A matrix differential Harnack estimate for a class of ultraparabolic equations *

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Abstract

Let u be a positive solution of the ultraparabolic equation

$$\partial_t u = \sum_{i=1}^{n} \partial_{x_i}^2 u + \sum_{i=1}^{k} x_i \partial_{x_{n+i}} u \qquad \text{on} \quad \mathbb{R}^{n+k} \times (0,T),$$

where $1 \leq k \leq n$ and $0 < T \leq +\infty$. Assume that u and its derivatives (w.r.t. the space variables) up to the second order are bounded on any compact subinterval of $(0,T)$. Then the difference $H(\log u) - H(\log f)$ of the Hessian matrices of $\log u$ and of $\log f$ (both w.r.t. the space variables) is non-negatively definite, where f is the fundamental solution of the above equation with pole at the origin $(0,0)$. The estimate in the case $n = k = 1$ is due to Hamilton. As a corollary we get that $\Delta l + \frac{n+3k}{2t} + \frac{6k}{t^3} \geq 0$, where $l = \log u$, and $\Delta = \sum_{i=1}^{n+k} \partial_{x_i}^2$.

Key words: ultraparabolic equation, matrix differential Harnack estimate, maximum principle.

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1 Introduction

Since the seminal work of Li and Yau [12], there is extensive research on differential Harnack inequalities for parabolic equations, see for example [13] for a survey. In particular, Hamilton [5] obtained a remarkable matrix differential Harnack estimate for the Ricci flow, whose trace form is very important. Note that the trace version of Hamilton's Harnack estimate for Ricci flow is derived as a corollary of his matrix Harnack estimate, and so far there is no direct proof for it (without using the matrix estimate). (See also Cao [1] for a matrix Harnack estimate for the Kähler- Ricci flow.) In [6] Hamilton gave a matrix Harnack estimate for the heat equation on certain Riemannian manifolds, whose trace form recovers an estimate in [12]. (See

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also [2] for a related estimate. Recently Ni and his cooperator have further work in this direction.) This matrix Harnack estimate for the heat equation is useful for deriving monotonicity formulas, see for example [7].

Recently, in [8], among other things, Hamilton extended his matrix Harnack estimate in [6] to the simple ultraparabolic equation $f_t + xf_y = f_{xx}$. In this note we'll generalize this estimate of Hamilton in [8] to the following slightly more general class of ultraparabolic equations

$$\partial_t u = \sum_{i=1}^{n} \partial_{x_i}^2 u + \sum_{i=1}^{k} x_i \partial_{x_{n+i}} u \qquad \text{on} \quad \mathbb{R}^{n+k} \times (0,T), \tag{1.1}$$

where $1 \le k \le n$ and $0 < T \le +\infty$.

(1.1) is among a still more general class of ultraparabolic equations of Kolmogorov type satisfying the Hörmander condition ([9]); for some of the work on these equations, see for example [10] and the references therein. Harnack inequalities for positive solutions of these and some similar equations are extensively studied by Polidoro et al., see for example [11], [14], and more recently, [3] and [4].

The main motivation of our research is to find more matrix differential estimates, whose power is partially indicated above, and to pursue more similar properties that the heat equation shares with the Kolmogorov type equations satisfying the Hörmander condition, which are partially displayed in some of the references cited above and in some papers not cited here.

Using Hörmander [9], and Lanconelli- Polidoro [11], one finds that the fundamental solution of the equation (1.1) with pole at the origin (0,0) is

$$\begin{aligned} f(x,t) &= \frac{C}{t^{\frac{n+3k}{2}}} e^{-\frac{1}{t}(\sum_{i=1}^{k} x_i^2 + \frac{1}{4}\sum_{i=k+1}^{n} x_i^2) - \frac{3}{t^2}\sum_{i=1}^{k} x_i x_{n+i} - \frac{3}{t^3}\sum_{i=1}^{k} x_{n+i}^2} \\ &= \frac{C}{t^{\frac{n+3k}{2}}} e^{-\frac{1}{4t}\sum_{i=1}^{n} x_i^2 - \frac{3}{t^3}\sum_{i=1}^{k}(x_{n+i} + \frac{1}{2}t x_i)^2}, \end{aligned}$$

where C is a constant depending only on n and k. Then using [9], [11] again (see the formula (1.6) in [11]) one can easily derive the fundamental solution

$$\Gamma(x,t;\xi,\tau) = \frac{C}{(t-\tau)^{\frac{n+3k}{2}}} e^{-\frac{1}{4(t-\tau)}\sum_{i=1}^{n}(x_i-\xi_i)^2 - \frac{3}{(t-\tau)^3}\sum_{i=1}^{k}(x_{n+i}-\xi_{n+i}+\frac{1}{2}(x_i+\xi_i)(t-\tau))^2}$$

with pole at any point (ξ, τ) from f, where $t > \tau$; we let $\Gamma(x,t;\xi,\tau) = 0$ when $t \le \tau$.

