Invariant manifolds for Nonautonomous Delay Differential Equations

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Abstract

In this paper we give a direct proof of the existence of invariant manifolds, especially the center manifold, for nonautonomous delay differential equations with the form $\dot{x}(t) = L(t)x_t + f(t, x_t)$. We assume that the linear equation $\dot{x}(t) = L(t)x_t$ admits a exponential trichotomy and the nonlinear term f is a Lipschitz function with sufficiently small Lipschitz constant and $f(t, 0) = 0$. We prove the existence by constructing a contraction mapping on a functional space. In addition, by applying the fibre contraction theorem, we also show that the invariant manifolds are differentiable when the nonlinear term is of class C^k.

Keywords: nonautonomous delay equations, center manifold, invariant manifold, existence, differentiability

1. Introduction

Invariant manifolds theory plays a key role in the description and understanding of the dynamics of nonlinear systems. Especially for infinite dimensional systems it provides us with a very powerful tool. This method is widely applied to a majority of biological and economical systems. According to invariant manifolds theory, the stable and unstable manifolds theory describes the characteristics around the hyperbolic equilibrium point, and

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center manifolds theory deals with the case when an equilibrium point becomes non-hyperbolic. Compared with the stable and unstable manifolds theory, center manifolds theory finds a more widely application on high dimensional systems or on infinite dimensional systems.

The classical center manifolds theory was initiated during 1960's by V.A.Pliss [13] and A.Kelley [12] for a non-hyperbolic equilibrium z_0 of the system

$$z' = f(z), \quad z \in \mathbb{R}^n,$$

and had since then found widespread applications to many fields. In 1982, a comprehensive and complete version of the classical center manifolds theorem was given by J.Carr [5] for a system with the form

$$\begin{cases} \dot{x} = Ax + F(x, y), & x \in \mathbb{R}^n, \\ \dot{y} = Bx + G(x, y), & y \in \mathbb{R}^m, \end{cases}$$

where A and B are constant matrices such that all the eigenvalues of A have zero real parts while all the eigenvalues of B have negative real parts. By constructing contraction mappings, J.Carr proved not only the existence of center manifolds but also some properties of center manifolds, such as the reduction principle and the approximation of center manifold.

J.Carr's proof provided a basic method to study center manifolds, however, too many estimations were used in his proof. In order to avoid this shortage, A.Vanderbauwhede [17] made some modifications of J.Carr's results by constructing a sequence of functional spaces and extended the conclusions to the case which the characteristic equation has not only negative real characteristic roots but also positive real characteristic roots. Because of A.Vanderbauwhede's proof, the behaviour around the non-hyperbolic equilibrium point for finite dimensional systems is totally clear.

In order to apply center manifolds theory to more general systems, we attempt to extend center manifolds theory to nonautonomous cases. We refer the readers to B.Aulbach [1] for further references on center manifolds theory for nonautonomous finite dimensional system with the form

$$\begin{cases} \dot{x} = N(t)x + r_1(t, x, y), & x \in \mathbb{R}^n, \\ \dot{y} = V(t)y + r_2(t, x, y), & y \in \mathbb{R}^m, \end{cases}$$

$N(t)$ and $V(t)$ are the linear parts of the system and the assumptions of $N(t)$ and $V(t)$ are similar to the assumptions of J.Carr's proof. By providing a exponential estimation on $N(t)$ and $V(t)$, B.Aulbach obtained totally same conclusions for nonautonomous systems, including the existence of the center manifolds and relations between original system and the system on center manifold.

Besides generalizations to various cases of finite dimensional systems, center manifolds theory plays a more useful role in infinite dimensional systems. Along with the generalizations to infinite systems, center manifolds theory has already became a basic tool when we study infinite systems. The classical center manifolds theory had already been generalized by J.Carr to infinite dimensional systems in [5], but he omitted the detailed proof and gave the totally same assumptions to infinite systems. Therefore, for more general results, Th.Gallay weakened some assumptions of J.Carr and obtained the existence of center manifolds for the system with the form

$$\frac{\mathrm{d}}{\mathrm{d}t}z(t) = Az(t) + f(z(t)), \quad t \geq 0,$$

where $z \in E$ and E is a Banach space, A is a linear operator on E and the characteristic equation has negative, zero and positive real characteristic roots. Th.Gallay's results provided us a complete understanding of center manifolds on Banach spaces.

Be similar to the finite case, we expect to extend the conclusions to nonautonomous systems.A great deal of conclusions had already been obtained. B.Scarpellini [14] gave a direct proof of existence of center manifold for a special nonautonomous infinite system with the form

$$\begin{cases} \dot{y} = (L + B(t))y + g(t, z, y), & y \in Y, \\ \dot{z} = A(t)z + f(t, z, y), & z \in Z, \end{cases}$$

where Z, Y are Banach spaces such that $\dim(Z) < \infty$, $\dim(Y) \leq \infty$, L is a group generator on Y and $A(t), B(t)(t \in \mathbb{R})$ are families of bounded linear operators. He separated the infinite system into a finite part and a infinite part in order to simplify the results. Among all these conclusions, C.Chione and Y.Latushkin obtained results for the most general system in 1997.

C.Chione and Y.Latushkin [6] provided the conclusions about evolution equation of the form

$$x' = A(t)x + g(t, x(t)),$$

where $x \in X$, X is a Banach space and $A(t)(t \in \mathbb{R})$ is a linear operator on X. They made use of the method of evolution equations to study nonautonomous infinite systems and provided us basic theory and methods to study delay equations.

As so many methods have been developed to study infinite dimensional systems, we can apply these conclusions and methods to some specific infinite systems, such as delay equations. O.Diekmann [8] gave a comprehensive and complete proof of the center manifolds theory for autonomous delay equations. Furthermore, for some specific delay equations, the Taylor approximate expansion of center manifold for can be calculated by Maple [7].

For nonautonomous delay equations, L.Barreira [3] has already given a proof of stable manifold for system

$$\begin{cases} \dot{x}(t) = L(t)x_t + f(t, x_t, \lambda), \quad t \geq s \\ x_s = \varphi, \end{cases} \tag{1.1}$$

where $x_t(\theta)$ belongs to a continuous functional spaces, $L(t)$ admits a nonuniform exponential dichotomy and λ is a parameter.

Therefore, in this paper, we will make some modifications about the system (1.1) and will use a more direct traditional approach to prove the existence of center manifold for system

$$\begin{cases} \dot{x}(t) = L(t)x_t + f(t, x_t), \quad t \geq s, \\ x_s = \phi, \end{cases} \tag{1.2}$$

and by applying the fibre contraction theorem, we will prove the differentiability of center manifold about the initial value ϕ. Furthermore, we will also give a proof of stable and unstable manifolds on a subspace.

Our approach is straightforward. After stating in section 2 some basic notations and assumptions, we prove the existence of center manifold for system (1.2) in section 3. Section 4 contains the proof of the smoothness of center manifold and section 5 talks about the existence of stable and unstable manifolds.

2. Preliminaries

Given $r > 0$, let $\hat{\mathcal{B}} = C([-r, 0], \mathbb{R}^n)$ be the Banach space of continuous functions $\phi : [-r, 0] \to \mathbb{R}^n$ endowed with the norm

$$\|\phi\| = \sup_{\theta \in [-r,0]} |\phi(\theta)|.$$

It is also standard to consider the set $\mathcal{B}$ of all functions $\phi : [-r, 0] \to \mathbb{R}^n$ such that for each $s \in [-r, 0]$ the limits

$$\lim_{\theta \to s^-} \phi(\theta) \quad \text{and} \quad \lim_{\theta \to s^+} \phi(\theta)$$

exist and $\lim_{\theta \to s^+} \phi(\theta) = \phi(s)$. $\mathcal{B}$ is a Banach space when endowed with the norm in (2).

For each $(s, \phi) \in \mathbb{R} \times \hat{\mathcal{B}}$, consider the initial value problem

$$\begin{cases} \dot{x}(t) = L(t)x_t, \quad t \geq s, \\ \quad x_s = \phi, \end{cases} \tag{2.1}$$

where $x_t(\theta) = x(t+\theta)(\theta \in [-r, 0])$, $L(t)\phi$ in linear in ϕ and the map $(t, \phi) \mapsto L(t)\phi$ is continuous. Now let $\hat{T}(t, s) : \hat{\mathcal{B}} \to \hat{\mathcal{B}}$ be the evolution operator associated to Eq.(2.1), defined by

$$\hat{T}(t, s)\phi = x_t(\cdot, s, \phi), \quad t \geq s. \tag{2.2}$$

In order to extend $\hat{T}(t, s)$to the space $\mathcal{B}$, we write $L(t)$ in the form

$$L(t)\phi = \int_{-r}^{0} d_\theta[\eta(t, \theta)]\phi(\theta), \tag{2.3}$$

where $\eta(t, \theta)$ is an $n \times n$ matrix function and is measurable in $(t, \theta) \in \mathbb{R} \times [-r, 0]$. Moreover, $\eta(t, \theta)$ is continuous from the left in θ on $(-r, 0)$ and has bounded variation in θ on $[-r, 0]$ for each t.

