R/2}} \left(\mathbb{C}^0 \nabla v_1^1, \nabla v_1^2\right) dx\right| + C \leq \frac{C}{\epsilon^{1/4}}.$$

The proof of (4.13) is finished.

STEP 3. Proof of (4.14).

$$a_{11}^{\alpha 3} = a_{11}^{3\alpha} = \int_{\widetilde{\Omega}} \left(\mathbb{C}^0 e\left(v_1^\alpha\right), e(v_1^3)\right) dx = \int_{\widetilde{\Omega}} \left(\mathbb{C}^0 \nabla v_1^\alpha, \nabla v_1^3\right) dx, \quad \alpha = 1, 2.$$

Similarly to the above, using (3.18) and (3.47), we have, for $\alpha = 1$,

$$\begin{aligned}
a_{11}^{13} &= \int_{\Omega_{R/2}} \left(\mathbb{C}^0 \nabla v_1^1, \nabla v_1^3\right) dx + O(1) \\
&= \int_{\Omega_{R/2}} \left(\mathbb{C}^0 \nabla \overline{u}_1^1, \nabla \overline{u}_1^3\right) dx + \int_{\Omega_{R/2}} \left(\mathbb{C}^0 \nabla \overline{u}_1^1, \nabla w_1^3\right) dx \\
&\quad + \int_{\Omega_{R/2}} \left(\mathbb{C}^0 \nabla \overline{u}_1^3, \nabla w_1^1\right) dx + \int_{\Omega_{R/2}} \left(\mathbb{C}^0 \nabla w_1^1, \nabla w_1^3\right) dx + O(1) \\
&= \int_{\Omega_{R/2}} \left(\mathbb{C}^0 \nabla \overline{u}_1^1, \nabla \overline{u}_1^3\right) dx + \int_{\Omega_{R/2}} \left(\mathbb{C}^0 \nabla \overline{u}_1^3, \nabla w_1^1\right) dx + \int_{\Omega_{R/2}} \left(\mathbb{C}^0 \nabla \overline{u}_1^1, \nabla w_1^3\right) dx + O(1) \\
&=: I + II + III + O(1).
\end{aligned}$$

By the definition of $\overline{u}_1^3 = (x_2\overline{u}, -x_1\overline{u})^T$, we have

$$\nabla \overline{u}_1^3 = \begin{pmatrix} x_2 \partial_{x_1} \overline{u} & \overline{u} + x_2 \partial_{x_2} \overline{u} \\ -\overline{u} - x_1 \partial_{x_1} \overline{u} & -x_1 \partial_{x_2} \overline{u} \end{pmatrix}.$$

Then

$$\begin{aligned}
\left(\mathbb{C}^0 \nabla \overline{u}_1^1, \nabla \overline{u}_1^3\right) &= \begin{pmatrix} (\lambda + 2\mu)\partial_{x_1} \overline{u} & \mu \partial_{x_2} \overline{u} \\ \mu \partial_{x_2} \overline{u} & \lambda \partial_{x_1} \overline{u} \end{pmatrix} : \begin{pmatrix} x_2 \partial_{x_1} \overline{u} & \overline{u} + x_2 \partial_{x_2} \overline{u} \\ -\overline{u} - x_1 \partial_{x_1} \overline{u} & -x_1 \partial_{x_2} \overline{u} \end{pmatrix} \\
&= (\lambda + 2\mu) x_2 \left(\partial_{x_1} \overline{u}\right)^2 + \mu\, x_2 \left(\partial_{x_2} \overline{u}\right)^2 - (\lambda + \mu) x_1 \partial_{x_1} \overline{u} \partial_{x_2} \overline{u}.
\end{aligned}$$

Hence, by (3.7),

$$\begin{aligned}
|I| &= \left| \int_{\Omega_{R/2}} \left(\mathbb{C}^0 \nabla \overline{u}_1^1, \nabla \overline{u}_1^3\right) dx \right| \\
&\leq C \left(\int_{\Omega_{R/2}} \frac{|x_2||x_1|^2}{(\epsilon + |x_1|^2)^2} dx + \int_{\Omega_{R/2}} \frac{|x_2|}{(\epsilon + |x_1|^2)^2} dx + \int_{\Omega_{R/2}} \frac{|x_1|^2}{(\epsilon + |x_1|^2)^2} dx \right) \\
&\leq C.
\end{aligned}$$

By (3.18) and (3.42),

$$|II| = \left| \int_{\Omega_{R/2}} \left(\mathbb{C}^0 \nabla \overline{u}_1^3, \nabla w_1^1\right) dx \right| \leq C \left(\int_{\Omega_{R/2}} \left|\nabla \overline{u}_1^3\right|^2 dx \right)^{1/2} \left(\int_{\Omega_{R/2}} \left|\nabla w_1^1\right|^2 dx \right)^{1/2} \leq C.$$

While, by (3.55),

$$|III| = \left| \int_{\Omega_{R/2}} \left(\mathbb{C}^0 \nabla \overline{u}_1^1, \nabla w_1^3\right) dx \right| \leq C \int_{\Omega_{R/2}} \left|\nabla \overline{u}_1^1\right| dx \leq C.$$

Therefore

$$\left|a_{11}^{13}\right| \leq C.$$

Similarly, using (3.18) and (3.47),

$$
\begin{aligned}
a_{11}^{23} &= \int_{\Omega_{R/2}} \left(\mathbb{C}^0\nabla v_1^2, \nabla v_1^3\right) dx + O(1) \\
&= \int_{\Omega_{R/2}} \left(\mathbb{C}^0\nabla \overline{u}_1^2, \nabla \overline{u}_1^3\right) dx + \int_{\Omega_{R/2}} \left(\mathbb{C}^0\nabla \overline{u}_1^2, \nabla w_1^3\right) dx \\
&\quad + \int_{\Omega_{R/2}} \left(\mathbb{C}^0\nabla \overline{u}_1^3, \nabla w_1^2\right) dx + \int_{\Omega_{R/2}} \left(\mathbb{C}^0\nabla w_1^2, \nabla w_1^3\right) dx + O(1) \\
&= \int_{\Omega_{R/2}} \left(\mathbb{C}^0\nabla \overline{u}_1^2, \nabla \overline{u}_1^3\right) dx + \int_{\Omega_{R/2}} \left(\mathbb{C}^0\nabla \overline{u}_1^2, \nabla w_1^3\right) dx + O(1).
\end{aligned}
$$

By the definition $\overline{u}_1^2$ and $\overline{u}_1^3$, we have

$$
\begin{aligned}
\left(\mathbb{C}^0\nabla \overline{u}_1^2, \nabla \overline{u}_1^3\right) &= \begin{pmatrix} \lambda\partial_{x_2}\overline{u} & \mu\partial_{x_1}\overline{u} \\ \mu\partial_{x_1}\overline{u} & (\lambda+2\mu)\partial_{x_2}\overline{u} \end{pmatrix} : \begin{pmatrix} x_2\partial_{x_1}\overline{u} & \overline{u} + x_2\partial_{x_2}\overline{u} \\ -\overline{u} - x_1\partial_{x_1}\overline{u} & -x_1\partial_{x_2}\overline{u} \end{pmatrix} \\
&= (\lambda+\mu)x_2\partial_{x_1}\overline{u}\partial_{x_2}\overline{u} - \mu\, x_1(\partial_{x_1}\overline{u})^2 - (\lambda+2\mu)x_1(\partial_{x_2}\overline{u})^2.
\end{aligned}
$$