Now let $l = \log f$ and $l_{x_i x_j} = \partial_{x_i}\partial_{x_j} l$. Then

$$\begin{aligned} & l_{x_i x_i} = -\tfrac{2}{t}, \ \ l_{x_i x_{n+i}} = l_{x_{n+i} x_i} = -\tfrac{3}{t^2}, \ \ l_{x_{n+i} x_{n+i}} = -\tfrac{6}{t^3} \quad \text{for } 1 \le i \le k, \\ & l_{x_i x_i} = -\tfrac{1}{2t} \quad \text{for } k+1 \le i \le n, \quad \text{and} \\ & l_{x_i x_j} = 0 \quad \text{for all other } i,j. \end{aligned}$$

Thus we get the Hessian matrix $H(\log f) = (l_{x_i x_j})_{i,j=1,\cdots,n+k}$ of $\log f$ w.r.t. the space variables. Note that the matrix $((\log \Gamma(x,t;\xi,0))_{x_i x_j}) = ((\log f(x,t))_{x_i x_j})$ for any $\xi \in \mathbb{R}^{n+k}$.

Then we consider a general positive solution u of the equation (1.1) with $1 \leq k \leq n$, and the Hessian matrix of log u w.r.t the space variables: $H(\log u) = ((\log u)_{x_i x_j})_{i,j=1,\cdots,n+k}$.

We'll use the maximum principle to show the following

Theorem 1.1. *Let u be a positive solution to the equation (1.1) with $1 \leq k \leq n$ and $0 < T \leq +\infty$. Assume that u and its derivatives (w.r.t. the space variables) up to the second order are bounded on any compact subinterval of $(0, T)$. Then the Hessian $H(\log u) \geq H(\log f)$, that is, the matrix $H(\log u) - H(\log f)$ is non-negatively definite. Here, f is the fundamental solution at the origin as above.*

This extends Theorem 4.1 in [8] which treats the case $n = k = 1$. (We also slightly weaken the assumption of Theorem 4.1 in [8], where the solution is assumed to be bounded with bounded derivatives for $t \geq 0$. Of course the fact that the assumption can be weakened in this way should be known to Hamilton, although he did not state it explicitly there.) The estimate in Theorem 1.1 is sharp since the equality holds trivially when $u = f$. (Note that the assumption in Theorem 1.1 on u is satisfied by the fundamental solution f.) This matrix estimate contains much information. It implies that all the principal submatrices of the matrix $H(\log u) - H(\log f)$ are non-negatively definite. In particular, we can get some control of $l_{x_{n+i}x_{n+i}}$ in the form $l_{x_{n+i}x_{n+i}} + \frac{6}{t^3} \geq 0$ for $i = 1, \cdots, k$, where $l = \log u$, even though these second order derivatives do not appear in the equation (1.1). Below we will give three more consequences, two of which were known before (see [14]), one of which may be new. First by partially tracing the above estimate we recover a special case of Proposition 4.2 in Pascucci- Polidoro [14].

Corollary 1.2. *([14]) With the same assumption as in Theorem 1.1 and letting l =log u, we have*

$$\sum_{i=1}^{n} l_{x_i x_i} + \frac{n+3k}{2t} \geq 0.$$

The original proof of Proposition 4.2 in [14] uses a representation formula for positive solutions of a class of Kolmogorov equations more general than (1.1). By integrating the estimate in Corollary 1.2 along some optimal path we recover a special case of Corollary 1.2 in [14] (see Theorem 1.2 in [3] for an even more general version).

Corollary 1.3. *([14]) With the same assumption as in Theorem 1.1, for any points $(p_1, \cdots, p_{n+k}, t_1)$ and $(q_1, \cdots, q_{n+k}, t_2)$ with $0 < t_1 < t_2 < T$ there holds*

$$u(q_1, \cdots, q_{n+k}, t_2) \geq \left(\frac{t_1}{t_2}\right)^{\frac{n+3k}{2}} e^{-\sum_{i=1}^{n} \frac{(q_i - p_i)^2}{4(t_2 - t_1)} - \frac{3}{(t_2 - t_1)^3} \sum_{i=1}^{k} [q_{n+i} - p_{n+i} + \frac{1}{2}(q_i + p_i)(t_2 - t_1)]^2} u(p_1, \cdots, p_{n+k}, t_1).$$

Comparing the fundamental solution $\Gamma(x, t; \xi, 0)$ above, one sees that the estimate in Corollary 1.3 is sharp. (This was already observed in [14].) This estimate in the case $n = k = 1$ also appeared in [8], see Corollary 4.2 there. (By the way, note that there are some misprints in the statement of Corollary 4.2 and some other places in [8].) By fully tracing the matrix estimate in Theorem 1.1 we get

Corollary 1.4. *With the same assumption as in Theorem 1.1 and letting $l =$log u, there holds*

$$\Delta l + \frac{n+3k}{2t} + \frac{6k}{t^3} \geq 0,$$

where $\Delta = \sum_{i=1}^{n+k} \partial_{x_i}^2$.

This corollary seems to be new. It is also sharp. Compare a similar estimate in [12] for the heat equation (see Theorem 1.1 there). Note that, as already said above, the second order derivatives $u_{x_{n+i}x_{n+i}}$ (and $l_{x_{n+i}x_{n+i}}$), for $i = 1, \cdots, k$, do not appear in the equation (1.1). For this reason, it may not be easy to recover Corollary 1.4 by using the method in [14]. It may also be difficult to derive Corollary 1.4 by applying the maximum principle to the scalar equation satisfied by Δl, instead of the matrix equation satisfied by the Hessian $H(l)$, since the scalar equation satisfied by Δl contains terms involving $l_{x_i x_j}$ for some $i \neq j$, as can be seen by tracing the equation (2.2) in Section 2 below. This may be another evidence for the advantage of the matrix estimates.

