

Hajłasz Gradients Are Upper Gradients

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Abstract Let (X, d, μ) be a metric measure space, with μ a Borel regular measure. In this paper, we prove that, if $u \in L^1_{\text{loc}}(X)$ and g is a Hajłasz gradient of u , then there exists \tilde{u} such that $\tilde{u} = u$ almost everywhere and $4g$ is a p -weak upper gradient of \tilde{u} . This result avoids a priori assumption on the quasi-continuity of u used in [Rev. Mat. Iberoamericana 16 (2000), 243-279]. As an application, an embedding of the Morrey-type function spaces based on Hajłasz-gradients into the corresponding function spaces based on upper gradients is obtained. We also introduce the notion of local Hajłasz gradient, and investigate the relations between local Hajłasz gradient and upper gradient.

1 Introduction

As a substitute for the classical gradient in the metric measure space setting, the Hajłasz gradients were first introduced by Hajłasz [6] in 1996. This opened the door to the study of Sobolev spaces on metric measure spaces. Let (X, d, μ) be a metric measure space with a nontrivial Borel regular measure μ , which is finite on bounded sets and positive on open sets.

Definition 1.1. Given a measurable function u on X , a non-negative measurable function g on X is called a *Hajłasz gradient* of u if there is a set $E \subset X$ with $\mu(E) = 0$ such that, for all $x, y \in X \setminus E$,

$$(1.1) \quad |u(x) - u(y)| \leq d(x, y)[g(x) + g(y)].$$

In [6] the above notion was employed to introduce the Sobolev space $M^{1,p}(X)$ for $p \in (1, \infty)$ on a metric measure space (X, d, μ) . The Hajłasz-Sobolev space $M^{1,p}(X)$ is defined to be the set of all functions $u \in L^p(X)$ having a Hajłasz gradient $g \in L^p(X)$. The norm of this space is given by

$$\|u\|_{M^{1,p}(X)} := \|u\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)},$$

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where the infimum is taken over all Hajłasz gradients of u .

In 1998, Heinonen and Koskela [12] introduced another type of gradients in metric measure spaces called upper gradients. Recall that a *rectifiable curve* γ is a continuous mapping from an interval into X with finite length. In what follows, $|\gamma|$ denotes the image of γ in X . The p -modulus of a collection Γ of curves is defined by

$$\text{Mod}_p(\Gamma) := \inf_{\rho \in F(\Gamma)} \|\rho\|_{L^p(X)}^p,$$

where

$$F(\Gamma) := \left\{ \rho : X \rightarrow [0, \infty] : \rho \text{ is Borel measurable and } \int_{\gamma} \rho(s) ds \geq 1 \text{ for all rectifiable } \gamma \in \Gamma \right\}.$$

Definition 1.2. Let u be a measurable function on X . A non-negative Borel function g on X is called an *upper gradient* of u if

$$(1.2) \quad |u(\gamma(a)) - u(\gamma(b))| \leq \int_{\gamma} g(s) ds$$

holds true for all non-constant rectifiable curves $\gamma : [a, b] \rightarrow X$. If the inequality (1.2) holds true for all non-constant rectifiable curves in X except a family of curves of p -modulus zero, then g is called a p -weak upper gradient of u .

Notice that in the definition of p -weak upper gradient, we may relax the Borel measurability of g to μ -measurability of g ; see, for instance, [2, Section 1.5]. This is because by the assumption that μ is Borel regular, we have that whenever g is μ -measurable, there is a Borel measurable function $\tilde{g} \geq g$ such that $\tilde{g} = g$ almost everywhere.

Using the notion of p -weak upper gradients introduced in [16], the Newton-Sobolev space (also called the Newtonian space) $N^{1,p}(X)$, $p \in [1, \infty)$, on a metric measure space X was introduced in [22]. Recall that the space $\tilde{N}^{1,p}(X)$ is defined to be the set of all functions $u \in L^p(X)$ having a p -weak upper gradient $g \in L^p(X)$, equipped with the quasi-norm

$$\|u\|_{\tilde{N}^{1,p}(X)} := \|u\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)},$$

where the infimum is taken over all p -weak upper gradients of u . The *Newton-Sobolev space* $N^{1,p}(X)$ is the quotient space $\tilde{N}^{1,p}(X)/\sim$, where the relation \sim is defined by $u \sim v$ if and only if $\|u - v\|_{\tilde{N}^{1,p}(X)} = 0$.

It is natural to compare these two notions of gradients. As shown by [22, Lemma 4.7], a Hajłasz gradient of a continuous function u , up to some modifications on a set of measure zero, is an upper gradient; see also [7, 14]. On the other hand, it is known that an upper gradient may not be a Hajłasz gradient, even if the underlying space is well connected. In general, one should think of a Hajłasz gradient of a function as the Hardy-Littlewood maximal function of an upper gradient, if the metric measure space supports a Poincaré inequality.

Based on this fact and the density of continuous functions in the Hajlasz-Sobolev spaces, it was shown in [22] that Hajlasz-Sobolev spaces are continuously embedded into Newton-Sobolev spaces (see also [7, Theorem 8.6]). Furthermore, the argument there can easily be generalized to show the embedding from function spaces defined via Hajlasz gradients to the corresponding spaces based on upper gradients, provided that *continuous functions are dense in the Hajlasz type spaces*; see, for example, [4, 7, 10, 23]. The above discussion holds for Hajlasz and Newtonian spaces based on the function spaces $L^p(X)$. The goal of this paper is to address the issue of whether Hajlasz and Newtonian spaces based on other types of quasi-Banach spaces, e. g. Morrey spaces, coincide.

Our main result below shows that, in general, Hajlasz gradients are upper gradients.

Theorem 1.3. *Let $u, g \in L^1_{\text{loc}}(X)$. Suppose that g is a Hajlasz gradient of u . Then there exist $\tilde{u}, \tilde{g} \in L^1_{\text{loc}}(X)$ such that $u = \tilde{u}$ and $g = \tilde{g}$ almost everywhere and $4\tilde{g}$ is an upper gradient of \tilde{u} .*

Let $\mathcal{F}(X)$ be a quasi-Banach function space, with $\|\cdot\|_{\mathcal{F}(X)}$ being its quasi-norm. The *Hajlasz type Sobolev space* $H\mathcal{F}(X)$ is defined to be the set of all functions $u \in \mathcal{F}(X)$ having a Hajlasz gradient $g \in \mathcal{F}(X)$, and its quasi-norm is given by

$$\|u\|_{H\mathcal{F}(X)} := \|u\|_{\mathcal{F}(X)} + \inf_g \|g\|_{\mathcal{F}(X)},$$

where the infimum is taken over all Hajlasz gradients of u . The *Newton type Sobolev space* $N\mathcal{F}(X)$ is defined as above with Hajlasz gradients replaced by upper gradients.