For each $(s, \phi) \in \mathbb{R} \times \mathcal{B}$ there exists a unique solution $t \mapsto x_t(\cdot, s, \phi) \subset \mathcal{B}$ of system (2.1). The corresponding evolution operator $T(t, s) : \mathcal{B} \to \mathcal{B}$ defined by

$$T(t, s)\phi = x_t(\cdot, s, \phi). \quad t \geq s. \tag{2.4}$$

we note that $T(t, s)|_{\hat{\mathcal{B}}} = \hat{T}(t, s)$ and that $T(t, s)\mathcal{B} \subset \hat{\mathcal{B}}$ for any $t \geq s + r$.

Furthermore, for $(s, \phi) \in \mathbb{R} \times \mathcal{B}$, we also consider the nonlinear system

$$\begin{cases} \dot{x}(t) = L(t)x_t + f(t, x_t), \quad t \geq s, \\ \quad x_s = \phi, \end{cases} \tag{2.5}$$

where $f : \mathbb{R} \times \mathcal{B} \to \mathbb{R}^n$ satisfies
(1) $f(t, 0) = 0$;

(2)for $t \in \mathbb{R}$, $\phi_1, \phi_2 \in \mathcal{B}$ and $\|\phi_1\|$ and $\|\phi_2\|$ sufficient small, there exist constant $L > 0$ such that

$$|f(t, \phi_1) - f(t, \phi_2)| \leq L\|\phi_1 - \phi_2\|. \tag{2.6}$$

According to the conclusion from J.Hale, the solution of system (2.5) satisfies the variation-of-parameter formula

$$x_t = T(t, s)\phi + \int_s^t T(t, \tau)X_0 f(\tau, x_\tau)d\tau, \tag{2.7}$$

where

$$X_0(\theta) = \begin{cases} 0, & -r \leq \theta < 0, \\ \mathrm{Id}, & \theta = 0. \end{cases} \tag{2.8}$$

We will consider the evolution family $\{T(t, s)\}_{t \geq s}$ that admits a splitting into "center" and "hyperbolic" parts. To be more precisely, there exists a bounded strongly continuous projection valued function $P(t)$ such that $T(t, s)P(s) = P(t)T(t, s)$ for all $t \geq s$. For $Q(t) := \mathrm{I} - P(t)$ define the subspaces $\mathcal{B}_c(t) := \mathrm{Im}P(t)$ and $\mathcal{B}_h(t) := \mathrm{Im}Q(t)$, and for $t \geq s$ define the restricted operators

$$T_c(t, s) := T(t, s)|_{\mathcal{B}_c(s)} : \mathcal{B}_c(s) \to \mathcal{B}_c(t), \tag{2.9}$$

$$T_h(t, s) := T(t, s)|_{\mathcal{B}_h(s)} : \mathcal{B}_h(s) \to \mathcal{B}_h(t). \tag{2.10}$$

For the center part, we assume that for all $t \geq s$ the operator $T_c(t, s)$ is invertible as an operator from $\mathcal{B}_c(s)$ to $\mathcal{B}_c(t)$ and define $T_c(s, t) := [T_c(t, s)]^{-1}$. Assume there exists two positive constants ω and $M_c = M_c(\omega)$, such that for all $(t, s) \in \mathbb{R}$,

$$\|T_c(t, s)\| \leq M_c e^{\omega|t-s|}. \tag{2.11}$$

For the hyperbolic part, we assume that $\{T_h(t, s)\}_{t \geq s}$ has an exponential dichotomy. To be more precisely, assume that there exist bounded strongly continuous projection valued functions $Q_\pm(t)$ such that $Q_+(t) + Q_-(t) = Q(t)$, and for all $t \geq s$,

$$T_h(t, s)Q_\pm(s) = Q_\pm(t)T_h(t, s).$$

In addition, consider the restrictions

$$T_h^\pm(t, s) = T_h(t, s)|_{\mathrm{Im}Q_\pm(s)} : \mathrm{Im}Q_\pm(s) \to \mathrm{Im}Q_\pm(t), \quad t \geq s.$$

We assume that the operator $T_h^-(t,s)$ is invertible and that there exist two positive constants β and $M_h = M_h(\beta)$, such that for all $t \geq s$,

$$\|T_h^+(t,s)\| \leq M_h e^{-\beta(t-s)}, \tag{2.12}$$

$$\|[T_h^-(t,s)]^{-1}\| \leq M_h e^{-\beta(t-s)}. \tag{2.13}$$

Our following discussions will base on a functional space X. Therefore for $p_1 > 0$ let X be the set of Lipschitz function $\Phi(t,\cdot) : \mathcal{B}_c(t) \to \mathcal{B}_h(t)$ with Lipschitz constant p_1 for $t \in \mathbb{R}$, and $\Phi(t,0) = 0$. With the supremum norm

$$|\Phi|_X = \sup\left\{\frac{\|\Phi(s,\phi)\|}{\|\phi\|} : s \in \mathbb{R}, \|\phi\| \neq 0, \phi \in \mathcal{B}_c(s)\right\},$$

X is a complete space.

According to the projection $P(t)$ and $Q(t)$, we can write (2.7) into

$$u_t = T_c(t,s)P(s)\phi + \int_s^t T_c(t,\tau)P(\tau)X_0 f(\tau, u_\tau + v_\tau)d\tau, \tag{2.14}$$

$$v_t = T_h(t,s)Q(s)\phi + \int_s^t T_h(t,\tau)Q(\tau)X_0 f(\tau, u_\tau + v_\tau)d\tau, \tag{2.15}$$

where $u_t = P(t)x_t$, $v_t = Q(t)x_t$. For $\Phi \in X$, we consider the graph

$$W = \{(s, \varphi + \Phi(s,\varphi)) : s \in \mathbb{R}, \varphi \in \mathcal{B}_c(s)\}. \tag{2.16}$$

We will look for an element $\Phi \in X$ such that for initial value sufficiently small, the W is an invariant manifold for (2.7), that is for each $(s, \varphi + \Phi(s,\varphi)) \in W$, the solution of (2.7) $(t, x_t(\cdot, s, \varphi + \Phi(s,\varphi))) \in W$. We call the invariant manifold W the center manifold. We prove the existence of the center manifold by constructing a contraction mapping on the functional space X. In order to simplify the calculation, we define $F(t,x_t) = X_0 f(t,x_t)$, $F_c(t,x_t) = P(t)F(t,x_t)$, $F_h(t,x_t) = Q(t)F(t,x_t)$, and rewrite (2.14) and (2.15) into

$$u_t = T_c(t,s)\varphi + \int_s^t T_c(t,\tau)F_c(\tau, u_\tau + v_\tau)d\tau, \tag{2.17}$$

$$v_t = T_h(t,s)\psi + \int_s^t T_h(t,\tau)F_h(\tau, u_\tau + v_\tau)d\tau, \tag{2.18}$$

where $\varphi = P(s)\phi \in \mathcal{B}_c(s)$, $\psi = Q(s)\phi \in \mathcal{B}_h(s)$.

3. Existence of the center manifold

We now formulate our main result.

Theorem 3.1. *Assume that system* (2.5) *admits the above assumptions, then provided that the constant* L *is sufficiently small and* $\xi > 0$*, for* $\varphi \in \mathcal{B}_c(s)$ *and* $\|\varphi\| \leq \xi$*, there exist a unique function* $\Phi \in X$ *such that* W *is an invariant manifold for system* (2.5)*.*

Before giving a detailed proof of the theorem, we first give an introduction about the approaches of the proof. For each $\Phi \in X$, let functions u_t and v_t are the solutions of (2.17) and (2.18) with initial value $(s, \varphi + \Phi(s, \varphi)) \in W$. In order to prove that W is an invariant manifold, we need to prove that there exist $\Phi \in X$ such that $v_t = \Phi(t, u_t)$. To be more precisely, $\exists \Phi \in X$ such that

$$u_t = T_c(t,s)\varphi + \int_s^t T_c(t,\tau) F_c(\tau, u_\tau + \Phi(\tau, u_\tau)) d\tau, \tag{3.1}$$

$$\Phi(t, u_t) = T_h(t,s)\Phi(s,\varphi) + \int_s^t T_h(t,\tau) F_h(\tau, u_\tau + \Phi(\tau, u_\tau)) d\tau. \tag{3.2}$$

We prove this conclusion by constructing a contraction mapping on X. Firstly, in Lemma 3.2, for $\forall \Phi \in X$, $\varphi \in \mathcal{B}_c(s)$, $s \in \mathbb{R}$, we prove that $\exists u_t \in \mathcal{B}_c(t)$ satisfies (3.1). Then we will prove that $\exists \Phi \in X$ such that (3.2) holds. In order to prove this conclusion, we need to rewrite (3.2) into another form, the detailed form will be given in Lemma 3.3. Hence, we give a directly proof of Lemma 3.2 and Lemma 3.3 first.