While the equation (1.1) is very special, we expect similar matrix differential Harnack estimates should hold for a more general class of ultraparabolic equations of Kolmogorov type satisfying the Hörmander condition. See Section 4 for a more precise statement.

In the next two sections we'll prove Theorem 1.1 and Corollary 1.3 respectively, following [8] with some necessary modifications. In Section 4 we state two conjectures related to our results above.

2 Proof of Theorem 1.1

We may and will assume that $T < +\infty$. First we claim that we can reduce the proof of Theorem 1.1 to the case that the positive solution u and its derivatives (w.r.t. the space variables) up to the second order are uniformly bounded on $\mathbb{R}^{n+k} \times (0, T)$. The proof of this claim is an application of a standard trick: Suppose that u is a positive solution of the equation (1.1) such that u and its derivatives (w.r.t. the space variables) up to the second order are bounded on any compact subinterval of $(0, T)$. Fix $(x_0, t_0) \in \mathbb{R}^{n+k} \times (0, T)$. Let $v_\varepsilon(x, t) = u(x, t+\varepsilon)$ for $0 < \varepsilon < \frac{T-t_0}{2}$. Then v_ε is a positive solution of the same equation on $\mathbb{R}^{n+k} \times (-\varepsilon, T - \varepsilon)$ such that v_ε and its derivatives (w.r.t. the space variables) up to the second order are uniformly bounded on $\mathbb{R}^{n+k} \times [0, T - 2\varepsilon]$. If we have proven Theorem 1.1 in the case that the solution and its derivatives (w.r.t. the space variables) up to the second order are uniformly bounded, then the matrix Harnack estimate holds for v_ε at the point (x_0, t_0). Note that the conclusion of the matrix Harnack estimate is independent of the bounds of v_ε and its derivatives (w.r.t. the space variables) up to the second order. Then letting $\varepsilon \to 0$ one sees that the matrix Harnack inequality also holds for u at (x_0, t_0).

So in the proof below we assume that the positive solution u and its derivatives (w.r.t. the space variables) up to the second order are uniformly bounded on $\mathbb{R}^{n+k} \times (0, T)$. Let $l = \log u$, and $M = H(\log u) - H(\log f) = H(l) - H(\log f)$, where f is the fundamental solution of (1.1) at the origin (see Section 1). We decompose the matrix M into blocks:

$$M = \begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix},$$

where the $n \times n$ matrix

$$M_1 = \begin{pmatrix} l_{x_1x_1} + \frac{2}{t} & \cdots & l_{x_1x_k} & l_{x_1x_{k+1}} & \cdots & l_{x_1x_n} \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ l_{x_kx_1} & \cdots & l_{x_kx_k} + \frac{2}{t} & l_{x_kx_{k+1}} & \cdots & l_{x_kx_n} \\ l_{x_{k+1}x_1} & \cdots & l_{x_{k+1}x_k} & l_{x_{k+1}x_{k+1}} + \frac{1}{2t} & \cdots & l_{x_{k+1}x_n} \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ l_{x_nx_1} & \cdots & l_{x_nx_k} & l_{x_nx_{k+1}} & \cdots & l_{x_nx_n} + \frac{1}{2t} \end{pmatrix},$$

the $n \times k$ matrix

$$M_2 = \begin{pmatrix} l_{x_1x_{n+1}} + \frac{3}{t^2} & \cdots & l_{x_1x_{n+k}} \\ \cdot & \cdots & \cdot \\ l_{x_kx_{n+1}} & \cdots & l_{x_kx_{n+k}} + \frac{3}{t^2} \\ l_{x_{k+1}x_{n+1}} & \cdots & l_{x_{k+1}x_{n+k}} \\ \cdot & \cdots & \cdot \\ l_{x_nx_{n+1}} & \cdots & l_{x_nx_{n+k}} \end{pmatrix},$$

the $k \times n$ matrix

$$M_3 = M_2^T,$$

and the $k \times k$ matrix

$$M_4 = \begin{pmatrix} l_{x_{n+1}x_{n+1}} + \frac{6}{t^3} & \cdots & l_{x_{n+1}x_{n+k}} \\ \cdot & \cdots & \cdot \\ l_{x_{n+k}x_{n+1}} & \cdots & l_{x_{n+k}x_{n+k}} + \frac{6}{t^3} \end{pmatrix}.$$

By a direct computation one sees that l satisfies the equation

$$l_t = \sum_{i=1}^{n} (l_{x_ix_i} + l_{x_i}^2) + \sum_{i=1}^{k} x_i l_{x_{n+i}}, \tag{2.1}$$

and M satisfies the equation

$$M_t = \sum_{i=1}^{n} (M_{x_ix_i} + 2l_{x_i} M_{x_i}) + \sum_{i=1}^{k} x_i M_{x_{n+i}} + N, \tag{2.2}$$

where N is some matrix; actually N is the matrix obtained from $\tilde{N}$ below (see (2.3) and below) by replacing $\tilde{l}$ by l and setting $\sigma = 0, \alpha = 1, \beta = 1, \gamma = 1$. We want to use the maximum principle to show that the matrix M is non-negatively definite. But to deal with the noncompact situation we need to apply the maximum principle to a slightly modified equation (see (2.3) below), instead of the equation (2.2) above. So we modify the solution u to

$$\tilde{u} = u + \varepsilon\{t^2 \sum_{i=1}^{k} x_i^2 + \sum_{i=1}^{n+k} x_i^2 + 2t(\sum_{i=1}^{k} x_i x_{n+i} + n) + \frac{2k}{3} t^3\}$$

with ε a small positive constant, which is also a positive solution.