Theorem 1.3 implies that, if $\mathcal{F}(X) \subset L^1_{\text{loc}}(X)$, then $H\mathcal{F}(X) \hookrightarrow N\mathcal{F}(X)$, without requiring the density of continuous functions in $H\mathcal{F}(X)$. Notice that, while Lipschitz functions and hence continuous functions are dense in the Hajlasz-Sobolev spaces ($\mathcal{F} = L^p$), Lipschitz functions are not in general dense in the Hajlasz type Morrey-Sobolev space $HM^p_q(X)$ ($\mathcal{F} = M^p_q$; see [18, Remark 4.8]), and we do not know whether continuous functions are dense in $HM^p_q(X)$ or not.

The proof of Theorem 1.3 is given in Section 2 and, in Section 3, we first introduce the notion of local Hajlasz gradients (see Definition 3.1) and further show that local Hajlasz gradients are upper gradients. The key tool used to prove Theorem 1.3 is an extension property of Lipschitz functions defined on subsets of X (see Lemma 2.2 below). In Section 4, we apply Theorem 1.3 to several concrete settings, including Morrey-Sobolev spaces in which Lipschitz functions are not dense; see [18].

2 Proof of Theorem 1.3

To prove Theorem 1.3, we need two technical lemmas. The following one follows from an easy argument; see [22] and [14, Lemma 9.2.2].

Lemma 2.1 (Refinement of Hajlasz gradients). *Let $u, g \in L^1_{\text{loc}}(X)$. Suppose that g is a Hajlasz gradient of u . Then there exists a Borel function $\hat{g} \in L^1_{\text{loc}}(X)$ such that $g = \hat{g}$ almost everywhere and, for all $x, y \in X$,*

$$|u(x) - u(y)| \leq d(x, y)[\hat{g}(x) + \hat{g}(y)].$$

Proof. From the definition of Hajlasz gradient, there is a set $E \subset X$ with $\mu(E) = 0$ such that, for all $x, y \in X \setminus E$,

$$|u(x) - u(y)| \leq d(x, y)[g(x) + g(y)].$$

Let $\tilde{g}(x) := g(x)$ for each $x \in X \setminus E$, and $\tilde{g}(x) := \infty$ for each $x \in E$. Then \tilde{g} is measurable and $\tilde{g} = g$ a.e. Moreover, it is obvious that for all $x, y \in X$,

$$|u(x) - u(y)| \leq d(x, y)[\tilde{g}(x) + \tilde{g}(y)].$$

By [2, Proposition 1.1], there exists a Borel function \hat{g} such that $\hat{g} \geq \tilde{g}$ and $\hat{g} = \tilde{g} = g$ a.e., which is as desired and completes the proof. \square

The following result can be found in [11, Theorem 6.2] and its proof; see also [21]. Recall that a real-valued function f on a metric space (X, d) is said to be *L-Lipschitz* if there exists a constant $L \in [1, \infty)$ such that, for all x, y in X ,

$$|f(x) - f(y)| \leq Ld(x, y).$$

Lemma 2.2 (Lipschitz extension). *Let $A \subset X$ and $L \geq 1$. Suppose that $f : A \rightarrow \mathbb{R}$ is an L -Lipschitz function. Then there exists an L -Lipschitz function $F : X \rightarrow \mathbb{R}$, given by setting, for all $x \in X$,*

$$F(x) := \inf_{y \in A} \{f(y) + Ld(x, y)\},$$

such that $F = f$ on A .

Now we are ready to prove Theorem 1.3.

Proof of Theorem 1.3. Due to Lemma 2.1, it is not restrictive to assume that g is Borel measurable and

$$(2.1) \quad |u(x) - u(y)| \leq d(x, y)[g(x) + g(y)]$$

holds true for all $x, y \in X$.

For each $k \in \mathbb{N} := \{1, 2, \dots\}$, denote $\{x \in X : g(x) \leq 2^k\}$ by E_k . Let

$$E := \bigcup_{k \in \mathbb{N}} E_k$$

and $F := X \setminus E$. Since $g \in L^1_{\text{loc}}(X)$, we know that $\mu(F) = 0$.

We complete the proof of Theorem 1.3 via two steps.

Step 1. For each non-constant rectifiable curve $\gamma : [a, b] \rightarrow X$ with $\gamma(a), \gamma(b) \in E$, we now show that

$$(2.2) \quad |u(\gamma(a)) - u(\gamma(b))| \leq 4 \int_{\gamma} g(s) ds.$$

To see this, notice that, for all $x, y \in E_k$,

$$|u(x) - u(y)| \leq d(x, y)[g(x) + g(y)] \leq 2^{k+1}d(x, y),$$

that is, u is 2^{k+1} -Lipschitz on E_k . By Lemma 2.2, we extend u to a 2^{k+1} -Lipschitz function u_k on X , defined by setting, for all $x \in X$,

$$(2.3) \quad u_k(x) := \inf_{y \in E_k} \left\{ u(y) + 2^{k+1}d(x, y) \right\}.$$

Let

$$g_k(x) := g(x)\chi_{E_k}(x) + 2^{k+1}\chi_{X \setminus E_k}(x), \quad x \in X.$$

By Lemma 2.2, we know that $u_k = u$ on E_k and u_k is a 2^{k+1} -Lipschitz function on X . Moreover, we claim that, for all $x, y \in X$,

$$(2.4) \quad |u_k(x) - u_k(y)| \leq d(x, y)[g_k(x) + g_k(y)].$$

Indeed, if $x, y \in E_k$, then, $u_k(x) = u(x)$, $u_k(y) = u(y)$, $g_k(x) = g(x)$ and $g_k(y) = g(y)$ and hence, by (2.1), we obtain (2.4). In the case that one of x and y belongs to $X \setminus E_k$, without loss of generality, we may assume that $x \in X \setminus E_k$. Then, by the fact that u_k is 2^{k+1} -Lipschitz and $g_k(x) = 2^{k+1}$, we conclude that

$$|u_k(x) - u_k(y)| \leq 2^{k+1}d(x, y) = d(x, y)g_k(x) \leq d(x, y)[g_k(x) + g_k(y)],$$

which completes the proof of the above claim.