Hence, using (3.7), we have

$$
\begin{aligned}
&\int_{\Omega_{R/2}} \left(\mathbb{C}^0\nabla \overline{u}_1^2, \nabla \overline{u}_1^3\right) dx \\
&= -(\lambda+2\mu)\int_{\Omega_{R/2}} x_1(\partial_{x_2}\overline{u})^2 dx + O(1) \\
&= -(\lambda+2\mu)\int_{|x_1|<R/2} x_1\left(\frac{1}{\epsilon + h_1(x_1) - h_2(x_1)} - \frac{1}{\epsilon + \frac{1}{2}(h_1''(0) - h_2''(0))x_1^2}\right) dx_1 + O(1) \\
&= O(1).
\end{aligned}
$$

Therefore

$$
\left|a_{11}^{23}\right| \leq C.
$$

Lemma 4.4 is proved. □

Proof of Proposition 4.2. By (4.7), Lemma 4.3 and Lemma 4.4,

$$
C_1^1 - C_2^1 = \frac{1}{\det a_{11}}\left(\left(p^1 a_{11}^{22}a_{11}^{33} - p^3 a_{11}^{22}a_{11}^{13}\right) + O(\frac{1}{\epsilon^{1/4}})\right).
$$

Therefore

$$
\left|C_1^1 - C_2^1\right| \leq C\sqrt{\epsilon}.
$$

Similarly, using (4.8),

$$
C_1^2 - C_2^2 = \frac{1}{\det a_{11}}\left(\left(p^2 a_{11}^{11}a_{11}^{33} - p^3 a_{11}^{11}a_{11}^{23}\right) + O(\frac{1}{\epsilon^{1/4}})\right).
$$

Therefore

$$
\left|C_1^2 - C_2^2\right| \leq C\sqrt{\epsilon}.
$$

The proof is completed. □

Proof of Proposition 2.1. Estimates (2.5) and (2.6) have been proved in Lemma 3.1; estimate (2.7) has been proved in Corollary 3.3; estimate (2.8) has been proved in Lemma 3.4 and Lemma 3.6; estimate (2.9) has been proved in Lemma 4.1; and estimate (2.10) has been proved in Proposition 4.2. The proof of Proposition 2.1 is completed. □

5 More general D_1 and D_2

As mentioned in the introduction, the strict convexity assumption on ∂D_1 and ∂D_2 can be weakened. In fact, our proof of Theorem 1.1 applies, with minor modification, to more general situations.

In $\mathbb{R}^2$, under the same assumptions in the beginning of Section 3 except for the strict convexity condition, ∂D_i near P_i can be represented by the graphs of $x_2 = \frac{\epsilon}{2} + h_1(x_1)$, and $x_2 = -\frac{\epsilon}{2} + h_2(x_1)$, for $|x_1| < 2R$. We assume that $h_1, h_2 \in C^2([-2R, 2R])$ and (3.1) still holds. Instead of the convexity assumption, we assume that

$$\Lambda_0 |x_1|^m \leq \; h_1(x_1) - h_2(x_1) \leq \Lambda_1 |x_1|^m, \quad \text{for } \; |x_1| < 2R, \tag{5.1}$$

and

$$|h_i'(x_1)| \leq \; C|x_1|^{m-1}, \quad |h_i''(x_1)| \leq \; C|x_1|^{m-2}, \quad i = 1, 2, \quad \text{for } \; |x_1| < 2R, \tag{5.2}$$

for some $\epsilon-$independent constants $0 < \Lambda_0 < \Lambda_1$, and $m \geq 2$. Define $\delta := \delta(z_1)$ as (3.14). Clearly,

$$\frac{1}{C}(\epsilon + |z_1|^m) \leq \delta(z_1) \leq \; C(\epsilon + |z_1|^m). \tag{5.3}$$

Then

Theorem 5.1. *Under the above assumptions with $m \geq 2$, let $u \in H^1(\Omega; \mathbb{R}^2) \cap C^1(\overline{\widetilde{\Omega}}; \mathbb{R}^2)$ be a solution to* (1.5)*. Then for $0 < \epsilon < 1$, we have*

$$|\nabla u(x)| \leq \begin{cases} C\dfrac{\epsilon^{1-\frac{1}{m}} + \mathrm{dist}(x, \overline{P_1P_2})}{\epsilon + \mathrm{dist}^m(x, \overline{P_1P_2})} \|\varphi\|_{C^{1,\gamma}(\partial\Omega;\mathbb{R}^2)}, & x \in \widetilde{\Omega}, \\ \\ C\|\varphi\|_{C^{1,\gamma}(\partial\Omega;\mathbb{R}^2)}, & x \in D_1 \cup D_2. \end{cases} \tag{5.4}$$

where C is a universal constant. In particular,

$$\|\nabla u\|_{L^\infty(\Omega)} \leq C\epsilon^{\frac{1}{m}-1} \|\varphi\|_{C^{1,\gamma}(\partial\Omega;\mathbb{R}^2)}. \tag{5.5}$$

In the following, we only list the main differences. We define $\bar{u}$ by (3.5) as before. A calculation gives

$$|\partial_{x_1}\bar{u}(x)| \leq \frac{C|x_1|^{m-1}}{\epsilon + |x_1|^m}, \quad |\partial_{x_2}\bar{u}(x)| \leq \frac{C}{\epsilon + |x_1|^m}, \quad x \in \Omega_R, \tag{5.6}$$

by (3.3), we have

$$|\partial_{x_1x_1}\bar{u}(x)| \leq \frac{C|x_1|^{m-2}}{\epsilon + |x_1|^m}, \;\; |\partial_{x_1x_2}\bar{u}(x)| \leq \frac{C|x_1|^{m-1}}{(\epsilon + |x_1|^m)^2}, \;\; \partial_{x_2x_2}\bar{u}(x) = 0, \;\; x \in \Omega_R. \tag{5.7}$$

Define $\overline{u}_i^\alpha$, $i, \alpha = 1,2$ as in (3.9) and (3.10). By (1.6), (5.6) and (5.7), we have

$$|\mathcal{L}_{\lambda,\mu}\overline{u}_i^\alpha(x)| \leq \frac{C|x_1|^{m-2}}{\epsilon + |x_1|^m} + \frac{C|x_1|^{m-1}}{(\epsilon + |x_1|^m)^2}, \quad i, \alpha = 1,2, \quad x \in \Omega_R. \tag{5.8}$$