Let $\tilde{l} = \log \tilde{u}$. Then $\tilde{l}_{x_i x_j} \to 0$, for $i, j = 1, \cdots, n+k$, as $|x| \to \infty$ uniformly in t, since now we are assuming that u and its derivatives (w.r.t. the space variables) up to the second order are uniformly bounded on $\mathbb{R}^{n+k} \times (0, T)$. Note that $\tilde{l}$ satisfies the equation

$$\tilde{l}_t = \sum_{i=1}^{n} (\tilde{l}_{x_i x_i} + \tilde{l}_{x_i}^2) + \sum_{i=1}^{k} x_i \tilde{l}_{x_{n+i}}.$$

Let $\alpha = 1 + \sigma\delta_0, \beta = 1 + \sigma\theta_0, \gamma = 1 + \sigma\eta_0$, where σ is a small positive constant, and $\delta_0, \theta_0, \eta_0$ are constants which will be chosen later, and let

$$\tilde{M} = \begin{pmatrix} \tilde{M}_1 & \tilde{M}_2 \\ \tilde{M}_3 & \tilde{M}_4 \end{pmatrix},$$

where the $n \times n$ matrix

$$\tilde{M}_1 = \begin{pmatrix} \tilde{l}_{x_1x_1} + \frac{2\alpha}{t} & \cdots & \tilde{l}_{x_1x_k} & \tilde{l}_{x_1x_{k+1}} & \cdots & \tilde{l}_{x_1x_n} \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \tilde{l}_{x_kx_1} & \cdots & \tilde{l}_{x_kx_k} + \frac{2\alpha}{t} & \tilde{l}_{x_kx_{k+1}} & \cdots & \tilde{l}_{x_kx_n} \\ \tilde{l}_{x_{k+1}x_1} & \cdots & \tilde{l}_{x_{k+1}x_k} & \tilde{l}_{x_{k+1}x_{k+1}} + \frac{1+\sigma}{2t} & \cdots & \tilde{l}_{x_{k+1}x_n} \\ \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \tilde{l}_{x_nx_1} & \cdots & \tilde{l}_{x_nx_k} & \tilde{l}_{x_nx_{k+1}} & \cdots & \tilde{l}_{x_nx_n} + \frac{1+\sigma}{2t} \end{pmatrix},$$

the $n \times k$ matrix

$$\tilde{M}_2 = \begin{pmatrix} \tilde{l}_{x_1x_{n+1}} + \frac{3\beta}{t^2} & \cdots & \tilde{l}_{x_1x_{n+k}} \\ \cdot & \cdots & \cdot \\ \tilde{l}_{x_kx_{n+1}} & \cdots & \tilde{l}_{x_kx_{n+k}} + \frac{3\beta}{t^2} \\ \tilde{l}_{x_{k+1}x_{n+1}} & \cdots & \tilde{l}_{x_{k+1}x_{n+k}} \\ \cdot & \cdots & \cdot \\ \tilde{l}_{x_nx_{n+1}} & \cdots & \tilde{l}_{x_nx_{n+k}} \end{pmatrix},$$

the $k \times n$ matrix

$$\tilde{M}_3 = \tilde{M}_2^T,$$

and the $k \times k$ matrix

$$\tilde{M}_4 = \begin{pmatrix} \tilde{l}_{x_{n+1}x_{n+1}} + \frac{6\gamma}{t^3} & \cdots & \tilde{l}_{x_{n+1}x_{n+k}} \\ \cdot & \cdots & \cdot \\ \tilde{l}_{x_{n+k}x_{n+1}} & \cdots & \tilde{l}_{x_{n+k}x_{n+k}} + \frac{6\gamma}{t^3} \end{pmatrix}.$$

Claim 1 There is a positive constant σ_0 such that for any $0 < \sigma < \sigma_0$, $\tilde{M}$ is positive definite when $t > 0$ is sufficiently small.

Claim 1 follows from the fact that there is a positive constant σ_0 such that for any $0 < \sigma < \sigma_0$, all the leading principal minors of $\tilde{M}$ are positive when $t > 0$ is sufficiently small. The fact itself can be shown by a direct check: For the i-th leading principal minor, where $1 \le i \le n$, it is trivial since now $\tilde{l}_{x_j x_l}$ (for $j, l = 1, \cdots, n+k$) are uniformly bounded; for the $(n+i)$-th leading principal minor, where $1 \le i \le k$, it follows from that for $t > 0$ sufficiently small, the term