Lemma 3.2. *Given* L *sufficiently small and* $\forall (s, \varphi, \Phi) \in \mathbb{R} \times \mathcal{B}_c(s) \times X$*, there exists a unique function* $u : (-\infty, +\infty) \mapsto \mathbb{R}^n$ *with* $u_s = \varphi$ *such that* $u_t \in \mathcal{B}_c(t)$ *and* (3.1) *holds for every* $t, s \in \mathbb{R}$*. Moreover, for* $\varphi_1, \varphi_2 \in \mathcal{B}_c(s)$*,*

$$\|u_t^1 - u_t^2\| \leq M_c \|\varphi_1 - \varphi_2\| e^{[LM_c(1+p_1)+\omega]|t-s|} \tag{3.3}$$

where u_t^1, u_t^2 *are the functions satisfy* (3.1) *respectively for* (s, φ_1, Φ) *and* (s, φ_2, Φ)*.*

Proof. We claim that it suffices to prove the lemma for $t \geq s$. Indeed, for $t \leq s$, then $-t \geq -s$, if we define $u_t^{'} = u_{-t}$, $T_c^{'}(t,s) = T_c(-t,-s)$,

$F_c^{'}(t,x_t) = -F_c(-t,x_t)$, $\Phi^{'}(t,u_t) = \Phi(-t,u_{-t})$, then by replacing t by $-t$ and s by $-s$ in (3.1), we obtain

$$u_t^{'} = T_c^{'}(t,s)\varphi + \int_s^t T_c^{'}(t,\tau)F_c^{'}(\tau, u_\tau^{'} + \Phi^{'}(\tau,u_\tau))d\tau. \tag{3.4}$$

Since $\{T_c^{'}(t,s)\}_{t,s\in\mathbb{R}}$ satisfies (2.11), $F_c^{'}$ satisfies (2.6) and $\Phi^{'} \in X$. Therefore we can prove this case completely similar to the case $t \geq s$.

Define $\mathcal{L} : \mathcal{B}_c(t) \to \mathcal{B}_c(t)$ by

$$(\mathcal{L}u)_t = T_c(t,s)\varphi + \int_s^t T_c(t,\tau)F_c(\tau, u_\tau + \Phi(\tau,u_\tau))d\tau. \tag{3.5}$$

First we will assume that t belongs to $[s,T]$, $T > s$. Denoting by $\|u\|_\infty$ the norm of u_t as an element of $\mathcal{B}_c(t)(s \leq t \leq T)$, it follows readily from the definition of $\mathcal{L}$ that

$$\|(\mathcal{L}u^1)_t - (\mathcal{L}u^2)_t\| \leq \frac{M_c L(1+p_1)}{\omega} e^{\omega(t-s)} \|u^1 - u^2\|_\infty \leq \frac{M_c L(1+p_1)}{\omega} e^{\omega T} \|u^1 - u^2\|_\infty, \tag{3.6}$$

where $u_t^1, u_t^2 \in \mathcal{B}_c(t)$. Using (3.5) (3.6) and the induction method on n it follows easily that

$$\|(\mathcal{L}^n u^1)_t - (\mathcal{L}^n u^2)_t\| \leq \frac{(M_c L(1+p_1))^n}{\omega} e^{\omega(t-s)} \|u^1 - u^2\|_\infty \frac{1}{(n-1)!}(t-s)^{n-1},$$

that is

$$\|(\mathcal{L}^n u^1)_t - (\mathcal{L}^n u^2)_t\| \leq \frac{(M_c L(1+p_1))^n}{\omega} e^{\omega T} \|u^1 - u^2\|_\infty \frac{1}{(n-1)!} T^{n-1}. \tag{3.7}$$

For n large enough such that $\frac{(M_c L(1+p_1))^n T^{n-1}}{\omega(n-1)!} e^{\omega T} < 1$ and by a well known extension of the contraction principle, $\mathcal{L}$ has a unique fixed point $u_t \in \mathcal{B}_c(t)$. This fixed point is the desired solution of (3.1).

The uniqueness of u_t and the proof of (3.3) are consequences of the following arguments. Rewrite the Eq. (3.1) into

$$u_{t+s} = T_c(t+s,s)\varphi + \int_0^t T_c(t+s,\tau+s)F_c(\tau+s, u_{\tau+s} + \Phi(\tau+s, u_{\tau+s}))d\tau. \tag{3.8}$$

We also prove the conclusions only for $t \geq 0$. For $\varphi_1, \varphi_2 \in \mathcal{B}_c(s)$, u^1_{t+s}, u^2_{t+s} are the functions satisfy (3.1) respectively for (s, φ_1, Φ) and (s, φ_2, Φ), it follows from (2.11) and (3.8) that

$$\|u^1_{t+s} - u^2_{t+s}\| \leq M_c e^{\omega t}\|\varphi_1 - \varphi_2\| + \int_0^t M_c e^{\omega(t-\tau)} L(1+p_1)\|u^1_{\tau+s} - u^2_{\tau+s}\| d\tau,$$

that is

$$e^{-\omega t}\|u^1_{t+s} - u^2_{t+s}\| \leq M_c\|\varphi_1 - \varphi_2\| + \int_0^t M_c e^{-\omega\tau} L(1+p_1)\|u^1_{\tau+s} - u^2_{\tau+s}\| d\tau.$$

An application of Gronwall's inequality yields

$$e^{-\omega t}\|u^1_{t+s} - u^2_{t+s}\| \leq M_c\|\varphi_1 - \varphi_2\| e^{M_c L(1+p_1)t}, \quad t \geq 0,$$

that is

$$\|u^1_{t+s} - u^2_{t+s}\| \leq M_c\|\varphi_1 - \varphi_2\| e^{[LM_c(1+p_1)+\omega]|t|}, \quad t \in \mathbb{R}, \tag{3.9}$$

which yields the uniqueness of u_t.

We now prove the prolongation of the solution u_t. We start by showing that for every $s \in \mathbb{R}$, $\varphi \in \mathcal{B}_c(s)$, the integral equation (3.1) has a unique solution u_t on an interval $[s, s_1]$ whose length is bounded below by

$$|s - s_1| = \delta = \min\left\{1, \frac{\|\varphi\|}{K(s)L(1+p_1)}\right\} \tag{3.10}$$

where $M(s) = \sup\{\|T_c(t,s)\| : s \leq t \leq s+1\}$, $K(s) = 2M(s)\|\varphi\|$. The mapping $\mathcal{L}$ defined by (3.5) maps the ball of radius $K(s)$ centered at 0 of $\mathcal{B}_c(t)$ into itself. This follows from the estimate

$$\begin{aligned}
\|\mathcal{L}u(t)\| &\leq M(s)\|\varphi\| + M(s)L(1+p_1)\int_s^t \|u_\tau\| d\tau, \\
&\leq M(s)\|\varphi\| + M(s)L(1+p_1)K(s)\delta, \\
&\leq 2M(s)\|\varphi\| = K(s).
\end{aligned}$$

In this ball, $\mathcal{L}$ satisfies a uniform Lipschitz condition with constant L and thus it possesses a unique fixed point u_t in the ball. This fixed point is the desired solution on the interval $[s, s_1]$. From what we have just proved, it follows that if u_t is a mild solution of (3.1) on the interval $[s, T]$, it can be extended to the interval $[T, T + \delta]$ with $\delta > 0$ defined by (3.10). Because of this conclusion, we set the maximum interval of existence of u_t as $[s, t_{max}]$, according to (3.3), we can obtain that if $t_{max} < \infty$, then $\lim_{t \to t_{max}} \|u_t\| < \infty$. Thus we can extend the existence interval to $[s, \infty)$. This completes the proof of Lemma 3.2. $\square$

Lemma 3.3. *Given $\Phi \in X$ and $\varphi \in \mathcal{B}_c(s), \|\varphi\| \leq \xi$, denote u_t the unique function given by Lemma 3.2, the following equations are equivalent,*

$$\Phi(t, u_t) = T_h(t, s)\Phi(s, \varphi) + \int_s^t T_h(t, \tau)F_h(\tau, u_\tau + \Phi(\tau + u_\tau))d\tau, \tag{3.11}$$

and

$$\Phi(s, \varphi) = \int_{-\infty}^{+\infty} K(s, \tau)F_h(\tau, u_\tau + \Phi(\tau, u_\tau))d\tau, \tag{3.12}$$

where

$$K(t, s) = \begin{cases} T_h^+(t, s), & t \geq s, \\ -[T_h^-(t, s)]^{-1}, & t < s. \end{cases} \tag{3.13}$$

Proof. According to the assumption of $T_h(t, s)$ and (2.12) (2.13),the following estimate holds:

$$\|K(t, s)\| \leq M_h e^{-\beta|t-s|}, \qquad (t, s) \in \mathbb{R}^2. \tag{3.14}$$