Instead of Proposition 2.1, we have

Proposition 5.2. *Under the hypotheses of Theorem 5.1 and a normalization* $\|\varphi\|_{C^{1,\gamma}(\partial\Omega)} = 1$*, we have, for* $0 < \epsilon < 1$*,*

$$\|\nabla v_3\|_{L^\infty(\widetilde{\Omega})} \leq C; \tag{5.9}$$

$$\|\nabla v_1^\alpha + \nabla v_2^\alpha\|_{L^\infty(\widetilde{\Omega})} \leq C, \quad \alpha = 1,2,3; \tag{5.10}$$

$$|\nabla v_i^\alpha(x)| \leq \frac{C}{\epsilon + \mathrm{dist}^m(x, \overline{P_1P_2})}, \quad i, \alpha = 1,2, \quad x \in \widetilde{\Omega}; \tag{5.11}$$

$$|\nabla v_i^3(x)| \leq C\frac{\epsilon + \mathrm{dist}(x, \overline{P_1P_2})}{\epsilon + \mathrm{dist}^m(x, \overline{P_1P_2})}, \quad i = 1,2, \quad x \in \widetilde{\Omega}; \tag{5.12}$$

and

$$|C_i^\alpha| \leq C, \quad i = 1,2, \ \alpha = 1,2,3; \tag{5.13}$$

$$|C_1^\alpha - C_2^\alpha| \leq C\epsilon^{1-\frac{1}{m}}, \quad \alpha = 1,2. \tag{5.14}$$

Denote

$$w_i^\alpha := v_i^\alpha - \overline{u}_i^\alpha, \quad i = 1,2, \ \alpha = 1,2,3.$$

Then, instead of Proposition 3.2, we have

Proposition 5.3. *Assume the above, let* $v_i^\alpha \in C^2(\widetilde{\Omega}; \mathbb{R}^2) \cap C^1(\overline{\widetilde{\Omega}}; \mathbb{R}^2)$ *be the weak solution of* (2.2)*. Then, for* $i, \alpha = 1,2$*,*

$$\int_{\widetilde{\Omega}} \left|\nabla w_i^\alpha\right|^2 dx \leq C, \tag{5.15}$$

$$\int_{\widehat{\Omega}_\delta(z_1)} \left|\nabla w_i^\alpha\right|^2 dx \leq \begin{cases} C\left(\epsilon^{2m-2} + |z_1|^{2m-2}\right), & |z_1| \leq \sqrt[m]{\epsilon}, \\ C|z_1|^{2m-2}, & \sqrt[m]{\epsilon} < |z_1| \leq R, \end{cases} \tag{5.16}$$

and

$$\left|\nabla w_i^\alpha(x)\right| \leq \begin{cases} C\frac{\epsilon^{m-1} + |x_1|^{m-1}}{\epsilon}, & |x_1| \leq \sqrt[m]{\epsilon}, \\ \frac{C}{|x_1|}, & \sqrt[m]{\epsilon} < |x_1| \leq R. \end{cases} \tag{5.17}$$

Proof. The proof of (5.15) is the same as that of (3.18). We only list the main differences from STEP 2 and STEP 3 in the proof of Proposition 3.2.
STEP 2. Proof of (5.16).
Case 1. For $\sqrt[m]{\epsilon} \leq |z_1| \leq R/2$**.**

Note that for $0 < s < \frac{2|z_1|}{3}$, we have

$$\begin{aligned} \int_{\widehat{\Omega}_s(z_1)} |w|^2 dx &\leq \int_{|x_1 - z_1| \leq s} (\epsilon + h_1(x_1) - h_2(x_1))^2 \int_{-\frac{\epsilon}{2} + h_2(x_1)}^{\frac{\epsilon}{2} + h_1(x_1)} |\partial_{x_2} w(x_1, x_2)|^2 dx_2 dx_1 \\ &\leq C|z_1|^{2m} \int_{\widehat{\Omega}_s(z_1)} |\nabla w|^2 dx, \end{aligned} \tag{5.18}$$

By (5.8), we have

$$\begin{aligned}\int_{\widehat{\Omega}_s(z_1)} \left|\mathcal{L}_{\lambda,\mu}\overline{u}_1^1\right|^2 dx &\leq \int_{\widehat{\Omega}_s(z_1)} \left(\frac{C|x_1|^{m-2}}{\epsilon+|x_1|^m}+\frac{C|x_1|^{m-1}}{(\epsilon+|x_1|^m)^2}\right)^2 dx\\ &\leq \frac{C|z_1|^m s}{|z_1|^{2(m+1)}} \leq \frac{Cs}{|z_1|^{m+2}}, \qquad 0<s<\frac{2|z_1|}{3}.\end{aligned} \tag{5.19}$$

As before, it follows from the above and (3.30) that

$$\widehat{F}(t) \leq \left(\frac{C_0|z_1|^m}{s-t}\right)^2 \widehat{F}(s) + C(s-t)^2\frac{s}{|z_1|^{m+2}}, \quad \forall\ 0<t<s<\frac{2|z_1|}{3}, \tag{5.20}$$

where C_0 is also a universal constant.

Let $t_i = 2C_0 i\,|z_1|^m$, $i = 1, 2, \cdots$. Then

$$\frac{C_0|z_1|^m}{t_{i+1}-t_i} = \frac{1}{2}.$$

Let $k = \left[\frac{1}{4C_0|z_1|^{m-1}}\right]$. Then by (5.20) with $s = t_{i+1}$ and $t = t_i$, we have

$$\widehat{F}(t_i) \leq \frac{1}{4}\widehat{F}(t_{i+1}) + \frac{C(t_{i+1}-t_i)^2 t_{i+1}}{|z_1|^{m+2}} \leq \frac{1}{4}\widehat{F}(t_{i+1}) + C(i+1)|z_1|^{2m-2},$$

After k iterations, we have, using (5.15),

$$\begin{aligned}\widehat{F}(t_1) &\leq \left(\frac{1}{4}\right)^k \widehat{F}(t_{k+1}) + C|z_1|^{2m-2}\sum_{l=1}^{k}\left(\frac{1}{4}\right)^{l-1}(l+1)\\ &\leq C|z_1|^{2m-2}.\end{aligned}$$

This implies that

$$\int_{\widehat{\Omega}_\delta(z_1)} |\nabla w|^2 dx \leq C|z_1|^{2m-2}.$$

Case 2. For $|z_1| \leq \sqrt[m]{\epsilon}$.