$$\begin{aligned} &\begin{vmatrix} \frac{2}{t} I_k & 0 & \frac{3}{t^2} J \\ 0 & \frac{1}{2t} I_{n-k} & 0 \\ \frac{3}{t^2} J^T & 0 & \frac{6}{t^3} I_i \end{vmatrix} \\ &= \begin{vmatrix} \mathrm{diag}(\frac{1}{2t}, \cdots, \frac{1}{2t}, \frac{2}{t}, \cdots, \frac{2}{t}) & 0 & \frac{3}{t^2} J \\ 0 & \frac{1}{2t} I_{n-k} & 0 \\ 0 & 0 & \frac{6}{t^3} I_i \end{vmatrix} \\ &= (\tfrac{1}{2t})^i (\tfrac{2}{t})^{k-i} (\tfrac{1}{2t})^{n-k} (\tfrac{6}{t^3})^i \end{aligned}$$

dominates the other terms in the expansion of the $(n+i)$-th leading principal minor of $\tilde{M}$, where the $k \times i$ matrix $J = \begin{pmatrix} I_i \\ 0 \end{pmatrix}$, and I_i is the $i \times i$ identity matrix.

Now we choose $(\delta_0, \theta_0, \eta_0)^T$ with $2\theta_0 \ge \eta_0$ to be an eigenvector of the matrix

$$C_0 := \begin{pmatrix} -8 & 10 & -3 \\ 10 & -14 & 5 \\ -3 & 5 & -2 \end{pmatrix}$$

corresponding to a positive eigenvalue. Note that the matrix C_0 does have a positive eigenvalue since its determinant is 2.

Let $F := (4\alpha^2 - \alpha - 3\beta)(\beta^2 - \gamma) - (2\alpha\beta - \beta - \gamma)^2$. Then

$$\begin{aligned} F &= (-4\delta_0^2 + 10\delta_0\theta_0 - 7\theta_0^2 - 3\delta_0\eta_0 + 5\theta_0\eta_0 - \eta_0^2)\sigma^2 + \cdots \\ &= \tfrac{1}{2}(\delta_0 \;\; \theta_0 \;\; \eta_0) C_0 (\delta_0 \;\; \theta_0 \;\; \eta_0)^T \sigma^2 + \cdots, \end{aligned}$$

where we have omitted the terms of order (w.r.t. σ) greater than 2. Clearly there is a positive constant σ_1 such that for any $0 < \sigma < \sigma_1$, $F > 0$.

Claim 2 With the above choice of δ_0, θ_0 and η_0, and assuming that $0 < \sigma < \min\{\sigma_0, \sigma_1\}$, the matrix $\tilde{M}$ is positive definite for all $t > 0$.

Theorem 1.1 follows from Claim 2 by first letting $\sigma \to 0$, then letting $\varepsilon \to 0$.

Before proving Claim 2 we note that $\tilde{M}$ satisfies the following equation

$$\tilde{M}_t = \sum_{i=1}^{n}(\tilde{M}_{x_i x_i} + 2\tilde{l}_{x_i}\tilde{M}_{x_i}) + \sum_{i=1}^{k} x_i \tilde{M}_{x_{n+i}} + \tilde{N}, \tag{2.3}$$

where

$$\tilde{N} = \begin{pmatrix} \tilde{N}_1 & \tilde{N}_2 \\ \tilde{N}_3 & \tilde{N}_4 \end{pmatrix},$$

where the $n \times n$ matrix

$$\tilde{N}_1 = \begin{pmatrix} \tilde{P}_1 & \tilde{P}_2 \\ \tilde{P}_3 & \tilde{P}_4 \end{pmatrix}$$

with the $k \times k$ matrix

$$\tilde{P}_1 = \begin{pmatrix} 2\sum_{i=1}^n \tilde{l}^2_{x_1 x_i} + 2\tilde{l}_{x_{n+1}x_1} - \frac{2\alpha}{t^2} & \cdots & 2\sum_{i=1}^n \tilde{l}_{x_1 x_i}\tilde{l}_{x_k x_i} + \tilde{l}_{x_{n+1}x_k} + \tilde{l}_{x_1 x_{n+k}} \\ \cdot & \cdots & \cdot \\ 2\sum_{i=1}^n \tilde{l}_{x_k x_i}\tilde{l}_{x_1 x_i} + \tilde{l}_{x_{n+k}x_1} + \tilde{l}_{x_k x_{n+1}} & \cdots & 2\sum_{i=1}^n \tilde{l}^2_{x_k x_i} + 2\tilde{l}_{x_{n+k}x_k} - \frac{2\alpha}{t^2} \end{pmatrix},$$

the $k \times (n-k)$ matrix

$$\tilde{P}_2 = \begin{pmatrix} 2\sum_{i=1}^n \tilde{l}_{x_1 x_i}\tilde{l}_{x_{k+1}x_i} + \tilde{l}_{x_{n+1}x_{k+1}} & \cdots & 2\sum_{i=1}^n \tilde{l}_{x_1 x_i}\tilde{l}_{x_n x_i} + \tilde{l}_{x_{n+1}x_n} \\ \cdot & \cdots & \cdot \\ 2\sum_{i=1}^n \tilde{l}_{x_k x_i}\tilde{l}_{x_{k+1}x_i} + \tilde{l}_{x_{n+k}x_{k+1}} & \cdots & 2\sum_{i=1}^n \tilde{l}_{x_k x_i}\tilde{l}_{x_n x_i} + \tilde{l}_{x_{n+k}x_n} \end{pmatrix},$$