As $\Phi \in X$ is a bounded function, it follows from (2.12) and (3.11) that

$$Q^+(t)\Phi(t, u_t) = T_h^+(t, s)Q^+(s)\Phi(s, \varphi) + \int_s^t T_h^+(t, \tau)Q^+(\tau)F_h(\tau, u_\tau + \Phi(\tau, u_\tau))d\tau, \tag{3.15}$$

and provided that L is sufficiently small such that $LM_c(1+p_1)+\omega < \beta$, then

$$\begin{aligned} \|T_h^+(t, s)Q^+(s)\Phi(s, \varphi)\| &\leq M_h e^{-\beta(t-s)}\|\Phi(s, \varphi)\|, \\ &\leq M_h e^{-\beta(t-s)} p_1\|\varphi\| \longrightarrow 0, (s \to -\infty). \end{aligned} \tag{3.16}$$

Let $s \to -\infty$ in (3.15),we get

$$Q^+(t)\Phi(t, u_t) = \int_{-\infty}^t T_h^+(t, \tau)Q^+(\tau)F_h(\tau, u_\tau + \Phi(\tau, u_\tau))d\tau,$$

and let $t = s$, then

$$Q^+(s)\Phi(s,\varphi) = \int_{-\infty}^{s} T_h^+(s,\tau)Q^+(\tau)F_h(\tau, u_\tau + \Phi(\tau,u_\tau))d\tau. \tag{3.17}$$

In a similar way, it follows from (2.13) and (3.11) that

$$[T_h^-(t,s)]^{-1}Q^-(t)\Phi(t,u_t) = Q^-(s)\Phi(s,\varphi) + \int_s^t T_h^-(s,\tau)Q^-(\tau)F_h(\tau,u_\tau+\Phi(\tau,u_\tau))d\tau, \tag{3.18}$$

and

$$\begin{aligned} \|[T_h^-(t,s)]^{-1}Q^-(t)\Phi(t,u_t)\| &\leq M_h e^{-\beta(t-s)}\|\Phi(t,u_t)\| \\ &\leq M_h e^{-\beta(t-s)}p_1\|u_t\| \\ &\leq M_h e^{-\beta(t-s)}p_1\|\varphi\|e^{[LM_c(1+p_1)+\omega](t-s)} \longrightarrow 0, \quad (t \to +\infty). \end{aligned}$$

Therefore, let $t \to +\infty$ in (3.18), we get

$$Q^-(s)\Phi(s,\varphi) = -\int_s^{+\infty} T_h^-(s,\tau)Q^-(\tau)F_h(\tau,u_\tau + \Phi(\tau,u_\tau))d\tau. \tag{3.19}$$

In conclusion, we prove the sufficiency.

To prove the necessity, according to equation (3.17), we obtain that

$$\begin{aligned} T_h^+(t,s)Q^+(s)\Phi(s,\varphi) &= \int_{-\infty}^{s} Q^+(\tau)T_h^+(t,\tau)F_h(\tau,u_\tau + \Phi(\tau,u_\tau))d\tau, \\ &= \int_{-\infty}^{t} T_h^+(t,\tau)Q^+(\tau)F_h(\tau,u_\tau + \Phi(\tau,u_\tau))d\tau \\ &\quad + \int_t^s T_h^+(t,\tau)Q^+(\tau)F_h(\tau,u_\tau + \Phi(\tau,u_\tau))d\tau, \\ &= Q^+(t)\Phi(t,u_t) - \int_s^t T_h^+(t,\tau)Q^+(\tau)F_h(\tau,u_\tau + \Phi(\tau,u_\tau))d\tau, \end{aligned}$$

that is

$$Q^+(t)\Phi(t,u_t) = T_h^+(t,s)Q^+(s)\Phi(s,\varphi) + \int_s^t T_h^+(t,\tau)Q^+(\tau)F_h(\tau,u_\tau+\Phi(\tau,u_\tau))d\tau.$$

We can use the same way to prove that

$$Q^-(t)\Phi(t,u_t) = T_h^-(t,s)Q^-(s)\Phi(s,\varphi)+\int_s^t T_h^-(t,\tau)Q^-(\tau)F_h(\tau,u_\tau+\Phi(\tau,u_\tau))d\tau.$$

This completes the proof of Lemma 3.3. $\square$

Proof of the Theorem 3.1 We note that in Lemma 3.2 we establish the existence of a unique function u_t satisfying (3.1). Now we establish the existence of a unique function Φ satisfying (3.11). According to Lemma 3.3, we will construct an appropriate integral operator $\mathcal{F}$ such that W is an invariant set for (2.7) whenever Φ is a fixed point of $\mathcal{F}$ in X.

Define

$$(\mathcal{F}\Phi)(s,\varphi) = \int_{-\infty}^{+\infty} K(s,\tau+s)F_h(\tau+s,u_{\tau+s}+\Phi(\tau+s,u_{\tau+s}))d\tau. \quad (3.20)$$

We show that $\mathcal{F}$ is a contraction on X. It easily proves that $\mathcal{F}(t.0)=0$. According to (2.6) and (3.3), given $\varphi_1,\varphi_2\in\mathcal{B}_c(s)$, $\|\varphi_1\|\le\xi, \|\varphi_2\|\le\xi$,

$$\begin{aligned}
\|(\mathcal{F}\Phi)(s,\varphi_1)-(\mathcal{F}\Phi)(s,\varphi_2)\| \;&\le\; M_h\int_{-\infty}^{+\infty} e^{-\beta|\tau|}(\|F_h(\tau+s,u^1_{\tau+s}+\Phi(\tau+s,u^1_{\tau+s}))-\\
&\quad\; F_h(\tau+s,u^2_{\tau+s}+\Phi(\tau+s,u^2_{\tau+s}))\|)d\tau\\
&\le\; M_hL(1+p_1)\int_{-\infty}^{+\infty} e^{-\beta|\tau|}\|u^1_{\tau+s}-u^2_{\tau+s}\|d\tau\\
&\le\; M_hL(1+p_1)M_c\|\varphi_1-\varphi_2\|\int_{-\infty}^{+\infty} e^{[LM_c(1+p_1)+\omega-\beta]|\tau|}d\tau\\
&=\; \frac{2M_hL(1+p_1)M_c}{\beta-[LM_c(1+p_1)+\omega]}\|\varphi_1-\varphi_2\|
\end{aligned}$$

. (3.21)

Provided that L is sufficiently small, we get

$$\|(\mathcal{F}\Phi)(s,\varphi_1)-(\mathcal{F}\Phi)(s,\varphi_2)\|\le p_1\|\varphi_1-\varphi_2\|. \quad (3.22)$$

This proves that $\mathcal{F}$ maps X into X.

Next we prove that $\mathcal{F}$ is a contraction. Given $\Phi_1,\Phi_2\in X$, let u^1_{t+s}, u^2_{t+s} are the functions given by Lemma 3.2 respectively for (s,φ,Φ_1) and (s,φ,Φ_2). Using similar arguments to those in Lemma 3.2, we only discuss the case for $t\ge 0$.

We note that

$$\begin{aligned}
\|\Phi_1(\tau+s,u^1_{\tau+s})-\Phi_2(\tau+s,u^2_{\tau+s})\| &= \|\Phi_1(\tau+s,u^1_{\tau+s})-\Phi_1(\tau+s,u^2_{\tau+s})\\
&\quad +\Phi_1(\tau+s,u^2_{\tau+s})-\Phi_2(\tau+s,u^2_{\tau+s})\|\\
&\le p_1\|u^1_{\tau+s}-u^2_{\tau+s}\|+\|u^2_{\tau+s}\||\Phi_1-\Phi_2|_X,
\end{aligned}$$

we obtain

$$\begin{aligned}
\|u^1_{t+s}-u^2_{t+s}\| &\le \int_0^t M_cL(1+p_1)\|u^1_{\tau+s}-u^2_{\tau+s}\|e^{\omega(t-\tau)}d\tau+\int_0^t M_cL|\Phi_1-\Phi_2|_X\|u^2_{\tau+s}\|e^{\omega(t-\tau)}d\tau,\\
&\le \frac{M_c\|\varphi\||\Phi_1-\Phi_2|_X}{1+p_1}e^{[\omega+LM_c(1+p_1)]t}+\int_0^t M_cL(1+p_1)\|u^1_{\tau+s}-u^2_{\tau+s}\|e^{\omega(t-\tau)}d\tau,
\end{aligned}$$

then

$$e^{-\omega t}\|u^1_{t+s}-u^2_{t+s}\| \le \frac{M_c\|\varphi\||\Phi_1-\Phi_2|_X}{1+p_1}e^{LM_c(1+p_1)t}+\int_0^t M_cL(1+p_1)\|u^1_{\tau+s}-u^2_{\tau+s}\|e^{-\omega\tau}d\tau.$$