We now show that g_k acts like an upper gradient of u_k on γ . To do so we adopt the method found in the proof of [7, Theorem 8.6]. Recall that we assume $\gamma(a), \gamma(b) \in E_k$, and hence $u(\gamma(a)) = u_k(\gamma(a))$ and $u(\gamma(b)) = u_k(\gamma(b))$. Let $\gamma : [a, b] \rightarrow X$ be parameterized by its arc-length. By Luzin's theorem there is a set $D \subset [a, b]$ of full measure such that for each $t \in D$ there exists a sequence $h_n \rightarrow 0$ such that $g_k \circ \gamma(t + h_n) \rightarrow g_k \circ \gamma(t)$. As the function $u_k \circ \gamma$ is Lipschitz from $[a, b]$ into X , for almost every $t \in D$ we have

$$\begin{aligned} \left| \frac{d}{dt} u_k \circ \gamma(t) \right| &= \left| \lim_{n \rightarrow \infty} \frac{u_k \circ \gamma(t + h_n) - u_k \circ \gamma(t)}{h_n} \right| \\ &\leq \limsup_{n \rightarrow \infty} \left| \frac{d(\gamma(t + h_n), \gamma(t))}{h_n} \right| [g_k \circ \gamma(t + h_n) + g_k \circ \gamma(t)] \\ &\leq 2g_k \circ \gamma(t). \end{aligned}$$

The Fundamental Theorem of Calculus ensures that

$$(2.5) \quad |u_k(\gamma(a)) - u_k(\gamma(b))| \leq \int_a^b \left| \frac{d}{dt} u_k \circ \gamma(t) \right| dt \leq 2 \int_a^b g_k \circ \gamma(t) dt.$$

On the other hand, observe that, if $y \in E_k$, then $g_k(y) = g(y)$ and, if $y \in X \setminus E_k$, then $g_k(y) = 2^{k+1} < 2g(y)$. Hence we obtain

$$|u(\gamma(a)) - u(\gamma(b))| \leq 4 \int_{\gamma} g(s) ds,$$

completing the proof of Step 1.

Step 2. We need to show that, for the remaining cases, namely, if $\gamma(a)$ or $\gamma(b)$ lies in F , inequality (2.2) also holds true.

Indeed, without loss of generality, we may assume that both $\gamma(a)$ and $\gamma(b)$ lie in $X \setminus E$. The proof for the other cases that only one of $\gamma(a)$ and $\gamma(b)$ belongs to $X \setminus E$ is similar but simpler, the details being omitted.

We let $\tilde{u}(x) := u(x)$ when $x \in E$ and, otherwise,

$$\tilde{u}(x) := \limsup_{k \rightarrow \infty} u_k(x),$$

where u_k is as in (2.3). Obviously, $\tilde{u} = u$ almost everywhere.

If $\int_{\gamma} g(s) ds$ is infinite, then (2.2) holds true trivially. Hence we may assume that $\int_{\gamma} g(s) ds < \infty$. Notice that $g = \infty$ on $X \setminus E$ and hence in this case

$$\mathcal{H}^1(|\gamma| \cap F) = 0,$$

since $\mathcal{H}^1(|\gamma| \cap F) > 0$ implies that $\int_{\gamma} g(s) ds = \infty$, where \mathcal{H}^1 denotes the 1-dimensional Hausdorff measure. Thus, we can find a point $t \in (a, b)$ such that $\gamma(t) \in E \cap |\gamma|$. Inequality (2.1) implies that $|u(\gamma(t))| < \infty$. For each $k \in \mathbb{N}$, by (2.5), we conclude that

$$|u_k(\gamma(a)) - u_k(\gamma(t))| \leq 2 \int_{\gamma|_{[a,t]}} g_k(s) ds \leq 4 \int_{\gamma|_{[a,t]}} g(s) ds.$$

Since $\gamma(t) \in E$, it follows that there exists $k_0 \in \mathbb{N}$ such that $u_k(\gamma(t)) = u(\gamma(t))$ for each $k \geq k_0$. This further implies that, for each $k \geq k_0$,

$$(2.6) \quad |u_k(\gamma(a)) - u(\gamma(t))| = |u_k(\gamma(a)) - u_k(\gamma(t))| \leq 4 \int_{\gamma|_{[a,t]}} g(s) ds$$

and hence

$$|u_k(\gamma(a))| \leq |u(\gamma(t))| + 4 \int_{\gamma|_{[a,t]}} g(s) ds < \infty.$$

Therefore, by (2.6), we obtain

$$\begin{aligned} |\tilde{u}(\gamma(a)) - \tilde{u}(\gamma(t))| &= \left| \limsup_{k \rightarrow \infty} u_k(\gamma(a)) - u(\gamma(t)) \right| \\ &\leq \limsup_{k \rightarrow \infty} |u_k(\gamma(a)) - u(\gamma(t))| \\ &\leq 4 \int_{\gamma|_{[a,t]}} g(s) ds. \end{aligned}$$

Similarly,

$$|\tilde{u}(\gamma(b)) - \tilde{u}(\gamma(t))| \leq 4 \int_{\gamma|_{[t,b]}} g(s) ds$$

and hence

$$|\tilde{u}(\gamma(a)) - \tilde{u}(\gamma(b))| \leq |\tilde{u}(\gamma(a)) - \tilde{u}(\gamma(t))| + |\tilde{u}(\gamma(b)) - \tilde{u}(\gamma(t))| \leq 4 \int_{\gamma} g(s) ds,$$

which is the desired inequality. The proof of Theorem 1.3 is then completed by combining the above two steps. \square

3 Local Hajlasz gradients

In this section, we show that, if g is only a local Hajlasz gradient (see the following definition) of u , then the conclusion of Theorem 1.3 still holds true.