the $(n-k) \times k$ matrix

$$\tilde{P}_3 = \tilde{P}_2^T,$$

the $(n-k) \times (n-k)$ matrix

$$\tilde{P}_4 = \begin{pmatrix} 2\sum_{i=1}^n \tilde{l}^2_{x_{k+1}x_i} - \frac{1+\sigma}{2t^2} & \cdots & 2\sum_{i=1}^n \tilde{l}_{x_{k+1}x_i}\tilde{l}_{x_n x_i} \\ \cdot & \cdots & \cdot \\ 2\sum_{i=1}^n \tilde{l}_{x_n x_i}\tilde{l}_{x_{k+1}x_i} & \cdots & 2\sum_{i=1}^n \tilde{l}^2_{x_n x_i} - \frac{1+\sigma}{2t^2} \end{pmatrix},$$

the $n \times k$ matrix

$$\tilde{N}_2 = \begin{pmatrix} 2\sum_{i=1}^n \tilde{l}_{x_1 x_i}\tilde{l}_{x_{n+1}x_i} + \tilde{l}_{x_{n+1}x_{n+1}} - \frac{6\beta}{t^3} & \cdots & 2\sum_{i=1}^n \tilde{l}_{x_1 x_i}\tilde{l}_{x_{n+k}x_i} + \tilde{l}_{x_{n+1}x_{n+k}} \\ \cdot & \cdots & \cdot \\ 2\sum_{i=1}^n \tilde{l}_{x_k x_i}\tilde{l}_{x_{n+1}x_i} + \tilde{l}_{x_{n+k}x_{n+1}} & \cdots & 2\sum_{i=1}^n \tilde{l}_{x_k x_i}\tilde{l}_{x_{n+k}x_i} + \tilde{l}_{x_{n+k}x_{n+k}} - \frac{6\beta}{t^3} \\ 2\sum_{i=1}^n \tilde{l}_{x_{k+1}x_i}\tilde{l}_{x_{n+1}x_i} & \cdots & 2\sum_{i=1}^n \tilde{l}_{x_{k+1}x_i}\tilde{l}_{x_{n+k}x_i} \\ \cdot & \cdots & \cdot \\ 2\sum_{i=1}^n \tilde{l}_{x_n x_i}\tilde{l}_{x_{n+1}x_i} & \cdots & 2\sum_{i=1}^n \tilde{l}_{x_n x_i}\tilde{l}_{x_{n+k}x_i} \end{pmatrix},$$

the $k \times n$ matrix

$$\tilde{N}_3 = \tilde{N}_2^T,$$

and finally, the $k \times k$ matrix

$$\tilde{N}_4 = \begin{pmatrix} 2\sum_{i=1}^n \tilde{l}^2_{x_{n+1}x_i} - \frac{18\gamma}{t^4} & \cdots & 2\sum_{i=1}^n \tilde{l}_{x_{n+1}x_i}\tilde{l}_{x_{n+k}x_i} \\ \cdot & \cdots & \cdot \\ 2\sum_{i=1}^n \tilde{l}_{x_{n+k}x_i}\tilde{l}_{x_{n+1}x_i} & \cdots & 2\sum_{i=1}^n \tilde{l}^2_{x_{n+k}x_i} - \frac{18\gamma}{t^4} \end{pmatrix}.$$

Now we prove Claim 2 by contradiction. Note that $\tilde{M} \geq cI_{n+k}$ as $|x| \to \infty$, where c is a positive constant which is uniform in t. Suppose for some $0 < \sigma < \min \{\sigma_0, \sigma_1\}$ Claim 2 is not true. Fix one such σ. Then by Claim 1 and the behavior of $\tilde{M}$ as $|x| \to \infty$, there would be a smallest $t_0 \in (0, T)$ such that there exist a point $x_0 \in \mathbb{R}^{n+k}$ and a nonzero vector $V = (v_1, \cdots, v_{n+k})^T \in \mathbb{R}^{n+k}$ with $\tilde{M}(x_0, t_0)V = 0$. Then at the space-time point (x_0, t_0),

$$\sum_{j=1}^{n+k} \tilde{l}_{x_ix_j} v_j = -\frac{2\alpha}{t} v_i - \frac{3\beta}{t^2} v_{n+i} \quad \text{for } 1 \leq i \leq k,$$
$$\sum_{j=1}^{n+k} \tilde{l}_{x_ix_j} v_j = -\frac{1+\sigma}{2t} v_i \quad \text{for } k+1 \leq i \leq n, \text{ and}$$
$$\sum_{j=1}^{n+k} \tilde{l}_{x_{n+i}x_j} v_j = -\frac{3\beta}{t^2} v_i - \frac{6\gamma}{t^3} v_{n+i} \quad \text{for } 1 \leq i \leq k.$$