For $0<t<s<\sqrt[m]{\epsilon}$, estimate (5.18) becomes

$$\int_{\widehat{\Omega}_s(z_1)} |w|^2 dx \leq C\epsilon^2 \int_{\widehat{\Omega}_s(z_1)} |\nabla w|^2 dx, \quad 0<s<\sqrt[m]{\epsilon}; \tag{5.21}$$

Estimate (5.19) becomes

$$\begin{aligned}\int_{\widehat{\Omega}_s(z_1)} |\mathcal{L}_{\lambda,\mu}\overline{u}_1^1|^2 dx &\leq \int_{\widehat{\Omega}_s(z_1)} \left(\frac{C|x_1|^{m-2}}{\epsilon+|x_1|^m}+\frac{C|x_1|^{m-1}}{(\epsilon+|x_1|^m)^2}\right)^2 dx\\ &\leq \frac{Cs}{\epsilon} + \frac{C(|z_1|^{2m-2}+s^{2m-2})s}{\epsilon^3}, \quad \text{for } 0<s<\sqrt[m]{\epsilon};\end{aligned} \tag{5.22}$$

Estimate (5.20) becomes, in view of (3.30),

$$\widehat{F}(t) \leq \left(\frac{C_0\epsilon}{s-t}\right)^2 \widehat{F}(s) + C(s-t)^2 s\left(\frac{1}{\epsilon}+\frac{|z_1|^{2m-2}}{\epsilon^3}+\frac{s^{2m-2}}{\epsilon^3}\right), \quad \forall\ 0<t<s<\sqrt[m]{\epsilon}. \tag{5.23}$$

Let $t_i = 2C_0 i\epsilon$, $i = 1, 2, \cdots$. Then

$$\frac{C_0\epsilon}{t_{i+1} - t_i} = \frac{1}{2}.$$

Let $k = \left[\frac{1}{4C_0\epsilon^{1-\frac{1}{m}}}\right]$. Then by (3.36) with $s = t_{i+1}$ and $t = t_i$, we have

$$\widehat{F}(t_i) \le \frac{1}{4}\widehat{F}(t_{i+1}) + Ci^3\left(\epsilon^{2m-2} + |z_1|^{2m-2}\right).$$

After k iterations, we have, using (5.15),

$$\begin{aligned}
\widehat{F}(t_1) &\le \left(\frac{1}{4}\right)^k \widehat{F}(t_{k+1}) + C\sum_{l=1}^{k}\left(\frac{1}{4}\right)^{l-1} l^3\left(\epsilon^{2m-2} + |z_1|^{2m-2}\right)\\
&\le C\left(\frac{1}{4}\right)^{\frac{1}{C\epsilon^{1-\frac{1}{m}}}} + C\left(\epsilon^{2m-2} + |z_1|^{2m-2}\right) \le C\left(\epsilon^{2m-2} + |z_1|^{2m-2}\right).
\end{aligned}$$

This implies that

$$\int_{\widehat{\Omega}_\delta(z_1)} |\nabla w|^2 dx \le C\left(\epsilon^{2m-2} + |z_1|^{2m-2}\right).$$

STEP 3. Proof of (5.17).

Using a change of variables (3.37), define Q_r', $\hat{h}_1$, and $\hat{h}_2$ as in the proof of Proposition 3.2. Then by (5.2),

$$|\hat{h}_1'(0)| + |\hat{h}_2'(0)| \le C|z_1|^{m-1}, \quad |\hat{h}_1''(0)| + |\hat{h}_2''(0)| \le C\delta|z_1|^{m-2}.$$

Since R is small, $\|\hat{h}_1\|_{C^{1,1}((-1,1))}$ and $\|\hat{h}_2\|_{C^{1,1}((-1,1))}$ are small and $\frac{1}{2}Q_1'$ is essentially a unit square as far as applications of Sobolev embedding theorems and classical L^p estimates for elliptic systems are concerned. By the same argument as in the proof of Proposition 3.2, (3.40) still holds. We divide into two cases to proceed.

Case 1. For $\sqrt[m]{\epsilon} \le |z_1| \le R/2$.

By (5.16),

$$\int_{\widehat{\Omega}_\delta(z_1)} \left|\nabla w_1^1\right|^2 dx \le C|z_1|^{2m-2}.$$

By (5.8),

$$\delta^2\left|\mathcal{L}_{\lambda,\mu}\overline{u}_1^1\right| \le \delta^2\left(\frac{C}{|z_1|^2} + \frac{C}{|z_1|^{m+1}}\right) \le C|z_1|^{m-1}, \qquad \text{in } \widehat{\Omega}_\delta(z_1).$$

We deduce from (3.40) that

$$\left|\nabla w_1^1(z_1, x_2)\right| = \frac{C|z_1|^{m-1}}{\delta} \le \frac{C}{|z_1|}, \qquad \forall\ -\frac{\epsilon}{2} + h_2(z_1) < x_2 < \frac{\epsilon}{2} + h_1(z_1).$$

Case 2. For $|z_1| \le 2\sqrt[m]{\epsilon}$.

By (5.16),

$$\int_{\widehat{\Omega}_\delta(z_1)} \left|\nabla w_1^1\right|^2 dx \le C(\epsilon^{2m-2} + |z_1|^{2m-2}).$$

By (5.8),

$$\delta^2 \left|\mathcal{L}_{\lambda,\mu}\vec{u}_1^1\right| \leq C\delta^2 \left(\frac{(\epsilon + |z_1|)^{m-2}}{\epsilon} + \frac{(\epsilon + |z_1|)^{m-1}}{\epsilon^2}\right) \leq C(\epsilon + |z_1|)^{m-1}, \quad \text{in } \widehat{\Omega}_\delta(z_1). \tag{5.24}$$

We deduce from (3.40) that

$$\left|\nabla w_1^1(z_1, x_2)\right| = \frac{C}{\delta}\left(\epsilon^{m-1} + |z_1|^{m-1}\right) \leq C\frac{\epsilon^{m-1} + |z_1|^{m-1}}{\epsilon}, \qquad \forall -\frac{\epsilon}{2}+h_2(z_1) < x_2 < \frac{\epsilon}{2}+h_1(z_1).$$

Proposition 5.3 is established. □

Define $\vec{u}_i^3$, $i = 1, 2$ by (3.41). Using (5.1), (5.2) and (5.6), we have

$$\left|\nabla\vec{u}_i^3(x)\right| \leq \frac{C(\epsilon + |x_1|)}{\epsilon + |x_1|^m}, \quad i = 1, 2, \quad x \in \Omega_R, \tag{5.25}$$

and

$$\left|\nabla\vec{u}_i^3(x)\right| \leq C, \quad i = 1, 2, \quad x \in \widetilde{\Omega} \setminus \Omega_R. \tag{5.26}$$

It follows from (1.6), (5.6) and (5.7) that

$$\left|\mathcal{L}_{\lambda,\mu}\vec{u}_i^3\right| \leq \frac{C}{\epsilon + |x_1|^m}, \quad i = 1, 2, \quad x \in \Omega_R. \tag{5.27}$$

Then Lemma 3.4 still holds, while Lemma 3.5 and Lemma 3.6 become

Lemma 5.4. *With $\delta = \delta(z_1)$ in* (3.14), *we have, for $i = 1, 2$,*

$$\int_{\widehat{\Omega}_\delta(z_1)} \left|\nabla w_i^3\right|^2 dx \leq \begin{cases} C\epsilon^2, & |z_1| < \sqrt[m]{\epsilon}, \\ C|z_1|^{2m}, & \sqrt[m]{\epsilon} \leq |z_1| < R/2. \end{cases} \tag{5.28}$$

Proof. The proof is very similar to that of Lemma 3.5. By the same argument, we still have (3.50) holds.