Definition 3.1 (Local Hajlasz gradient). Let u be a measurable function on X . A non-negative measurable function g on X is called a *local Hajlasz gradient* of u , if for each $z \in X$ there exists an open set $U_z \ni z$ and $E_z \subset U_z$ with $\mu(E_z) = 0$ such that for all $x, y \in U_z \setminus E_z$,

$$(3.1) \quad |u(x) - u(y)| \leq d(x, y)[g(x) + g(y)].$$

For measurable functions u, g , we shall call g is a *strong Hajlasz gradient* of u in the set $U \subset X$, if for all $x, y \in U$, it holds true that

$$|u(x) - u(y)| \leq d(x, y)[g(x) + g(y)].$$

Obviously, the Hajlasz gradients of a measurable function are local Hajlasz gradients of that function. From the following Corollaries 3.3 and 3.5, it turns out that local Hajlasz gradients can serve as a connection between Hajlasz gradients and upper gradients.

Theorem 3.2. *Suppose that X is a separable space. Let $u, g \in L^1_{\text{loc}}(X)$. Suppose that g is a local Hajlasz gradient of u . Then there exist $\tilde{u}, \tilde{g} \in L^1_{\text{loc}}(X)$ such that $u = \tilde{u}$ and $g = \tilde{g}$ almost everywhere and $4\tilde{g}$ is an upper gradient of \tilde{u} .*

Proof. For each $x \in X$, we fix an open set $U_x \ni x$ such that g is a Hajlasz gradient of u in U_x . The collection $\{U_x : x \in X\}$ covers X ; hence by the separability of X we find a countable subcover $\{U_j\}_j$ of X .

For each j , there exists $D_j \subset U_j$ with $\mu(D_j) = 0$ such that, for all $x, y \in U_j \setminus D_j$,

$$|u(x) - u(y)| \leq d(x, y)[g(x) + g(y)].$$

Let $D := \bigcup_j D_j$, $\hat{g}(x) := g(x)$ for each $x \in X \setminus D$ and $\hat{g}(x) := \infty$ for each $x \in D$. By [2, Proposition 1.1], there exists a Borel function \tilde{g} such that $\tilde{g} \geq \hat{g}$ and $\tilde{g} = \hat{g} = g$ a.e. Notice that for each j , \tilde{g} is a strong Hajlasz gradient of u in U_j .

Let us refine u . Denote $E := \{x \in X : \tilde{g}(x) < \infty\}$ and $F := X \setminus E$. Notice that since $\tilde{g} = g$ a.e. and $g \in L^1_{\text{loc}}(X)$, we have $\mu(F) = 0$; moreover, $D \subset F$. For each j and each $x \in E \cap U_j$, we define

$$\tilde{u}_j(x) := u(x).$$

For each $x \in F \cap U_j$, we extend u on the set $\{y \in U_j : \tilde{g}(y) \leq 2^k\}$ to $u_{k,j}$ on X for each $k \in \mathbb{N}$, as in the proof of Theorem 1.3, finally set

$$\tilde{u}_j(x) := \limsup_{k \rightarrow \infty} u_{k,j}(x).$$

From the proof of Theorem 1.3 again, we know that the restriction of $4\tilde{g}$ to U_j is an upper gradient of \tilde{u}_j in U_j .

Denote by $Z(j, m)$ the set of points in $U_j \cap U_m$ where \tilde{u}_j is not equal to \tilde{u}_m , and let $Z := \bigcup_{m,j} Z(j, m)$. Notice that $Z \subset F$ and hence $\mu(Z) = 0$.

Now for each $x \in X \setminus Z$, there exists j such that $x \in U_j$, and we refine u by setting

$$\tilde{u}(x) := \tilde{u}_j(x),$$

and for each $x \in Z$, define

$$\tilde{u}(x) := 0.$$

Then \tilde{u} is well defined since, if $x \in (X \setminus Z) \cap U_j \cap U_m$, then $\tilde{u}_j(x) = \tilde{u}_m(x)$ by the definition of Z . Moreover, for each $x \in E$, we have $\tilde{u}(x) = u(x)$, and hence $\tilde{u} = u$ a.e.

Let us show that $4\tilde{g}$ is an upper gradient of \tilde{u} on U_j for each j . Suppose that γ is a nonconstant compact rectifiable curve that maps $[a, b]$ into U_j . If $\gamma(a), \gamma(b) \in U_j \setminus Z$, then since the restriction of $4\tilde{g}$ to U_j is an upper gradient of \tilde{u}_j in U_j , we deduce that

$$|\tilde{u}(\gamma(a)) - \tilde{u}(\gamma(b))| = |\tilde{u}_j(\gamma(a)) - \tilde{u}_j(\gamma(b))| \leq 4 \int_{\gamma} \tilde{g} ds.$$

Let us assume that $\gamma(a) \in U_j \cap Z$, the remaining cases can be proved similarly. Since $\gamma(a) \in U_j \cap Z$, there exists $m \neq j$ such that $\gamma(a) \in U_m$ and $\tilde{u}_m(x) \neq \tilde{u}_j(x)$. Moreover, since $U_m \cap U_j$ is open, there exists $c \in (a, b)$ such that $\gamma([a, c]) \subset U_m \cap U_j$.

Our aim will be attended by showing $\int_{\gamma} \tilde{g} ds = \infty$. Suppose towards a contradiction that $\int_{\gamma} \tilde{g} ds < \infty$. Then we have

$$\mathcal{H}^1(|\gamma| \cap F) = 0,$$

which implies that $|\gamma| \cap E$ is a dense subset of $|\gamma|$. Take $\epsilon := \frac{1}{16} |\tilde{u}_m(x) - \tilde{u}_j(x)|$. Then from the absolute continuity of integral, we deduce that there exists $d \in (a, c)$ such that $\gamma(d) \in U_m \cap U_j \cap E$ and

$$\int_{\gamma|_{[a,d]}} \tilde{g} ds < \epsilon.$$

On the other hand, since $\gamma([a, d]) \subset U_m \cap U_j$ and $\gamma(d) \in U_m \cap U_j \cap E$, we have

$$\begin{aligned} |\tilde{u}_m(\gamma(a)) - \tilde{u}_j(\gamma(a))| &\leq |\tilde{u}_m(\gamma(a)) - \tilde{u}_j(\gamma(d))| + |\tilde{u}_j(\gamma(a)) - \tilde{u}_j(\gamma(d))| \\ &= |\tilde{u}_m(\gamma(a)) - \tilde{u}_m(\gamma(d))| + |\tilde{u}_j(\gamma(a)) - \tilde{u}_j(\gamma(d))| \\ &\leq \int_{\gamma|_{[a,d]}} 4\tilde{g} ds + \int_{\gamma|_{[a,d]}} 4\tilde{g} ds \leq 8\epsilon \\ &\leq \frac{1}{2} |\tilde{u}_m(x) - \tilde{u}_j(x)|, \end{aligned}$$

which is a contradiction, and hence $\int_{\gamma} \tilde{g} ds = \infty$ as desired.