It follows that

$$\begin{aligned}
&\tilde{N}(x_0, t_0)(V, V) \\
&= 2\sum_{i=1}^n (\sum_{j=1}^{n+k} \tilde{l}_{x_ix_j} v_j)^2 + 2\sum_{i=1}^k v_i (\sum_{j=1}^{n+k} \tilde{l}_{x_{n+i}x_j} v_j) \\
&-\frac{2\alpha}{t^2}\sum_{i=1}^k v_i^2 - \frac{12\beta}{t^3}\sum_{i=1}^k v_i v_{n+i} - \frac{1+\sigma}{2t^2}\sum_{i=k+1}^n v_i^2 - \frac{18\gamma}{t^4}\sum_{i=1}^k v_{n+i}^2 \\
&= 2\sum_{i=1}^k (\frac{2\alpha}{t} v_i + \frac{3\beta}{t^2} v_{n+i})^2 + 2\sum_{i=k+1}^n (\frac{1+\sigma}{2t} v_i)^2 - 2\sum_{i=1}^k v_i (\frac{3\beta}{t^2} v_i + \frac{6\gamma}{t^3} v_{n+i}) \\
&-\frac{2\alpha}{t^2}\sum_{i=1}^k v_i^2 - \frac{12\beta}{t^3}\sum_{i=1}^k v_i v_{n+i} - \frac{1+\sigma}{2t^2}\sum_{i=k+1}^n v_i^2 - \frac{18\gamma}{t^4}\sum_{i=1}^k v_{n+i}^2 \\
&= 2\{\frac{4\alpha^2-\alpha-3\beta}{t^2}\sum_{i=1}^k v_i^2 + \frac{6(2\alpha\beta-\beta-\gamma)}{t^3}\sum_{i=1}^k v_i v_{n+i} + \frac{9(\beta^2-\gamma)}{t^4}\sum_{i=1}^k v_{n+i}^2\} \\
&+\frac{\sigma^2+\sigma}{2t^2}\sum_{i=k+1}^n v_i^2.
\end{aligned}$$

By our choice of $\delta_0, \theta_0, \eta_0$ and σ, $\beta^2 - \gamma = \sigma^2\theta_0^2 + \sigma(2\theta_0 - \eta_0) > 0$ (noting that $2\theta_0 - \eta_0$ and θ_0 can not both be zero since $(1, 0, 0)^T$ is not an eigenvector of the matrix C_0), and $F = (4\alpha^2 - \alpha - 3\beta)(\beta^2 - \gamma) - (2\alpha\beta - \beta - \gamma)^2 > 0$. On the other hand, by Cauchy-Schwarz inequality $(\sum_{i=1}^k v_i v_{n+i})^2 \leq \sum_{i=1}^k v_i^2 \sum_{i=1}^k v_{n+i}^2$. It follows that $\tilde{N}(x_0, t_0)(V, V) > 0$.

Now we arrive at a contradiction by applying the equation (2.3) at (x_0, t_0) to (V, V): $\tilde{M}_t(x_0, t_0)(V, V) \leq 0$, $\tilde{M}_{x_i}(x_0, t_0)(V, V) = 0$, $\tilde{M}_{x_ix_i}(x_0, t_0)(V, V) \geq 0$, for $1 \leq i \leq n+k$, but $\tilde{N}(x_0, t_0)(V, V) > 0$. This completes the proof of Claim 2 and Theorem 1.1.

Remark From the proof above we see that if we assume that $l_{x_ix_j} \to 0$ $(i, j = 1, \cdots, n+k$, where $l = \log u)$ as $|x| \to \infty$ uniformly for t in any compact subinterval of $(0, T)$, instead of assuming that u and its derivatives (w.r.t. the space variables) up to the second order are bounded on any compact subinterval of $(0, T)$ as in the original statement of Theorem 1.1, then the result of Theorem 1.1 also holds true.

3 Proof of the Corollary 1.3

We follow closely Hamilton [8]. From the equation (2.1) satisfied by l and Corollary 1.2 we get that

$$l_t \geq -\frac{n+3k}{2t} + \sum_{i=1}^{n} l_{x_i}^2 + \sum_{i=1}^{k} x_i l_{x_{n+i}}.$$

Along any path with $\frac{dx_{n+i}}{dt} = -x_i$ $(1 \leq i \leq k)$ we compute

$$\frac{dl}{dt} = l_t + \sum_{i=1}^{n+k} l_{x_i}\frac{dx_i}{dt} \geq \sum_{i=1}^{n}(l_{x_i}^2 + l_{x_i}\frac{dx_i}{dt}) - \frac{n+3k}{2t} \geq -\frac{n+3k}{2t} - \frac{1}{4}\sum_{i=1}^{n}(\frac{dx_i}{dt})^2.$$

We integrate along such path and get

$$l(q_1, \cdots, q_{n+k}, t_2) \geq l(p_1, \cdots, p_{n+k}, t_1) - \frac{n+3k}{2}\log\frac{t_2}{t_1} - \frac{1}{4}\int_{t_1}^{t_2}\sum_{i=1}^{n}(\frac{dx_i}{dt})^2 dt. \quad (3.1)$$