An application of Gronwall's inequality, we get

$$e^{-\omega t}\|u^1_{t+s}-u^2_{t+s}\| \le \frac{M_c\|\varphi\|}{1+p_1}|\Phi_1-\Phi_2|_X e^{2M_cL(1+p_1)t},$$

that is

$$\|u^1_{t+s}-u^2_{t+s}\| \le \frac{M_c\|\varphi\|}{1+p_1}|\Phi_1-\Phi_2|_X e^{[2M_cL(1+p_1)+\omega]t}.$$

For $t\in\mathbb{R}$, we obtain that

$$\|u^1_{t+s}-u^2_{t+s}\| \le \frac{M_c\|\varphi\|}{1+p_1}|\Phi_1-\Phi_2|_X e^{[2M_cL(1+p_1)+\omega]|t|}. \tag{3.23}$$

According to the definition of $\mathcal{F}$ and (3.3), (3.14), (3.23), we obtain that

$$\begin{aligned}
\|(\mathcal{F}\Phi_1)(s,\varphi)-(\mathcal{F}\Phi_2)(s,\varphi)\| &\le M_hL\int_{-\infty}^{+\infty}e^{-\beta|\tau|}[\|u^1_{\tau+s}-u^2_{\tau+s}\|\\
&\quad +\|\Phi_1(\tau+s,u^1_{\tau+s})-\Phi_2(\tau+s,u^2_{\tau+s})\|]d\tau,\\
&\le M_hL\int_{-\infty}^{+\infty}e^{-\beta|\tau|}[(1+p_1)\|u^1_{\tau+s}-u^2_{\tau+s}\|\\
&\quad +\|\Phi_1(\tau+s,u^1_{\tau+s})-\Phi_2(\tau+s,u^2_{\tau+s})\|]d\tau,\\
&\le M_hL\int_{-\infty}^{+\infty}e^{-\beta|\tau|}[(1+p_1)\frac{M_c\|\varphi\|}{1+p_1}|\Phi_1-\Phi_2|_X e^{[2M_cL(1+p_1)+\omega]|\tau|}\\
&\quad +M_c\|\varphi\||\Phi_1-\Phi_2|_X e^{[M_cL(1+p_1)+\omega]\tau}d\tau,\\
&\le H\|\varphi\||\Phi_1-\Phi_2|_X,
\end{aligned} \tag{3.24}$$

where $H = \left(\frac{2LM_cM_h}{\beta - 2LM_c(1+p_1)-\omega} + \frac{2LM_cM_h}{\beta - LM_c(1+p_1)-\omega} \right)$, provided that L is sufficiently small such that

$$|(\mathcal{F}\Phi_1)(s,\varphi) - (\mathcal{F}\Phi_2)(s,\varphi)|_X \leq |\Phi_1 - \Phi_2|_X. \tag{3.25}$$

Therefore, $\mathcal{F}$ is a contraction on X, the fixed point $\Phi \in X$ is the desired solution and this completes the proof of the theorem. □

We will illustrate Theorem 3.1 with an example.

Example Consider the delay equation

$$\begin{cases} x' = x \sin t + (2 \sin t) y(t-1)^2, \\ y' = (-9 - \sin t) y + (\cos t) z(t-1)^2, \\ z' = (9 + \sin t) z - (\sin t) x(t-1)^2. \end{cases} \tag{3.26}$$

For each $\phi = (\phi_{\mathbf{1}}, \phi_{\mathbf{2}}, \phi_{\mathbf{3}}) \in \mathcal{B}$, let $f(t,\phi) = ((2\sin t)\phi_2(-1)^2, (\cos t)\phi_3(-1)^2, (-\sin t)\phi_1(-1)^2)$. Eq.(3.26) is obtained from perturbing by f a linear equation with evolution operator

$$T(t,s) = \begin{pmatrix} U(t,s) & 0 & 0 \\ 0 & V^+(t,s) & 0 \\ 0 & 0 & V^-(t,s) \end{pmatrix},$$

where

$$U(t,s) = e^{\cos s - \cos t},$$

and

$$V^+(t,s) = e^{-9(t-s)+\cos s - \cos t}.$$

and

$$V^-(t,s) = e^{9(t-s)+\cos t - \cos s}.$$

Now let $P(t)(x,y,z) = x$, $Q^+(t) = y$, $Q^-(t) = z$. It is easy to verify that for any $\omega > 0$

$$\|T(t,s)P(s)\| = \|U(t,s)\| \leq e^2 \leq M_c e^{\omega|t-s|},$$

where $M_c = e^2$, and

$$\|T(t,s)Q^+(s)\| = \|V^+(t,s)\| \leq e^2 e^{-9(t-s)}.$$

and

$$\|T(t,s)Q^-(s)\| = \|[V^-(t,s)]^{-1}\| \leq e^2 e^{-9(t-s)}.$$

This shows that the linear equation admits the assumptions of Theorem 3.1. Moreover, $f(t,0)=0$, and for any $\|\phi\|$ and $\|\psi\|$ sufficiently small, there exists $L>0$ sufficiently small satisfies

$$|f(t,\phi)-f(t,\psi)| \le L\|\phi-\psi\|.$$

Therefore, it follows from Theorem 3.1 that Eq. (3.26) has a center manifold.

4. Smoothness of the center manifold

After proving the existence of center manifold, it is naturally to consider the differentiability of center manifold about the initial value φ. In classical center manifold theory, J.Carr has already claimed in [5] that when the nonlinear term f is of class C^k, then center manifold has the same property. For the nonautonous case in finite space, B.Aulbach also claimed in [1] that the smoothness of center manifold depends to the smoothness of nonlinear term. For autonomous delay equations, Diekmann proved that center manifold is of class C^k when the nonlinear form is of class C^k. More generally, for infinite dimensional nonautonomous differential equations, C.Chicone and Y.Latushkin gave a directly proof of the smoothness of center manifold in [6].

Hence, in this section, we will discuss the smoothness of center manifold for nonautonomous delay equation. Except for the assumptions in Theorem 3.1, we will also assume that the nonlinear term f is of class C^k, and for $f\in C_b^k(\mathbb{R}\times\mathcal{B},\mathbb{R}^n)$, $F=X_0f$, let $|F|_1=\sup\limits_{\phi\in\mathcal{B},s\in\mathbb{R}}\|DF(s,\phi)\|$.

Theorem 4.1. *Suppose that assumptions in Theorem 3.1 hold, then if $f\in C_b^k(\mathbb{R}\times\mathcal{B},\mathbb{R}^n)$ and $|F|_1$ sufficiently small, center manifold W of (2.7) is of class C^k, more precisely, the mapping $\Phi(s,\cdot)$ given by 3.1 belongs under the foregoing conditions to the space $C_b^k(\mathcal{B}_c(s),\mathcal{B}_h(s))$.*

In order to make the proof more clearly, we give some notations first. According to (3.1), (3.2), then

$$u_t+\Phi(t,u_t)=T_c(t,s)\varphi+\int_s^t T_c(t,\tau)F_c(\tau,u_\tau+\Phi(\tau,u_\tau))d\tau+\int_\infty^\infty K(t,\tau)F_h(\tau,u_\tau+\Phi(\tau,u_\tau))d\tau.$$

Define $n(t,u_t)=u_t+\Phi(t,u_t)$, $n(t,\cdot):\mathcal{B}_c(t)\to\mathcal{B}(t)$, then we get

$$n(t,u_t)=T_c(t,s)\varphi+\int_s^t T_c(t,\tau)F_c(\tau,n(\tau,u_\tau)d\tau+\int_\infty^\infty K(t,\tau)F_h(\tau,n(\tau,u_\tau)d\tau, \tag{4.1}$$

as we have already given the function Φ, the solution $n(t, u_t)$ is a function of the initial value φ, in order to make the proof more clearly, we also need the function $m(t, \cdot) : \mathcal{B}_c(t) \to \mathcal{B}(t)$, let $m(t, T_c(t, s)\varphi) = n(t, u_t)$, then

$$\begin{aligned} m(t, T_c(t, s)\varphi) &= T_c(t, s)\varphi + \int_s^t T_c(t, \tau)F_c(\tau, m(\tau, T_c(\tau, s)\varphi)d\tau \\ &\quad + \int_\infty^\infty K(t, \tau)F_h(\tau, n(\tau, T_c(\tau, s)\varphi)d\tau, \end{aligned} \tag{4.2}$$

It is easy to know that $\Phi(s, \varphi) = Q(s)m(s, \varphi)$ and the operator $T_c(t, s)$ is a linear operator, it is sufficient to show that the mapping $\Phi(s, \cdot)$ and $m(s, \cdot)$ has the same differentiability. In order to simplify the calculation, we give the notation $m_t(\cdot) = m(t, \cdot)$. In order to prove the theorem, we will need the fibre contraction lemma cited and proved by A.Vanderbauwhede in [16].