Finally we show that $4\tilde{g}$ is an upper gradient of \tilde{u} .

It only remains to show that inequality (2.2) holds whenever γ is a nonconstant compact rectifiable curve in X that may not lie entirely in some U_j . Let γ be a non-constant compact rectifiable curve in X . Since $\{U_j\}_j$ is a cover of X , there is a subcover $\{U_{j_k}\}_{k=1}^{N(\gamma)}$ covering

the trajectory of γ . Since γ is a continuous curve and each U_{j_k} is an open set, it follows that we can cut γ up into $M(\gamma)$ sub-curves, denoted by γ_k , $k \in \{1, \dots, M(\gamma)\}$, such that for each k the subcurve $\gamma_k \subset U_{j_k}$. Denoting the end points of γ_k by x_k, y_k , and noticing that $y_k = x_{k+1}$ for $k \in \{1, \dots, M(\gamma) - 1\}$, we have

$$|\tilde{u}(\gamma(a)) - \tilde{u}(\gamma(b))| = |\tilde{u}(x_1) - \tilde{u}(y_{M(\gamma)})| \leq \sum_{k=1}^{M(\gamma)} |\tilde{u}(x_k) - \tilde{u}(y_k)|.$$

Since $\gamma_k \subset U_{j_k}$, we are allowed to apply inequality (2.2) to each γ_k . Hence we obtain

$$|\tilde{u}(\gamma(a)) - \tilde{u}(\gamma(b))| \leq \sum_{k=1}^{M(\gamma)} \int_{\gamma_k} 4\tilde{g} ds = \int_{\gamma} 4g ds.$$

It follows that we have (2.2) even for curves γ that do not lie entirely in some U_j . Thus $4\tilde{g}$ is an upper gradient of \tilde{u} in X . This finishes the proof of Theorem 3.2. \square

Corollary 3.3. *Suppose $u \in L^1_{\text{loc}}(X)$ and let $D(u)$ be the collection of all local Hajlasz gradients of u and $p \in (1, \infty)$. Then the closure of $D(u)$ in $L^p(X)$ is contained in the class of all weak upper gradients of u (up to a factor of 4).*

Recall that a measure μ on X is said to be *locally doubling*, if for each $R_0 \in (0, \infty)$ there exists a positive constant $C_d(R_0)$ such that for each $r \in (0, R_0/2)$ and all $x \in X$,

$$\mu(B(x, 2r)) \leq C_d(R_0)\mu(B(x, r)).$$

We say that μ is *globally doubling* if the above inequality holds with a positive constant that is independent of R_0 .

We also say that X *supports a local p -Poincaré inequality*, if for each $R_0 \in (0, \infty)$ there exists a positive constant $C_P(R_0)$ such that for each ball $B := B(x, r)$ with $r \in (0, R_0)$ and a function-upper gradient pair (u, g) in X , there exists $\lambda \in [1, \infty)$ so that

$$(3.2) \quad \frac{1}{\mu(B)} \int_B |u - u_B| d\mu \leq C_P(R_0) r \left[\frac{1}{\mu(\lambda B)} \int_{\lambda B} g^p d\mu \right]^{1/p}.$$

Here,

$$u_B := \frac{1}{\mu(B)} \int_B u d\mu$$

is the average of u on the ball B , and λB the ball concentric with B but with λ -times the radius of B . We then say that (X, d, μ) *supports a p -Poincaré inequality*, if (3.2) holds with a uniform constant C_P for all $R_0 \in (0, \infty)$.

Recall also that the *restricted Hardy-Littlewood maximal function* is defined for each $f \in L^1_{\text{loc}}(X)$ by

$$M_R f(x) := \sup_{0 < t \leq R} \frac{1}{\mu(B(x, t))} \int_{B(x, t)} f d\mu, \quad x \in X.$$

Proposition 3.4. *Let X be complete, μ locally doubling, and X support a local p -Poincaré inequality for some $p \in (1, \infty)$. Suppose that the restricted Hardy-Littlewood maximal function M_1 is bounded on $L^t(X)$ for all $t > 1$. Let $u \in L^1_{\text{loc}}(X)$ and $D(u)$ be the collection of all local Hajłasz gradients of u . Then the closure of $D(u)$ in $L^p(X)$ contains all the p -weak upper gradients of u modulo a positive constant multiple C .*

Proof. By the results of [15], we know that there exists some $q \in (1, p)$ such that X supports a local q -Poincaré inequality.

Given the assumptions of the proposition, let ρ be an upper gradient of u (since p -weak upper gradients of u can be approximated in $L^p(X)$ by upper gradients (see [16]), it suffices to show that upper gradients in $L^p(X)$ can be approximated by local Hajłasz gradients).

Because of the doubling property of μ and the support of the q -Poincaré inequality, we know that there exists a positive constant C such that, whenever $x, y \in X$ are Lebesgue points of u ,

$$|u(x) - u(y)| \leq C d(x, y) \left\{ [M_{2\lambda d(x,y)}(\rho^q)(x)]^{1/q} + [M_{2\lambda d(x,y)}(\rho^q)(y)]^{1/q} \right\};$$

see, for example, [8]. It follows that, for $r \in (0, \infty)$, $Cg_r := [M_r(\rho^q)]^{1/q}$ is a local Hajłasz gradient of u , with $U_z = B(z, r/(4\lambda))$ for each $z \in X$. Because $q < p$, we have $Cg_r \in D(u) \cap L^p(X)$ based on the assumption, and by the Lebesgue differentiation theorem we know that $g_r \rightarrow \rho$ almost everywhere in X as $r \rightarrow 0$. It follows from the monotone convergence theorem that $g_r \rightarrow \rho$ in $L^p(X)$, that is, $C^{-1}\rho$ is in the L^p -closure of $D(u)$. This finishes the proof of Proposition 3.4. \square

If the measure is doubling, then the Hardy-Littlewood maximal function is bounded on $L^q(X)$ for all $q \in (1, \infty]$; see for example [11]. Hence, a direct corollary of the previous result is as following.