Case 1. $\sqrt[m]{\epsilon} < |z_1| < R/2$.

We still have (5.18) for $0 < s < \frac{2|z_1|}{3}$. Instead of (5.19), we have, using (5.27),

$$\int_{\widehat{\Omega}_s(z_1)} \left|\mathcal{L}_{\lambda,\mu}\vec{u}_1^3\right|^2 dx \leq \frac{Cs}{|z_1|^m}. \tag{5.29}$$

Instead of (5.20), we have

$$\widehat{F}(t) \leq \left(\frac{C_0|z_1|^m}{s-t}\right)^2 \widehat{F}(s) + C(s-t)^2\frac{s}{|z_1|^m}, \quad \forall\, 0 < t < s < \frac{2|z_1|}{3}. \tag{5.30}$$

We define $\{t_i\}$, k and iterate as in the proof of (5.16), right below formula (5.20), to obtain, using (3.47),

$$\widehat{F}(t_1) \leq \left(\frac{1}{4}\right)^k \widehat{F}(\frac{2|z_1|}{3}) + C|z_1|^{2m}\sum_{l=1}^{k}\left(\frac{1}{4}\right)^l l \leq C|z_1|^{2m}.$$

This implies that

$$\int_{\widehat{\Omega}_\delta(z_1)} |\nabla w|^2 dx \leq C|z_1|^{2m}.$$

Case 2. $|z_1| < \sqrt[m]{\epsilon}$**.**

Estimate (5.21) remains the same. Estimate (5.22) becomes

$$\int_{\widehat{\Omega}_s(z_1)} \left|\mathcal{L}_{\lambda,\mu}\bar{u}_1^3\right|^2 dx \leq \frac{Cs}{\epsilon}, \quad 0 < s < \sqrt[m]{\epsilon}. \tag{5.31}$$

Estimate (5.23) becomes

$$\widehat{F}(t) \leq \left(\frac{C_0\epsilon}{s-t}\right)^2 \widehat{F}(s) + \frac{C(s-t)^2 s}{\epsilon}, \quad \forall\ 0 < t < s < \sqrt[m]{\epsilon}. \tag{5.32}$$

Define $\{t_i\}$, k and iterate as in the proof of (3.19), right below formula (3.36), to obtain

$$\widehat{F}(t_1) \leq \left(\frac{1}{4}\right)^k \widehat{F}(t_{k+1}) + C\sum_{l=1}^{k}\left(\frac{1}{4}\right)^{l-1} l\epsilon^2 \leq C\epsilon^2.$$

This implies that

$$\int_{\widehat{\Omega}_\delta(z_1)} |\nabla w|^2 dx \leq C\epsilon^2.$$

□

It is not difficult to obtain

Lemma 5.5.

$$\|\nabla w_i^3\|_{L^\infty(\widetilde{\Omega})} \leq C, \quad i = 1, 2. \tag{5.33}$$

Consequently,

$$|\nabla v_i^3(x)| \leq \frac{C(\epsilon + |x_1|)}{\epsilon + |x_1|^m}, \quad i = 1, 2, \quad x \in \Omega_R. \tag{5.34}$$

The last main difference is the computation of $a_{11}^{\alpha\alpha}$, $\alpha = 1, 2$. In fact, By (1.8), (2.12), (2.7) and (5.15),

$$\begin{aligned}
a_{11}^{\alpha\alpha} &= \int_{\widetilde{\Omega}} \left(\mathbb{C}^0 e\left(v_1^\alpha\right), e\left(v_1^\alpha\right)\right) dx = \int_{\widetilde{\Omega}} \left(\mathbb{C}^0 \nabla v_1^\alpha, \nabla v_1^\alpha\right) dx \\
&\leq C\int_{\widetilde{\Omega}} \left|\nabla v_1^\alpha\right|^2 dx \leq C\int_{\widetilde{\Omega}} \left|\nabla \bar{u}_1^\alpha\right|^2 dx + C\int_{\widetilde{\Omega}} \left|\nabla w_1^\alpha\right|^2 dx \\
&\leq C\int_{-R}^{R} \frac{1}{\epsilon + h_1(x_1) - h_2(x_1)} dx_1 + C \\
&\leq C\int_0^R \frac{1}{\epsilon + |x_1|^m} dx_1 + C \\
&\leq C\epsilon^{\frac{1}{m}-1}, \qquad \alpha = 1, 2.
\end{aligned}$$

Using (5.15) again, we have

$$\begin{aligned} a_{11}^{11} &= \int_{\widetilde{\Omega}} \left(\mathbb{C}^0 e\left(v_1^1\right), e\left(v_1^1\right)\right) dx \geq \frac{1}{C} \int_{\widetilde{\Omega}} \left|e\left(v_1^1\right)\right|^2 dx \\ &\geq \frac{1}{2C} \int_{\widetilde{\Omega}} \left|e\left(\overline{u}_1^1\right)\right|^2 dx - C \int_{\widetilde{\Omega}} \left|e\left(w_1^1\right)\right|^2 dx \\ &\geq \frac{1}{2C} \int_{\widetilde{\Omega}} \left|e\left(\overline{u}_1^1\right)\right|^2 dx - C. \end{aligned}$$

In view of (4.16), we have

$$\begin{aligned} \int_{\widetilde{\Omega}} \left|e\left(\overline{u}_1^1\right)\right|^2 dx \geq \frac{1}{4} \int_{\widetilde{\Omega}} |\partial_{x_2} \overline{u}|^2 dx &\geq \frac{1}{C} \int_{\Omega_R} \frac{dx}{(\epsilon + h_1(x_1) - h_2(x_1))^2} \\ &\geq \frac{1}{C} \int_0^R \frac{1}{\epsilon + |x_1|^m} dx_1 + C \\ &\geq \frac{\epsilon^{\frac{1}{m}-1}}{C}. \end{aligned}$$

Thus

$$a_{11}^{11} \geq \frac{\epsilon^{\frac{1}{m}-1}}{C}.$$

Similarly, we have

$$a_{11}^{22} \geq \frac{\epsilon^{\frac{1}{m}-1}}{C}.$$

By the argument as in the proof of Lemma 4.4, we have

$$\frac{\epsilon^{\frac{2}{m}-2}}{C} \leq \det a_{11} \leq C\epsilon^{\frac{2}{m}-2}.$$

Then, we have

$$|C_1^\alpha - C_2^\alpha| \leq C\epsilon^{1-\frac{1}{m}}, \quad \alpha = 1, 2.$$

The proof of Theorem 5.1 is finished.