Lemma 4.2. *Let X and Y be complete metric spaces and $F : X \times Y \to X \times Y$ a mapping of the form*

$$F(x, y) = (F_1(x), F_2(x, y)), \quad \forall (x, y) \in X \times Y,$$

with the following properties:
(i) $F_1 : X \to X$ has an attractive fixed point $x_0 \in X$;
(ii) $F_2 : X \times Y \to Y$ is a uniform contraction;
(iii) the mapping $F_2(\cdot, y_0) : X \to Y$ is continuous, where $y_0 \in Y$ is the fixed point of $F_2(x_0, \cdot) : Y \to Y$.
Then (x_0, y_0) is an attractive fixed point for F.

A repeated application of Lemma 4.2 gives the following lemma.

Lemma 4.3. *Let $k \geq 1$, and let $X_0, X_1, ..., X_k$ be complete metric spaces. Let $F : X_0 \times X_1 \times \cdots \times X_k \to X_0 \times X_1 \times \cdots \times X_k$ be a mapping of the form*

$$F(x_0, x_1, ..., x_k) = (F_0(x_0), F_1(x_0.x_1), ..., F_k(x_0, x_1, ..., x_k)),$$

such that each $F_i : X_0 \times X_1 \times \cdots \times X_i \to X_i (0 \leq i \leq k)$ is a uniform contraction. Then F has a unique fixed point $\bar{x}_0, \bar{x}_1, ..., \bar{x}_k \in X_0 \times X_1 \times \cdots \times X_k$. If moreover each of the mappings $F_i(\cdot, \bar{x}_i) : X_0 \times X_1 \times \cdots \times X_{i-1} \to X_i (0 \leq i \leq k)$ is continuous, then this fixed point is attractive.

Before we study the function $m(t,\cdot)$, we talk about some properties of the solution on center manifold firstly. According to (3.3) and (3.21), we obtain that

$$\|u_t + \Phi(t,u_t)\| \le Ce^{\eta|t-s|} \tag{4.3}$$

where $\eta = [LM_c(1+p_1)+\omega]$, $C = \max\big(M_c\|\varphi\|, \frac{2M_h L(1+p_1)M_c}{\beta-[LM_c(1+p_1)+\omega]}\|\varphi\|\big)$. Therefore, we introduce the space $\mathcal{B}^\eta(t) := \{\psi \in \mathcal{B}(t) | \|\psi\|_\eta := \sup\limits_{t,s\in\mathbb{R}} e^{-\eta|t-s|}\|\psi\| < \infty\}$. As

$$\|T_c(t,s)\varphi\| \le M_c e^{\omega|t-s|}\|\varphi\|,$$

we can claim that $m(t,\cdot) : \mathcal{B}^\eta_c(t) \to \mathcal{B}^\eta(t)$, and for any $\tilde{\varphi} \in \mathcal{B}^\eta_c(t)$, we get

$$m(t,\tilde{\varphi}) = \tilde{\varphi} + \mathcal{U} \circ \mathcal{F}(m(t,\tilde{\varphi})), \tag{4.4}$$

where $\mathcal{F} : \mathcal{B}^\eta(t) \to \mathcal{B}^\eta(t)$ and for $\psi \in \mathcal{B}^\eta(t)$ satisfies

$$\mathcal{F}(t,\psi) = F(t,\psi) \tag{4.5}$$

and $\mathcal{U} : \mathcal{B}^\eta(t) \to \mathcal{B}^\eta(t)$ with

$$\mathcal{U}(t,\psi) = \int_s^t T_c(t,\tau)P(\tau)\psi d\tau + \int_{-\infty}^{+\infty} K(t,\tau)Q(\tau)\psi d\tau, \tag{4.6}$$

furthermore, the operator $\mathcal{U}$ satisfies that

$$\begin{aligned}
e^{-\eta|t-s|}\|\mathcal{U}(t,\psi)\| &\le M_c\int_s^t e^{\omega|t-\tau|}e^{-\eta|t-\tau|}\|\psi\|_\eta d\tau + \int_{-\infty}^{+\infty} M_h e^{-\beta|t-\tau|}e^{-\eta|t-\tau|}\|\psi\|_\eta d\tau,\\
&\le \|\psi\|_\eta \sup_{t,s\in\mathbb{R}}\Big(\int_s^t M_c e^{(\omega-\eta)|t-\tau|}d\tau + \int_{-\infty}^{+\infty} M_h e^{-(\beta+\eta)|t-\tau|}d\tau\Big),\\
&\le \|\psi\|_\eta\Big[\max\Big(\int_0^{+\infty} M_c e^{(\omega-\eta)|\tau'|}d\tau', \int_{-\infty}^{0} M_c e^{(\omega-\eta)|\tau'|}d\tau'\Big)\\
&\quad + M_h\int_{-\infty}^{+\infty} e^{-(\beta+\eta)|\tau'|}d\tau'\Big],\\
&= \|\psi\|_\eta\Big(\frac{1}{\eta-\omega} + \frac{1}{\beta+\eta}\Big).
\end{aligned} \tag{4.7}$$

To be more precisely, there exists constant $\gamma(\eta)$ such that $\|\mathcal{U}\|_\eta \le \gamma(\eta)$. Because of this conclusion, we obtain that

$$\|m_t(\tilde{\varphi}) - \tilde{\varphi}\| = \|\mathcal{U} \circ \mathcal{F}(m_t(\tilde{\varphi}))\| \le \|\mathcal{U}\|_\eta |F|_0 < \infty,$$

This motivates us to introduce the space

$$M_0 := \{m_t(\cdot) \in C(\mathcal{B}_c^\eta(t), \mathcal{B}^\eta(t)) : \sup_{\tilde{\varphi} \in \mathcal{B}_c^\eta(t), t \in \mathbb{R}} \|m_t(\tilde{\varphi}) - \tilde{\varphi}\| < \infty, t \in \mathbb{R}\}. \quad (4.8)$$

M_0 is a complete metric space when we use the metric

$$d_0(m_t, \tilde{m}_t) := \sup_{\tilde{\varphi} \in \mathcal{B}_c^\eta(t)} \|m_t(\tilde{\varphi}) - \tilde{m}_t(\tilde{\varphi})\|, \quad (4.9)$$

We also define a mapping $F_0 : M_0 \to M_0$ by

$$F_0(m_t)(\tilde{\varphi}) = \tilde{\varphi} + \mathcal{U} \circ \mathcal{F}(m_t(\tilde{\varphi})), \qquad \forall \tilde{\varphi} \in \mathcal{B}_c^\eta(t), \forall m_t \in M_0. \quad (4.10)$$

For $m_{t,1}, m_{t,2} \in M_0$, taking L sufficient small and using the same way of the proof of (3.24), we can obtain that F_0 is a contraction mapping on M_0. For $1 \leq j \leq k$ we define

$$M_j := \{m_t^{(j)} = D^{(j)} m_t : \mathcal{B}_c^\eta(t) \to \mathcal{L}^n(\mathcal{B}), |m^{(j)}|_j := \sup_{\tilde{\varphi} \in \mathcal{B}_c^\eta(t), t \in \mathbb{R}} \|m_t^{(j)}(\tilde{\varphi})\| < \infty, t \in \mathbb{R}\} \quad (4.11)$$

$F_1 : M_0 \times M_1 \to M_1$ by

$$F_1(m_t, m_t^{(1)})(\tilde{\varphi}) = \mathrm{Id} + \mathcal{U} \circ \mathcal{F}^{(1)}(m_t(\tilde{\varphi})) m_t^{(1)}(\tilde{\varphi}), \quad (4.12)$$

and $F_j : M_0 \times M_1 \times ... \times M_j \to M_j$ by

$$\begin{aligned}
&F_j(m_t, m_t^{(1)}, ..., m_t^{(j)})(\tilde{\varphi})(\tilde{\varphi}_1, ..., \tilde{\varphi}_j) \\
&= \mathcal{U} \circ \mathcal{F}^{(1)}(m_t(\tilde{\varphi})) m_t^{(j)}(\tilde{\varphi}) \\
&+ \sum_{i=2}^{j} \sum_{r_1 + ... + r_i = j} \sum_{\{l\}} \mathcal{U} \circ \mathcal{F}^{(i)}(m_t(\tilde{\varphi}))(m_t^{r_1}(\tilde{\varphi})(\tilde{\varphi}_{l_1}, ..., \tilde{\varphi}_{l_{r_1}}), ..., \\
&m_t^{r_i}(\tilde{\varphi})(\tilde{\varphi}_{l_{r_1+...+r_{i-1}+1}}, ..., \tilde{\varphi}_{l_j})). \qquad (4.13)
\end{aligned}$$

Finally we define $F : M_0 \times M_1 \times \cdots \times M_k \to M_0 \times M_1 \times \cdots \times M_k$ by

$$F(m_t, m_t^{(1)}, ..., m_t^{(k)}) := (F_0(m_t), F_1(m_t, m_t^{(1)}), ..., F_k(m_t, m_t^{(1)}, ..., m_t^{(k)})). \quad (4.14)$$

We now check that the conditions of Lemma 4.3 are satisfied. We have already shown that F_0 is a contraction on M_0. It follows from that (4.12) and (4.13) that also $F_j(1 \le j \le k)$ is a uniform contraction on M_j with contraction constant

$$\|\mathcal{U}\|_\eta |F|_1 < 1.$$

The fixed point of F_0 is the mapping m_t. We now have all the ingredients to prove the theorem.