Corollary 3.5. *Suppose that X is complete, μ is doubling, and X supports a p -Poincaré inequality for some $p \in (1, \infty)$. If $u \in L^1_{\text{loc}}(X)$ and $D(u)$ is the collection of all local Hajłasz gradients of u , then the closure of $D(u)$ in $L^p(X)$ contains all the p -weak upper gradients of u up to a multiplicative positive constant C that depends only on the data of doubling and Poincaré inequalities.*

Invoking the local Hajłasz gradient, one can introduce the corresponding Hajłasz-Sobolev space in an obvious manner. Precisely, for $p \in (1, \infty)$, the *Hajłasz-Sobolev space* $m^{1,p}(X)$ is defined to be the set of all functions $u \in L^p(X)$ having a local Hajłasz gradient $g \in L^p(X)$. The norm of this space is given by

$$\|u\|_{m^{1,p}(X)} := \|u\|_{L^p(X)} + \inf_g \|g\|_{L^p(X)},$$

where the infimum is taken over all local Hajłasz gradients of u .

The followings follow from Theorem 3.2 and Proposition 3.4 immediately.

Corollary 3.6. *Let $p \in [1, \infty)$. Then $M^{1,p}(X) \hookrightarrow m^{1,p}(X) \hookrightarrow N^{1,p}(X)$.*

From Corollary 3.6, together with [22, Thoerem 4.9], it follows the following conclusions.

Corollary 3.7. *Let $p \in (1, \infty)$. If X is complete, μ is doubling, and X supports a p -Poincaré inequality, then the spaces $m^{1,p}(X)$, $M^{1,p}(X)$ and $N^{1,p}(X)$ coincide with equivalent norms.*

On an n -dimensional Riemannian manifold \mathcal{M} with Ricci curvature bounded from below, it is well known that the measure and the Poincaré inequalities hold only locally, we do not know whether $M^{1,p}(\mathcal{M})$ and $N^{1,p}(\mathcal{M})$ coincide or not, however, the coincidence of $m^{1,p}(\mathcal{M})$ and $N^{1,p}(\mathcal{M})$ still holds true. Notice that M_1 is bounded on $L^q(\mathcal{M})$ for all $q \in (1, \infty)$; see [1].

Proposition 3.8. *Let \mathcal{M} be an n -dimensional Riemannian manifold with Ricci curvature bounded from below by $-K$, where $n \geq 2$ and $K \geq 0$. Then the spaces $m^{1,p}(\mathcal{M})$ and $N^{1,p}(\mathcal{M})$ coincide with equivalent norms.*

Obviously, the above proposition gives a new local characterization of the Newton-Sobolev space $N^{1,p}(X)$.

4 Some applications

In recent years, there are many attempts to extend classical Sobolev type spaces to metric measure settings based on some more general spaces besides Lebesgue spaces. In [5] Durand-Cartagena extended the Hajlasz-Sobolev and the Newton-Sobolev spaces to the limit case $M^{1,\infty}(X)$ and $N^{1,\infty}(X)$, via replacing the underlying function space $L^p(X)$ with $p \in [1, \infty)$ in the definitions of $M^{1,p}(X)$ and $N^{1,p}(X)$ by $L^\infty(X)$. Tuominen in [23] considered the Newton-Orlicz-Sobolev spaces, namely, Newtonian type spaces associated with Orlicz spaces via replacing $L^p(X)$ with $p \in [1, \infty)$ in the definitions of $N^{1,p}(X)$ by reflexive Orlicz spaces $L^\Phi(X)$. Harjulehto, Hästö and Pere [10] further considered the Newtonian type spaces based on an Orlicz-Musielak variable exponent space whose exponent function is essentially bounded. Using Lorentz spaces instead of Lebesgue spaces, Costea and Miranda [4] introduced Newtonian-Lorentz type Sobolev spaces on metric measure spaces. In [18], Hajlasz-Morrey-Sobolev spaces and Newton-Morrey-Sobolev spaces were introduced via replacing Lebesgue spaces by Morrey spaces. The most general scale of Newtonian type spaces is due to Malý [19, 20], who studied the Newtonian type spaces associated with a general quasi-Banach function lattice \mathcal{B} on X .

Recall that a *quasi-Banach function lattice* \mathcal{B} is a quasi-Banach space of real-valued measurable functions on X satisfying, if $f \in \mathcal{B}$ and $|g| \leq |f|$ almost everywhere, then $g \in \mathcal{B}$ and $\|g\|_{\mathcal{B}} \leq \|f\|_{\mathcal{B}}$. Now we recall the Newtonian space based on \mathcal{B} introduced in [19, 20].

Definition 4.1. (i) Let $\tilde{N}^1\mathcal{B}(X)$ be the space of all measurable functions $u \in \mathcal{B}$ which have an upper gradient $g \in \mathcal{B}$ and, for all $u \in \tilde{N}^1\mathcal{B}(X)$, let

$$\|u\|_{\tilde{N}^1\mathcal{B}(X)} := \|u\|_{\mathcal{B}} + \inf_g \|g\|_{\mathcal{B}},$$

where the infimum is taken over all upper gradients g of f .

The *Newton-Sobolev space* $N^1\mathcal{B}(X)$ based on \mathcal{B} is then defined to be the quotient space

$$N^1\mathcal{B}(X) := \tilde{N}^1\mathcal{B}(X) / \sim$$

equipped with $\|\cdot\|_{N^1\mathcal{B}(X)} := \|\cdot\|_{\tilde{N}^1\mathcal{B}(X)}$, where \sim is an *equivalence relation* defined by setting, for all $u_1, u_2 \in \tilde{N}^1\mathcal{B}(X)$, $u_1 \sim u_2$ if and only if $\|u_1 - u_2\|_{\tilde{N}^1\mathcal{B}(X)} = 0$.