6 Appendix: Some results on the Lamé system with infinity coefficients

Assume that in $\mathbb{R}^d$, Ω and ω are bounded open sets with smooth boundaries satisfying

$$\overline{\omega} = \cup_{s=1}^m \overline{\omega}_s \subset \Omega,$$

where $\{\omega_s\}$ are connected components of ω. Clearly, $m < \infty$ and ω_s is open for all $1 \leq s \leq m$. Given $\varphi \in C^{1,\gamma}(\partial\Omega; \mathbb{R}^d)$, $0 < \gamma < 1$, $\mu > 0$, $d\lambda + 2\mu > 0$, and

$$\mu_n^{(s)} \to \infty, \;\; d\lambda_n^{(s)} + 2\mu_n^{(s)} \to \infty, \quad \text{as } \; n \to \infty.$$

We denote

$$\mathbb{C}_n^{(s)} := \lambda_n^{(s)} \delta_{ij}\delta_{kl} + \mu_n^{(s)} \left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \quad 1 \leq s \leq m,$$

$$\mathbb{C}^{(0)} := \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right),$$

and

$$\mathbb{C}_n(x) = \begin{cases} \mathbb{C}_n^{(s)}, & \text{in } \omega_s, \ 1 \leq s \leq m, \\ \mathbb{C}^{(0)}, & \text{in } \Omega \setminus \overline{\omega}. \end{cases}$$

Consider for every n

$$\begin{cases} \nabla \cdot (\mathbb{C}_n e(u_n)) = 0, & \text{in } \Omega, \\ u = \varphi, & \text{on } \partial\Omega. \end{cases} \tag{6.1}$$

Let Ψ be the linear space of rigid displacements of $\mathbb{R}^d$, i.e. the set of all vector -valued functions $\eta = (\eta^1, \cdots, \eta^d)^T$ such that $\eta = a + Ax$, where $a = (a_1, \cdots, a_d)^T$ is a vector with constant real components, A is a skew-symmetric $(d \times d)$-matrix with real constant elements. It is easy to see that Ψ is a linear space of dimension $d(d+1)/2$. Denote

$$\Psi = \operatorname{span}\left\{ \psi^\alpha \mid 1 \leq \alpha \leq \frac{d(d+1)}{2} \right\}.$$

Equation (6.1) can be rewritten in the following form to emphasize the transmission condition on $\partial\omega$:

$$\begin{cases} \nabla \cdot \left(\mathbb{C}_n^{(s)} e(u_n)\right) = 0, & \text{in } \omega_s, \ 1 \leq s \leq m, \\ \nabla \cdot \left(\mathbb{C}^{(0)} e(u_n)\right) = 0, & \text{in } \Omega \setminus \overline{\omega}, \\ \left.\dfrac{\partial u_n}{\partial \nu_0}\right|_+ \cdot \psi^\alpha = \left.\dfrac{\partial u_n}{\partial \nu_0}\right|_- \cdot \psi^\alpha, & \text{on } \partial\omega_s, \ 1 \leq s \leq m; \ 1 \leq \alpha \leq \frac{d(d+1)}{2}, \end{cases} \tag{6.2}$$

where

$$\left.\frac{\partial u_n}{\partial \nu_0}\right|_+ := \left(\mathbb{C}^{(0)} e(u)\right)\vec{n} = \lambda\,(\nabla \cdot u_n)\,\vec{n} + \mu\left(\nabla u_n + (\nabla u_n)^T\right)\vec{n}, \quad \text{on } \partial\omega_s,$$

$$\left.\frac{\partial u_n}{\partial \nu_0}\right|_- := \left(\mathbb{C}_n^{(s)} e(u)\right)\vec{n} = \lambda_n^{(s)}\,(\nabla \cdot u_n)\,\vec{n} + \mu_n^{(s)}\left(\nabla u_n + (\nabla u_n)^T\right)\vec{n}, \quad \text{on } \partial\omega_s,$$

and the subscript $\pm$ indicates the limit from outside and inside ω_s, respectively.

Theorem 6.1. *If $u_n \in H^1(\Omega; \mathbb{R}^d)$ is a solution of equation* (6.1)*, then $u_n \in C^1(\overline{\Omega \setminus \omega}; \mathbb{R}^d) \cap C^1(\overline{\omega}; \mathbb{R}^d)$ and satisfies equation* (6.2)*.*

If $u_n \in C^1(\overline{\Omega \setminus \omega}; \mathbb{R}^d) \cap C^1(\overline{\omega}; \mathbb{R}^d)$ is a solution of equation (6.2)*, then $u_n \in H^1(\Omega; \mathbb{R}^d)$ and satisfies equation* (6.1)*.*

Proof. The first part of the theorem follows from Proposition 1.4 of [27]. The proof of the rest is standard. □

Theorem 6.2. *There exists at most one solution $u_n \in H^1(\Omega; \mathbb{R}^d)$ to equation* (6.1)*.*

Proof. We only need to prove that if $\varphi = 0$ then a solution u_n of (6.1) is zero. Indeed it follows from (6.1) that

$$\int_\Omega (\mathbb{C}_n e(u_n), e(\psi))\, dx = 0, \quad \forall \ \psi \in C_c^\infty(\Omega; \mathbb{R}^d).$$

This implies by density of $C_c^\infty(\Omega;\mathbb{R}^d)$ in $H_0^1(\Omega;\mathbb{R}^d)$ that $\int_\Omega (\mathbb{C}_n e(u_n), e(u_n))\,dx = 0$. By the property of $\mathbb{C}_n$ and the first Korn's inequality, we have $\nabla u_n = 0$, and therefore $u_n = 0$. $\square$

Define the functional

$$I_n[v] := \frac{1}{2}\int_\Omega (\mathbb{C}_n(x)e(v), e(v))\,dx, \tag{6.3}$$

where v belongs to the set

$$H_\varphi^1(\Omega;\mathbb{R}^d) := \left\{ v \in H^1(\Omega;\mathbb{R}^d) \,\middle|\, v = \varphi, \text{ on } \partial\Omega \right\},$$

where $\varphi \in C^{1,\gamma}(\partial\Omega;\mathbb{R}^d)$, $0 < \gamma < 1$.

Theorem 6.3. *For every n, there exists a minimizer $u_n \in H_\varphi^1(\Omega;\mathbb{R}^d)$ satisfying*

$$I_n[u_n] := \min_{v \in H_\varphi^1(\Omega;\mathbb{R}^d)} I_n[v].$$

Moreover, $u_n \in H^1(\Omega;\mathbb{R}^d)$ is a solution of equation (6.1).

The proof of Theorem 6.3 is standard. The existence of a minimizer u_n follows from the lower semi-continuity property of the functional with respect to the weak convergence in $H^1(\Omega;\mathbb{R}^d)$ and the first Korn's inequality.