Proof of the theorem Choose $m_{t,0} \in M_0, m_{t,0} \in C^k$(for example $m_{t,0} =$ Id will do), then $F_0(m_{t,0}) \in C^k$, for $m_{t,0}^{(j)} \in M_j$ and define a sequence

$$\{(m_{t,n}, m_{t,n}^{(1)}, ..., m_{t,n}^{(k)}) | n \in \mathbb{N}^+ \cup 0\} \in M_0 \times M_1 \times \cdots \times M_k \tag{4.15}$$

by

$$(m_{t,n+1}, m_{t,n+1}^1, ..., m_{t,n+1}^k) = F(m_{t,n}, m_{t,n}^1, ..., m_{t,n}^k), \qquad n \ge 0. \tag{4.16}$$

Then each $m_{t,n} \in C^k$. For F is a contraction, and the fixed point is $(m_t, m_t^{(1)}, ..., m_t^{(k)})$, this implies that

$$\lim_{n\to\infty} \sup_{\tilde{\varphi}\in\mathcal{B}_c^\eta(t)} \|m_t(\tilde{\varphi}) - m_{t,n}(\tilde{\varphi})\| = 0 \tag{4.17}$$

and

$$\lim_{n\to\infty} \sup_{\tilde{\varphi}\in\mathcal{B}_c^\eta(t)} \|m_t^{(j)}(\tilde{\varphi}) - m_{t,n}^{(j)}(\varphi)\| = 0, \qquad 1 \le j \le k. \tag{4.18}$$

This proves that m_t is of class C^k. This completes the proof of the theorem.□

5. Existence of stable manifolds

When restricting the evolution operator $T(t,s)$ on $\mathcal{B}_h$, we can claim that the $T(t,s)$ admits a exponential dichotomy, therefore we can get the stable manifold when we consider the system on the $\mathcal{B}_h$. According to the definition of $P(t), Q(t)$, rewrite the (2.7) as

$$\begin{aligned} u_t &= T_c(t,s)\varphi + \int_s^t T_c(t,\tau)F_c(\tau, u_\tau + v_\tau^+ + v_\tau^-)d\tau, & (5.1)\\ v_t^+ &= T_h^+(t,s)\psi^+ + \int_s^t T_h^+(t,\tau)F_h^+(\tau, u_\tau + v_\tau^+ + v_\tau^-)d\tau, & (5.2)\\ v_t^- &= T_h^-(t,s)\psi^- + \int_s^t T_h^-(t,\tau)F_h^-(\tau, u_\tau + v_\tau^+ + v_\tau^-)d\tau, & (5.3) \end{aligned}$$

where $\varphi \in \mathcal{B}_c(s), \psi^+ \in \mathcal{B}_h^+(s), \psi^- \in \mathcal{B}_h^-(s)$. Be similar to the center manifold, for $p_2 > 0$ let Y be the set of Lipschitz function $\Psi(t,\cdot) : \mathcal{B}_h^+(t) \to \mathcal{B}_h^-(t)$ with Lipschitz constant p_2 for $t \in \mathbb{R}$, and when $u_t = 0$, $\Psi(t,0) = 0$. With the supremum norm

$$|\Psi|_Y = \sup\{\frac{\|\Psi(s,\psi^+)\|}{\|\psi^+\|}, s \in \mathbb{R}, \|\psi^+\| \neq 0, \psi^+ \in \mathcal{B}_h^+(s)\}$$

Y is a complete space. Then we define the manifold

$$S = \{(s, \psi^+ + \Psi(s,\psi^+)), s \in \mathbb{R}, \psi^+ \in \mathcal{B}_h^+(s)\} \tag{5.4}$$

as the stable manifold.

Theorem 5.1. *Assume that* $T(t,s)$ *and* f *satisfy the conditions* (2.12),(2.13)*and* (2.6)*, then provided that the constant* L *is sufficiently small and* $\mu > 0$*, for* $\psi^+ \in \mathcal{B}_h^+(s), \|\psi^+\| \leq \mu$*, there exists a unique function* $\Psi \in Y$ *such that* $S = \{(s, \psi^+ + \Psi(s,\psi^+)), \psi^+ \in \mathcal{B}_h^+\}$ *is a invariant manifold.*

Proof. The idea of the proof of the theorem is totally similar to the process of the center manifold. In order to prove S is an invariant manifold, for every $\psi^+ \in \mathcal{B}_h^+(s)$, $\|\psi^+\| \leq \mu$, we expect to prove that there exist $\Psi \in Y$ satisfies

$$v_t^+ = T_h^+(t,s)\psi^+ + \int_s^t T_h^+(t,\tau)F_h^+(\tau, u_\tau + v_\tau^+ + \Psi(\tau, v_\tau^+))d\tau, \tag{5.5}$$

$$\Psi(t, v_t^+) = T_h^-(t,s)\Psi(s,\psi^+) + \int_s^t T_h^-(t,\tau)F_h^-(\tau, u_\tau + v_\tau^+ + \Psi(\tau, v_\tau^+))d\tau. \tag{5.6}$$

The first step is to prove that for $\forall s \in \mathbb{R}, \Psi \in Y$, $\exists v_t^+ \in \mathcal{B}_h^+(t)$ satisfied (5.5) and the estimation

$$\|v_{t,1}^+ - v_{t,2}^+\| \leq M_h \|\psi_1^+ - \psi_2^+\| e^{[M_h L(1+p_2)-\beta](t-s)}, \tag{5.7}$$

holds, where $\psi_1^+, \psi_2^+ \in \mathcal{B}_h^+(s)$ and $v_{t,1}^+, v_{t,2}^+$ are functions satisfy (5.5) respectively for (s, ψ_1^+, Ψ) and (s, ψ_2^+, Ψ).

Be similar to Lemma 3.2, for $v_t^+ \in \mathcal{B}_h^+(t)$, we define $\|v\|_* = \frac{1}{M_h}\sup\{\frac{\|v_t^+\|}{e^{-\beta(t-s)}}, t \geq s\}$. Define

$$(\mathcal{L}v^+)(t) = T_h^+(t,s)\psi^+ + \int_s^t T_h^+(t,\tau)F_h^+(\tau, u_\tau + v_\tau^+ + \Psi(\tau, v_\tau^+))d\tau. \tag{5.8}$$

Firstly, we assume $t \in [s,T], T > 0$. For $v_{t,1}^+, v_{t,2}^+ \in \mathcal{B}_h^+(t)$, it follows readily from the definition of $\mathcal{L}$ and (2.12) that

$$\|\mathcal{L}^n v_1^+ - \mathcal{L}^n v_2^+\|_* \leq \frac{(M_h L(1+p_2))^n}{n!} \|v_1^+ - v_2^+\|_* (t-s)^n,$$

that is

$$\|\mathcal{L}^n v_1^+ - \mathcal{L}^n v_2^+\|_* \leq \frac{(M_h L(1+p_2)T)^n}{n!} \|v_1^+ - v_2^+\|_*. \tag{5.9}$$

For n large enough $\frac{(M_h L(1+p_2)T)^n}{n!} < 1$ and by a well known extension of the contraction principle $\mathcal{L}$ has a unique fixed point $v_t^+ \in \mathcal{B}_h^+(t)$. This fixed point is the desired solution of (5.5).

The uniqueness of v_t^+ and the proof of (5.7) are consequences of the following arguments. Rewrite Eq. (5.5) into

$$v_{t+s}^+ = T_h^+(t+s,s)\psi^+ + \int_0^t T_h^+(t+s,\tau+s) F_h^+(\tau+s, u_{\tau+s} + v_{\tau+s}^+ + \Psi(\tau+s, v_{\tau+s}^+)) d\tau. \tag{5.10}$$

For $\psi_1^+, \psi_2^+ \in \mathcal{B}_h^+(s)$, it easily follows from (2.12) and an application of Gronwall's inequality that

$$\|v_{t+s,1}^+ - v_{t+s,2}^+\| \leq M_h \|\psi_1^+ - \psi_2^+\| e^{[M_h L(1+p_2)-\beta]t}, \tag{5.11}$$

which yields (5.7) and the uniqueness of v_t^+.