(ii) The homogeneous version $\dot{N}^1\mathcal{B}(X)$ is defined via replacing the condition $u \in \mathcal{B}$ in the definition of $N^1\mathcal{B}(X)$ by $u \in L^1_{\text{loc}}(X)$ and the quasi-norm by

$$\|u\|_{\dot{N}^1\mathcal{B}(X)} := \inf_g \|g\|_{\mathcal{B}},$$

where the infimum is taken over all upper gradients g of u .

Recall that the \mathcal{B} -modulus of a collection Γ of curves is defined by

$$\text{Mod}_{\mathcal{B}}(\Gamma) := \inf_{\rho \in F(\Gamma)} \|\rho\|_{\mathcal{B}},$$

and g is called a \mathcal{B} -weak upper gradient of u , if inequality (1.2) holds true for all non-constant rectifiable curves in X except a family of curves of \mathcal{B} -modulus zero. Though the infimum in $\|u\|_{N^1\mathcal{B}(X)}$ is taken over all upper gradients g of u , it was proved in [19, Corollary 5.7] that $\|u\|_{N^1\mathcal{B}(X)}$ can be equivalently defined via the infimum over all \mathcal{B} -weak upper gradients g of u , namely,

$$\|u\|_{N^1\mathcal{B}(X)} = \|u\|_{\mathcal{B}} + \inf_g \|g\|_{\mathcal{B}},$$

where the infimum is taken over all \mathcal{B} -weak upper gradients of u . From this, we deduce that $N^1L^p(X) = N^{1,p}(X)$. Similar conclusion also holds true for homogeneous spaces.

Motivated by these definitions, one can also define the corresponding Hajlasz-Sobolev spaces based on \mathcal{B} .

Definition 4.2. (i) The *Hajlasz-Sobolev space* $M^1\mathcal{B}(X)$ based on \mathcal{B} is defined to be the space of all measurable functions $u \in \mathcal{B}$ which have a Hajlasz gradient $g \in \mathcal{B}$ and,

$$\|u\|_{M^1\mathcal{B}(X)} := \|u\|_{\mathcal{B}} + \inf_g \|g\|_{\mathcal{B}},$$

where the infimum is taken over all Hajlasz gradients g of u .

(ii) The homogeneous version $\dot{M}^1\mathcal{B}(X)$ is defined via replacing the condition $u \in \mathcal{B}$ in the definition of $M^1\mathcal{B}(X)$ by $u \in L^1_{\text{loc}}(X)$ and the quasi-norm by

$$\|u\|_{\dot{M}^1\mathcal{B}(X)} := \inf_g \|g\|_{\mathcal{B}},$$

where the infimum is taken over all Hajlasz gradients g of u .

Since Lebesgue spaces, Lorentz spaces, Orlicz spaces and Morrey spaces are all quasi-Banach lattices, these two scales of Sobolev type spaces, $M^1\mathcal{B}(X)$ and $\dot{M}^1\mathcal{B}(X)$, include the aforementioned Sobolev type spaces, $M^{1,p}(X)$ and $N^{1,p}(X)$ with $p \in [1, \infty]$,

Newton-Orlicz-Sobolev spaces, Newton-Lorentz-Sobolev spaces, Hajlasz-Morrey-Sobolev spaces and also Newton-Morrey-Sobolev spaces as special cases. Moreover, since \mathcal{B} is a general quasi-Banach function lattice, continuous functions might not be dense in some cases of the spaces $M^1\mathcal{B}(X)$, $\dot{M}^1\mathcal{B}(X)$, $N^1\mathcal{B}(X)$ and $\dot{N}^1\mathcal{B}(X)$; see, for example, [18].

As an immediate consequence of Theorem 1.3, we obtain the following relations between Hajlasz-Sobolev spaces and Newton-Sobolev spaces based on \mathcal{B} , the details being omitted.

Theorem 4.3. *Let \mathcal{B} be a quasi-Banach function lattice on X such that $\mathcal{B} \subset L^1_{\text{loc}}(X)$. Then*

$$M^1\mathcal{B}(X) \hookrightarrow N^1\mathcal{B}(X) \quad \text{and} \quad \dot{M}^1\mathcal{B}(X) \hookrightarrow \dot{N}^1\mathcal{B}(X).$$

In Theorem 4.3, \hookrightarrow denotes continuous embedding.

Comparing with the proof of $M^{1,p}(X) \hookrightarrow N^{1,p}(X)$ in [22, Theorem 4.8], the proof of Theorem 4.3 does not need the density of continuous functions in $M^1\mathcal{B}(X)$, $\dot{M}^1\mathcal{B}(X)$, $N^1\mathcal{B}(X)$ and $\dot{N}^1\mathcal{B}(X)$, which is the advantage of Theorem 4.3.

Let $p \in (1, \infty)$. Recall the notion of p -Poincaré inequality (3.2) from the previous section. It is well known that the Euclidean space supports a $(1, p)$ -Poincaré inequality. For more information on Poincaré inequalities, we refer the reader to [12, 13, 8, 17] and references therein.

In what follows, for all $r \in (0, \infty)$, $u \in L^r_{\text{loc}}(X)$ and $x \in X$, define

$$M_r(u)(x) := \sup_{B \ni x} \left\{ \frac{1}{\mu(B)} \int_B |u(y)|^r d\mu(y) \right\}^{1/r},$$

where the supremum is taken over all balls B of X containing x . Obviously, M_1 is just the classical Hardy-Littlewood maximal operator.

A measure μ on X is said to be *doubling* if there exists a positive constant C such that, for all balls B in X , $\mu(2B) \leq C\mu(B)$.

Theorem 4.4. *Let $p \in (1, \infty)$. Assume that \mathcal{B} is a quasi-Banach function lattice on X such that $\mathcal{B} \subset L^1_{\text{loc}}(X)$, X supports a weak $(1, p)$ -Poincaré inequality for some $p \in (1, \infty)$, and μ is doubling. If there exist $r \in (1, p)$ and a positive constant C such that, for all $f \in \mathcal{B}$,*

$$\|M_r(f)\|_{\mathcal{B}} \leq C\|f\|_{\mathcal{B}},$$

then

$$N^1\mathcal{B}(X) = M^1\mathcal{B}(X) \quad \text{and} \quad \dot{N}^1\mathcal{B}(X) = \dot{M}^1\mathcal{B}(X)$$

with equivalent quasi-norms.