Comparing equation (6.1), the Lamé system with infinity coefficients is

$$\begin{cases} \nabla \cdot \left(\mathbb{C}^{(0)} e(u)\right) = 0, & \text{in } \Omega \setminus \overline{\omega}, \\ u\big|_+ = u\big|_-, & \text{on } \partial\omega, \\ e(u) = 0, & \text{in } \omega, \\ \int_{\partial\omega_s} \frac{\partial u}{\partial \nu_0}\Big|_+ \cdot \psi^\alpha = 0, & 1 \le s \le m; \;\; 1 \le \alpha \le \frac{d(d+1)}{2}, \\ u = \varphi, & \text{on } \partial\Omega. \end{cases} \tag{6.4}$$

We have similar results:

Theorem 6.4. *If $u \in H^1(\Omega;\mathbb{R}^d)$ satisfies* (6.4) *except for the fourth line, then $u \in C^1(\overline{\Omega \setminus \omega};\mathbb{R}^d) \cap C^1(\overline{\omega};\mathbb{R}^d)$.*

Proof. By the third line of equation (6.4), u is a linear combination of $\{\psi^\alpha\}$, and therefore $u \in C^\infty(\partial\omega)$. Since $\nabla \cdot \left(\mathbb{C}^{(0)} e(u)\right) = 0$ on $\Omega \setminus \overline{\omega}$, the regularity of u in $\overline{\Omega \setminus \omega}$ follows from [2]. $\square$

Theorem 6.5. *There exists at most one solution $u \in H^1(\Omega;\mathbb{R}^d) \cap C^1(\overline{\Omega \setminus \omega};\mathbb{R}^d) \cap C^1(\overline{\omega};\mathbb{R}^d)$ of* (6.4).

Proof. It is equivalent to showing that if $\varphi = 0$, equation (6.4) only has the solution $u = 0$. We know from the third and the second lines of equation (6.4) that $u|_{\partial\omega_s}$ is a linear combination of $\{\psi^\alpha\}$. Multiplying the first line of equation (6.4) by u and integrating by parts leads to, using a version of the second Korn's inequality,

$$0 = \int_{\Omega\setminus\overline{\omega}} \left(\mathbb{C}^{(0)} e(u), e(u)\right) dx \ge \frac{1}{C}\int_{\Omega\setminus\overline{\omega}} |e(u)|^2 dx \ge \frac{1}{C}\int_{\Omega\setminus\overline{\omega}} |\nabla u|^2 dx.$$

It follows that $u = 0$. $\square$

The existence of a solution can be obtained by using the variational method. Define the energy functional

$$I_\infty[v] := \frac{1}{2}\int_{\Omega\setminus\overline{\omega}} \left(\mathbb{C}^{(0)}e(u), e(u)\right) dx, \tag{6.5}$$

where v belongs to the set

$$\mathcal{A} := \left\{ u \in H^1_\varphi(\Omega;\mathbb{R}^d) \,\middle|\, e(u) = 0 \ \text{ in } \omega \right\}.$$

Theorem 6.6. *There exists a minimizer $u \in \mathcal{A}$ satisfying*

$$I_\infty[u] = \min_{v\in\mathcal{A}} I_\infty[v].$$

Moreover, $u \in H^1(\Omega;\mathbb{R}^d) \cap C^1(\overline{\Omega\setminus\omega};\mathbb{R}^d) \cap C^1(\overline{\omega};\mathbb{R}^d)$ is a solution of equation (6.4).

Proof. By the lower semi-continuity of I_∞ and the weakly closed property of $\mathcal{A}$, it is not difficult to see that a minimizer $u \in \mathcal{A}$ exists and satisfies $\nabla\cdot(\mathbb{C}^{(0)}e(u)) = 0$ in $\Omega\setminus\overline{\omega}$. The only thing needs to shown is the fourth line of (6.4), i.e.

$$\int_{\partial\omega_s} \frac{\partial u}{\partial \nu_0}\Big|_+ \cdot \psi^\alpha = 0, \quad 1 \le s \le m.$$

Indeed, since u is a minimizer, for any $1 \le s \le m$, $1 \le \alpha \le d(d+1)/2$, and any $\phi \in C_c^\infty(\Omega;\mathbb{R}^d)$ satisfying $\phi \equiv \psi^\alpha$ on $\overline{\omega}_s$ and $\phi = 0$ on $\overline{\omega}_t$ $(t \neq s)$, let

$$i(t) := I_\infty[u + t\phi], \qquad t \in \mathbb{R},$$

we have

$$0 = i'(0) := \frac{di}{dt}\Big|_{t=0} = \int_{\Omega\setminus\overline{\omega}} \left(\mathbb{C}^{(0)}e(u), e(\phi)\right) dx.$$

Therefore

$$\begin{aligned} 0 &= -\int_{\Omega\setminus\overline{\omega}} \nabla\cdot\left(\mathbb{C}^{(0)}e(u)\right)\cdot\phi dx = \int_{\Omega\setminus\overline{\omega}} \left(\mathbb{C}^{(0)}e(u), e(\phi)\right) dx + \int_{\partial\omega_s} \frac{\partial u}{\partial\nu_0}\Big|_+ \cdot \phi \\ &= \int_{\partial\omega_s} \frac{\partial u}{\partial\nu_0}\Big|_+ \cdot \psi^\alpha. \end{aligned}$$

□

Finally, we give the relationship between u_n and u.

Theorem 6.7. *Let u_n and u in $H^1(\Omega;\mathbb{R}^d)$ be the solutions of equations* (6.2) *and* (6.4), *respectively. Then*

$$u_n \to u \quad \text{in } H^1(\Omega;\mathbb{R}^d), \text{ as } n \to \infty, \tag{6.6}$$

and

$$\lim_{n\to\infty} I_n[u_n] = I_\infty[u], \tag{6.7}$$

where I_n and I_∞ are defined by (6.3) *and* (6.5).

Proof. **Step 1.** Prove that $\{u_n\}$ weakly converges in $H^1(\Omega;\mathbb{R}^d)$ to a solution u of (6.4).

Due to the uniqueness of the solution to (6.4), we only need to show that after passing to a subsequence, $\{u_n\}$ weakly converges in $H^1(\Omega;\mathbb{R}^d)$ to a solution u of (6.4).

Let $\eta \in H^1_\varphi(\Omega;\mathbb{R}^d)$ be fixed and satisfy $\eta \equiv 0$ on $\overline{\omega}$. Since u_n is the minimizer of I_n in $H^1_\varphi(\Omega;\mathbb{R}^d)$, we have, for some constant C independent of n,

$$\frac{1}{C}\|e(u_n)\|^2_{L^2(\Omega)} \leq I_n[u_n] \leq I_n[\eta] = \frac{1}{2}\int_{\Omega\setminus\overline{\omega}} \left(\mathbb{C}^{(0)}e(\eta), e(\eta)\right) dx \leq C\|\eta\|^2_{H^1(\Omega)}.$$

Using the second Korn's inequality and the fact that $u_n = \varphi$ on $\partial\Omega$, we obtain

$$\|u_n\|_{H^1(\Omega)} \leq C,$$

and therefore, along a subsequence,

$$u_n \rightharpoonup u \quad \text{in } H^1_\varphi(\Omega;\mathbb{R}^d), \text{ as } n \to \infty.$$

Next we show that u is a solution of equation (6.4). In fact, we only need to prove the following three conditions:

$$\nabla\cdot\left(\mathbb{C}^{(0)}e(u)\right) = 0, \qquad \text{in } \Omega\setminus\overline{\omega}, \tag{6.8}$$

$$e(u) = 0, \qquad \text{in } \omega, \tag{6.9}$$

$$\int_{\partial\omega_s} \frac{\partial u}{\partial\nu_0}\Big|_+ \cdot \psi^\alpha = 0, \qquad 1 \leq s \leq m,\ 1 \leq \alpha \leq d(d+1)/2. \tag{6.10}$$