We now prove the prolongation of the solution. We start by showing that for every $s \in \mathbb{R}, \psi^+ \in \mathcal{B}_h^+(s)$, the integral equation (5.5) has a unique solution v_t^+ on an interval $[s, s_1]$ whose length is bounded below by

$$|s - s_1| = \delta = \min\{1, \frac{\|\psi^+\|}{K(s)L(1+p_2) + N(s)L}\}, \tag{5.12}$$

where $M(s) = \sup\{\|T_h^+(t,s)\| : s \leq t \leq s+1\}$, $N(s) = \sup\{\|u_t\|, t \in [s, s+1]\}$, $K(s) = 2M(s)\|\psi^+\|$. The mapping $\mathcal{L}$ defined by (5.8) maps the ball of radius $K(s)$ centered at 0 of $\mathcal{B}_h^+(t)$ into itself. This follows from the estimate

$$\begin{aligned}
\|\mathcal{L}u(t)\| &\leq M(s)\|\psi^+\| + M(s)L \int_s^t (\|u_\tau\| + (1+p_2)\|v_\tau^+\|) d\tau, \\
&\leq M(s)\|\psi^+\| + M(s)L(1+p_2)K(s)\delta + M(s)LN(s)\delta, \\
&\leq 2M(s)\|\psi^+\| = K(s).
\end{aligned}$$

In this ball, $\mathcal{L}$ satisfies a uniform Lipschitz condition with constant L and thus it possesses a unique fixed point v^+ in the ball. This fixed point is the desired solution on the interval $[s, s_1]$. From what we have just proved, it follows that if v_t^+ is a mild solution of (5.5) on the interval $[s, T]$, it can be extended to the interval $[T, T+\delta]$ with $\delta > 0$ defined by (5.12). Because of this conclusion, we set the maximum interval of existence of v_t^+ as $[s, t_{max}]$, according to (5.7), we can obtain that if $t_{max} < \infty$, then $\lim\limits_{t\to t_{max}} \|v_t^+\| < \infty$. Thus we can extend the existence interval to $[s, \infty)$.

The second step is to prove that (5.6) is equivalent to

$$\Psi(s, \psi^+) = -\int_s^\infty [T_h^-(\tau, s)]^{-1} F_h^-(\tau, u_\tau + v_\tau^+ + \Psi(\tau, v_\tau^+))d\tau. \tag{5.13}$$

This conclusion can be easily obtained from Lemma 3.3, therefore we omit the details.

The last step is to prove that there exists $\Psi \in Y$ such that (5.6) holds, that is $\exists \Psi \in Y$ satisfies (5.13). Define operator $\mathcal{F}$ on Y by

$$\mathcal{F}\Psi(s, \psi^+) = -\int_s^\infty [T_h^-(\tau, s)]^{-1} F_h^-(\tau, u_\tau + v_\tau^+ + \Psi(\tau, v_\tau^+))d\tau. \tag{5.14}$$

For $\psi_1^+, \psi_1^+ \in \mathcal{B}_h^+(s)$, according to (5.7), then

$$\begin{aligned}\|\mathcal{F}\Psi(s, \psi_1^+) - \mathcal{F}\Psi(s, \psi_2^+)\| &\leq \int_s^{+\infty} M_h e^{-\beta(\tau - s)} L(1+p_2) M_h \|\psi_1^+ - \psi_2^+\| \|v_\tau^+\| d\tau \\ &\leq \int_s^{+\infty} M_h e^{-\beta(\tau - s)} L(1+p_2) M_h \|\psi_1^+ - \psi_2^+\| e^{[M_h L(1+p_2) - \beta](\tau - s)} d\tau \\ &\leq \frac{L(1+p_2) M_h^2}{2\beta - M_h L(1+p_2)} \|\psi_1^+ - \psi_2^+\|, \end{aligned} \tag{5.15}$$

this proves that $\mathcal{F}$ maps Y into Y.

For different $\Psi_1, \Psi_2 \in Y$, $\psi^+ \in \mathcal{B}_h^+(s)$, according to (5.10)

$$\begin{aligned}
\|v^+_{t+s,1} - v^+_{t+s,2}\| &\le \int_0^t M_h e^{-\beta(t-\tau)} L(\|v^+_{t+s,1} - v^+_{t+s,2}\| + \|\Psi_1(\tau+s, v^+_{\tau+s,1}) - \Psi_2(\tau+s, v^+_{\tau+s,2})\|)d\tau,\\
&\le \int_0^t M_h e^{-\beta(t-\tau)} L(\|v^+_{t+s,1} - v^+_{t+s,2}\| + \|\Psi_1(\tau+s, v^+_{\tau+s,1}) - \Psi_1(\tau+s, v^+_{\tau+s,2})\|\\
&\quad +\|\Psi_1(\tau+s, v^+_{\tau+s,2}) - \Psi_2(\tau+s, v^+_{\tau+s,2})\|)d\tau,\\
&\le \int_0^t M_h e^{-\beta(t-\tau)} L(1+p_2)\|v^+_{t+s,1} - v^+_{t+s,2}\| d\tau\\
&\quad + \int_0^t M_h e^{-\beta(t-\tau)} L|\Psi_1 - \Psi_2|_Y \|v^+_{\tau+s,2}\| d\tau,\\
&\le \int_0^t M_h e^{-\beta(t-\tau)} L(1+p_2)\|v^+_{t+s,1} - v^+_{t+s,2}\| d\tau\\
&+ \frac{M_h\|\psi^+\|}{1+p_2}|\Psi_1 - \Psi_2|_Y e^{[M_h L(1+p_2)-\beta]t},
\end{aligned} \tag{5.16}$$

where $v^+_{t+s,1}, v^+_{t+s,2}$ are the solutions of (5.5) respectively to $(s, \psi^+, \Psi_1), (s, \psi^+, \Psi_2)$, then applies Gronwall's inequality to (5.16), we get

$$\|v^+_{t+s,1} - v^+_{t+s,2}\| \le \frac{M_h|\Psi_1 - \Psi_2|\|\psi^+\|}{1+p_2} e^{[2M_h L(1+p_2)-\beta]t}. \tag{5.17}$$

Using (5.15) and (5.16)

$$\begin{aligned}
\|\mathcal{F}\Psi_1(s,\psi^+) - \mathcal{F}\Psi_2(s,\psi^+)\| &\le \int_s^{+\infty} M_h e^{-\beta(\tau-s)} L((1+p_2)\|v^+_{\tau,1} - v^+_{\tau,2}\| + \|v^+_{\tau,2}\||\Psi_1 - \Psi_2|_Y)d\tau,\\
&\le M_h L(\int_s^{+\infty} e^{-\beta(\tau-s)}(1+p_2)\frac{M_h\|\psi^+\|}{1+p_2}|\Psi_1 - \Psi_2|_Y e^{[2\rho-\beta](\tau-s)} d\tau\\
&\quad + \int_s^{+\infty} e^{-\beta(\tau-s)} M_h\|\psi^+\| e^{[\rho-\beta](\tau-s)}|\Psi_1 - \Psi_2|_Y d\tau),\\
&= D|\Psi_1 - \Psi_2|_Y\|\psi^+\|,
\end{aligned} \tag{5.18}$$

where $\rho = M_h L(1+p_2)$, $D = \left(\frac{1}{2\beta - 2M_h L(1+p_2)} + \frac{1}{2\beta - M_h L(1+p_2)}\right) M_h^2 L$. Taking L sufficient small, we get

$$|\mathcal{F}\Psi_1 - \mathcal{F}\Psi_2|_Y < |\Psi_1 - \Psi_2|_Y, \tag{5.19}$$

this implies that $\mathcal{F}$ is a contraction mapping on Y. An application of fixed point theorem we can prove the theorem. $\square$

We complete the proof of existence of stable manifold. With the totally same way, we can get the unstable manifold. We only give the main result and omit the process of the proof. For $p_3 > 0$, let Z be the set of Lipschitz functions $\Psi' : \mathcal{B}_h^-(t) \to \mathcal{B}_h^+(t)$ with Lipschitz constant p_3 for $t \in \mathbb{R}$ and when $u_t = 0$, then $\Psi'(t, 0) = 0$. We define the manifold

$$U = \{(s, \Psi'(s, \psi^-) + \psi^-), s \in \mathbb{R}, \psi^- \in \mathcal{B}_h^-\},$$

as the unstable manifold.

Theorem 5.2. *Assume that* $T(t, s)$ *and* f *satisfy the condition* (2.12),(2.13)*and* (2.6)*, then provided that the constant* L *is sufficiently small and* $\varepsilon > 0$*, for* $\psi^- \in \mathcal{B}_h^-(s), \|\psi^-\| \leq \varepsilon$*, there exists a unique function* $\Psi' \in Z$ *such that* $U = \{(s, \Psi'(s, \psi^-) + \psi^-), s \in \mathbb{R}, \psi^- \in \mathcal{B}_h^-\}$ *is a invariant manifold.*

Meanwhile, we can also claim the smoothness of stable and unstable manifold, which are completely same with center manifold, that is when the nonlinear term f is of class C^k, then the stable and unstable manifold are also of class C^k.

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