Proof. By Theorem 4.3, it suffices to prove the embeddings $N^1\mathcal{B}(X) \hookrightarrow M^1\mathcal{B}(X)$ and $\dot{N}^1\mathcal{B}(X) \hookrightarrow \dot{M}^1\mathcal{B}(X)$. We only prove the first embedding due to similarity.

Let $u \in N^1\mathcal{B}(X)$. Then there exists an upper gradient g of u such that

$$\|u\|_{\mathcal{B}} + \|g\|_{\mathcal{B}} \sim \|u\|_{N^1\mathcal{B}(X)}$$

with the equivalent implicit positive constants independent of u and g . Since X supports a weak $(1, p)$ -Poincaré inequality, by [15, Theorem 1.0.1], we know that X also supports

a weak $(1, r)$ -Poincaré inequality and hence, by [8, Theorem 3.2], there exist a positive constant C and a set $E \subset X$ with $\mu(E) = 0$ such that, for all $x, y \in X \setminus E$,

$$|u(x) - u(y)| \leq Cd(x, y) [M_r(g)(x) + M_r(g)(y)].$$

Therefore, a positive constant multiple of $M_r(g)$ is a Hajlasz gradient of f . From the boundedness of M_r on \mathcal{B} , it follows that there exists a positive constant C such that, for all $u \in N^1\mathcal{B}(X)$,

$$\|u\|_{M^1\mathcal{B}(X)} \leq C \{\|u\|_{\mathcal{B}} + \|M_r(g)\|_{\mathcal{B}}\} \leq C \{\|u\|_{\mathcal{B}} + \|g\|_{\mathcal{B}}\} \leq C\|u\|_{N^1\mathcal{B}(X)},$$

and hence $N^1\mathcal{B}(X) \hookrightarrow M^1\mathcal{B}(X)$. This finishes the proof of Theorem 4.4. \square

We now present some examples of \mathcal{B} which satisfy the assumptions of Theorems 4.3 and 4.4.

Example 4.5. Lorentz spaces. Let $u : X \rightarrow [-\infty, \infty]$ be a μ -measurable function. We define the distribution μ_u by

$$\mu_u(t) := \mu(\{x \in X : |u(x)| > t\}), \quad t \in [0, \infty).$$

The nonincreasing rearrangement u^* of u is defined by

$$u^*(t) := \inf\{v \in [0, \infty) : \mu_u(v) \leq t\}, \quad t \in [0, \infty).$$

For all $p \in (1, \infty)$ and $q \in [1, \infty]$, the Lorentz space $L^{p,q}(X)$ is defined to be the set of all μ -measurable functions u on X such that $\|u\|_{L^{p,q}(X)} < \infty$, where

$$\|u\|_{L^{p,q}(X)} := \left\{ \int_0^\infty [t^{1/p} u^*(t)]^q \frac{dt}{t} \right\}^{1/q}$$

if $q < \infty$, and

$$\|u\|_{L^{p,\infty}(X)} := \sup_{t>0} t[\mu_u(t)]^{1/p} = \sup_{t>0} t^{1/p} u^*(t).$$

It is known that the Lorentz space is a quasi-Banach function lattice. Moreover, when $p \in (1, \infty)$, $q \in [1, p]$ and μ is doubling, the maximal operator M_r with $r \in [1, p)$ is bounded on $L^{p,q}(X)$; see, for example, [3] and [24].

Recall that the Newton-Lorentz-Sobolev space $N^1 L^{p,q}(X)$ was recently introduced by Costea and Miranda in [4]. The corresponding Hajlasz-Lorentz-Sobolev space $M^1 L^{p,q}(X)$ can be defined as in Definition 4.2. Then, by Theorems 4.3 and 4.4, we have the following conclusion on the relation between these two spaces.

Corollary 4.6. *Let $p \in (1, \infty)$ and $q \in [1, \infty]$.*

- (i) *It holds true that $M^1 L^{p,q}(X) \hookrightarrow N^1 L^{p,q}(X)$.*
- (ii) *If X supports a weak $(1, p)$ -Poincaré inequality and μ is doubling, then, for all $q \in [1, p]$,*

$$M^1 L^{p,q}(X) = N^1 L^{p,q}(X)$$

with equivalent quasi-norms.

Example 4.7. Morrey spaces. The Morrey-Sobolev space is our original motivation to compare Hajlasz gradients and upper gradients. It is shown by [18, Remark 4.9] that Lipschitz continuous functions are not dense in these spaces.

Let $0 < p \leq q \leq \infty$. The *Morrey space* $M_p^q(X)$ is defined to be the space of all measurable functions f on X such that

$$\|f\|_{M_p^q(X)} := \sup_{B \subset X} [\mu(B)]^{1/q-1/p} \left[\int_B |f(x)|^p d\mu(x) \right]^{1/p} < \infty,$$

where the supremum is taken over all balls in X . The space $M_p^q(X)$ is a quasi-Banach function lattice and becomes a Banach lattice if $1 \leq p \leq q \leq \infty$.

In [18], the Newton-Morrey-Sobolev space $N^1 M_p^q(X)$ and the Hajlasz-Morrey-Sobolev space $N^1 M_p^q(X)$ with $1 < p \leq q < \infty$ were introduced and studied. It was also showed therein that the set of Lipschitz functions may be not dense in these two spaces. Therefore, to show the coincidence between $N^1 M_p^q(X)$ and $N^1 M_p^q(X)$, we can not apply the approach used in [22] for the coincidence between Newton-Sobolev and Hajlasz-Sobolev spaces. However, notice that the Morrey space $M_p^q(X)$ supports the boundedness of the operator M_r when $r < p$. Then, by Theorems 4.3 and 4.4, we have the following conclusion.

Corollary 4.8. *Let $1 < p \leq q < \infty$.*

- (i) *It holds true that $M^1 M_p^q(X) \hookrightarrow N^1 M_p^q(X)$.*
- (ii) *If X supports a weak $(1, p)$ -Poincaré inequality and μ is doubling, then*

$$M^1 M_p^q(X) = N^1 M_p^q(X)$$

with equivalent norms.

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