(i) Since $u_n \in H^1(\Omega;\mathbb{R}^d)$ is a solution of equation (6.1) and $u_n \rightharpoonup u$ in $H^1_\varphi(\Omega;\mathbb{R}^d)$, we have, for any $\phi \in C^\infty_c(\Omega\setminus\overline{\omega};\mathbb{R}^d)$, that

$$0 = \int_{\Omega\setminus\overline{\omega}} \left(\mathbb{C}^{(0)}e(u_n), e(\phi)\right) dx \to \int_{\Omega\setminus\overline{\omega}} \left(\mathbb{C}^{(0)}e(u), e(\phi)\right) dx.$$

Therefore

$$\int_{\Omega\setminus\overline{\omega}} \left(\mathbb{C}^{(0)}e(u), e(\phi)\right) dx = 0, \quad \forall\, \phi \in C^\infty_c(\Omega\setminus\overline{\omega}),$$

that is (6.8).

(ii) Let $\eta \in H^1_\varphi(\Omega;\mathbb{R}^d)$ be fixed and satisfy $\eta \equiv 0$ on $\overline{\omega}$, then since u_n is a minimizer of I_n in $H^1_\varphi(\Omega;\mathbb{R}^d)$, we have

$$I_n[u_n] \leq I_n[\eta] \leq \frac{1}{2}\int_{\Omega\setminus\overline{\omega}} \left(\mathbb{C}^{(0)}e(\eta), e(\eta)\right) dx \leq C.$$

On the other hand,

$$I_n[u_n] \geq \sum_{s=1}^m \min\{2\mu_n^{(s)}, d\lambda_n^{(s)} + 2\mu_n^{(s)}\} \int_{\omega_s} |e(u_n)|^2 dx.$$

Since $\mu_n^{(s)} \to \infty$ and $d\lambda_n^{(s)} + 2\mu_n^{(s)} \to \infty$ as $n \to \infty$, we have

$$\|e(u_n)\|_{L^2(\omega)} \to 0, \quad \text{as } n \to \infty.$$

By (1), $u_n \rightharpoonup u$ in $H^1(\Omega;\mathbb{R}^d)$. Therefore

$$\|e(u)\|_{L^2(\omega)} = 0,$$

i.e. $e(u) = 0$ in ω, which is (6.9).

(iii) By (i) and (ii), u satisfies (6.8) and is a linear combination of $\{\psi^\alpha\}$ on each $\partial\omega_s$, and is equal to φ on $\partial\Omega$. Thus u is smooth on $\partial\omega$. By the elliptic regularity theorems, $u \in C^1(\overline{\Omega \setminus \omega};\mathbb{R}^d) \cap C^2(\Omega \setminus \overline{\omega};\mathbb{R}^d)$. For each $s = 1, 2, \cdots, m$, $1 \le \alpha \le d(d+1)/2$, we construct a function $\rho \in C^2(\overline{\Omega \setminus \omega};\mathbb{R}^d)$ such that $\rho = \psi^\alpha$ on $\partial\omega_s$, $\rho = 0$ on $\partial\omega_t$ for $t \neq s$, and $\rho = 0$ on $\partial\Omega$. By Green's identity, we have the following:

$$\begin{aligned}
0 &= -\int_{\Omega\setminus\overline{\omega}} \nabla\cdot\left(\mathbb{C}^{(0)}e(u_n)\right)\cdot\rho dx \\
&= \int_{\Omega\setminus\overline{\omega}} \left(\mathbb{C}^{(0)}e(u_n), e(\rho)\right)dx + \int_{\partial\omega_s} \frac{\partial u_n}{\partial\nu_0}\Big|_+ \cdot\psi^\alpha \\
&= \int_{\Omega\setminus\overline{\omega}} \left(\mathbb{C}^{(0)}e(u_n), e(\rho)\right)dx + \int_{\partial\omega_s} \frac{\partial u_n}{\partial\nu_0}\Big|_- \cdot\psi^\alpha \\
&= \int_{\Omega\setminus\overline{\omega}} \left(\mathbb{C}^{(0)}e(u_n), e(\rho)\right)dx.
\end{aligned}$$

Similarly,

$$0 = -\int_{\Omega\setminus\overline{\omega}} \nabla\cdot\left(\mathbb{C}^{(0)}e(u)\right)\cdot\rho dx = \int_{\Omega\setminus\overline{\omega}} \left(\mathbb{C}^{(0)}e(u), e(\rho)\right)dx + \int_{\partial\omega_s} \frac{\partial u}{\partial\nu_0}\Big|_+ \cdot\psi^\alpha.$$

Since $u_n \rightharpoonup u$ in $H^1(\Omega)$, it follows that

$$0 = \int_{\Omega\setminus\overline{\omega}} \left(\mathbb{C}^{(0)}e(u_n), e(\rho)\right)dx \to \int_{\Omega\setminus\overline{\omega}} \left(\mathbb{C}^{(0)}e(u), e(\rho)\right)dx.$$

Thus

$$\int_{\partial\omega_s} \frac{\partial u}{\partial\nu_0}\Big|_+ \cdot\psi^\alpha = 0, \qquad 1 \le s \le m,\ 1 \le \alpha \le d(d+1)/2.$$

Step 1 is completed.

Step 2. Prove (6.6) and (6.7).

Since u_n is a minimizer of I_n and $e(u) = 0$ in ω, we have

$$I_n[u_n] \le I_n[u] = I_\infty[u].$$

Thus

$$\limsup_{n\to\infty} I_n[u_n] \le I_\infty[u].$$

On the other hand, since $e(u) = 0$ and $u_n \rightharpoonup u$ in $H^1(\Omega;\mathbb{R}^d)$,

$$\begin{aligned}
I_\infty[u] &= \frac{1}{2}\int_{\Omega\setminus\overline{\omega}} \left(\mathbb{C}^{(0)}e(u), e(u)\right)dx \\
&\le \liminf_{n\to\infty} \frac{1}{2}\int_{\Omega\setminus\overline{\omega}} \left(\mathbb{C}^{(0)}e(u_n), e(u_n)\right)dx \\
&\le \liminf_{n\to\infty} \frac{1}{2}\int_{\Omega\setminus\overline{\omega}} \left(\mathbb{C}^{(0)}e(u_n), e(u_n)\right)dx + \limsup_{n\to\infty} \frac{1}{2}\sum_s \int_{\omega_s} \left(\mathbb{C}_n^{(s)}e(u_n), e(u_n)\right)dx \\
&\le \limsup_{n\to\infty} I_n[u_n].
\end{aligned}$$

With the help of the first Korn′s inequality, we easily deduce (6.7) and (6.6) from the above. The proof of Theorem 6.7 is completed. ◻

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