The optimal path will minimize the integral

$$\int_{t_1}^{t_2}\sum_{i=1}^{n}(\frac{dx_i}{dt})^2 dt$$

with the constraints that

$$\int_{t_1}^{t_2} x_i dt = -(q_{n+i} - p_{n+i}) \quad (1 \leq i \leq k).$$

The Euler-Lagrange equations give that along the optimal path $\frac{d^2x_i}{dt^2}$ are constants independent of t for $1 \leq i \leq k$, and $\frac{dx_i}{dt}$ are constants independent of t for $k+1 \leq i \leq n$. So such path should have the form

$$\begin{aligned}
&x_i = 3a_i t^2 + 2b_i t + c_i, \quad 1 \leq i \leq k,\\
&x_i = d_i t + e_i, \quad k+1 \leq i \leq n,\\
&x_{n+i} = -(a_i t^3 + b_i t^2 + c_i t + f_i), \quad 1 \leq i \leq k,
\end{aligned}$$

where a_i, b_i, $\cdots$, f_i are constants. As in Hamilton [8] we compute the optimal path from $(p_1, \cdots, p_{n+k}, t_1)$ to $(q_1, \cdots, q_{n+k}, t_2)$, using the substitution

$$x_i = \hat{x}_i + \frac{q_i - p_i}{t_2 - t_1}t + \frac{p_i t_2 - q_i t_1}{t_2 - t_1} \quad (1 \leq i \leq n).$$

Now the problem is reduced to minimize

$$\int_{t_1}^{t_2}\sum_{i=1}^{n}(\frac{d\hat{x}_i}{dt})^2 dt$$

with the constraints that

$$\int_{t_1}^{t_2} \hat{x}_i dt = -(q_{n+i} - p_{n+i}) - \frac{1}{2}(q_i + p_i)(t_2 - t_1) \quad (1 \le i \le k),$$

and the boundary conditions

$$\hat{x}_i = 0 \quad \text{at} \quad t = t_1 \quad \text{and} \quad \text{at} \quad t = t_2 \quad \text{for} \quad 1 \le i \le n.$$

The solution is given by

$$\hat{x}_i = \tfrac{6}{(t_2-t_1)^3}[-(q_{n+i} - p_{n+i}) - \tfrac{1}{2}(q_i + p_i)(t_2 - t_1)](t_2 - t)(t - t_1) \ \text{ for } \ 1 \le i \le k,$$
$$\hat{x}_i = 0 \quad \text{for} \quad k + 1 \le i \le n.$$

Now

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} (\frac{d\hat{x}_i}{dt})^2 dt = \frac{12}{(t_2 - t_1)^3} \sum_{i=1}^{k} [q_{n+i} - p_{n+i} + \frac{1}{2}(q_i + p_i)(t_2 - t_1)]^2,$$

and

$$\int_{t_1}^{t_2} \sum_{i=1}^{n} (\frac{dx_i}{dt})^2 dt = \sum_{i=1}^{n} \frac{(q_i - p_i)^2}{t_2 - t_1} + \frac{12}{(t_2 - t_1)^3} \sum_{i=1}^{k} [q_{n+i} - p_{n+i} + \frac{1}{2}(q_i + p_i)(t_2 - t_1)]^2.$$

Then we insert this in (3.1) and Corollary 1.3 follows by exponentiating.

4 Two conjectures

First we propose

Conjecture 1 Theorem 1.1 still holds true without assuming that u and its derivatives (w.r.t. the space variables) up to the second order are bounded on any compact subinterval of $(0, T)$.

Compare the last remark in Section 2. One (non-direct) evidence for this conjecture is that there is no such assumption in Corollary 1.2 in [14]. Perhaps one way to prove Conjecture 1 is to try to localize the estimate in Section 2 above. Note that there is a localized (non-matrix) differential Harnack estimate in Li-Yau [12] for the heat equation. But so far, even for the heat equation, the localized matrix differential Harnack estimate is missing.

To state our second conjecture let

$$A = \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$B=\begin{pmatrix} 0 & B_1 & 0 & \cdots & 0 \\ 0 & 0 & B_2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & \cdots & B_r \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix},$$

be two constant real $N \times N$ matrices (for some N), where A_0 is a positive definite symmetric $p_0 \times p_0$ matrix, and B_i is a $p_{i-1} \times p_i$ matrix of rank p_i for $i = 1, \cdots, r$, where

$$p_0 \geq p_1 \geq \cdots \geq p_r \geq 1 \qquad \text{and} \qquad \sum_{i=0}^{r} p_i = N.$$

Then let the operator

$$L = \operatorname{div}(AD) + < x, BD > -\partial_t, \qquad (x,t) \in \mathbb{R}^N \times (0,T),$$

where $D = (\partial_{x_1}, \cdots, \partial_{x_N})^T$, and div is the divergence in $\mathbb{R}^N$.

Note that for the corresponding operator in our equation (1.1), $N = n+k$, $r = 1$, A_0 is the $n \times n$ identity matrix I_n, and B_1 is the $n \times k$ matrix

$$\begin{pmatrix} I_k \\ 0 \end{pmatrix}.$$

From [11] we know that the operator L satisfies Hörmander's hypoellipticity condition ([9]). In [14] (see also [3]) a Harnack estimate is obtained for the equation $Lu = 0$. Now we propose

Conjecture 2 The same result as in Theorem 1.1 still holds for the more general equation $Lu = 0$ for L defined above; of course now the fundamental solution f of (1.1) should be replaced by that of the equation $Lu = 0$.

If Conjecture 2 should be true then it would recover the Harnack estimate in [14] at least when the solution and its derivatives (w.r.t. the space variables) up to the second order are bounded on any compact subinterval of $(0,T)$. I hope that when $r \leq 2$ Conjecture 2 could be attacked by an argument similar to that in this paper. This will be checked in our future study. In general I expect the geometry of the operator L will play some role.

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