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Improvement of Hermite-Fujiwara Theorem for the Inertia of Polynomials*

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Abstract. In this paper, we first present an improvement of the well-known Hermite-Fujiwara theorem for the inertia of polynomials, and then use it to deduce some explicit formulas for the number of roots of a complex polynomial located in a given interval of the real axis in terms of the inertia of Bezout or Hankel matrices.

Key words: inertia of polynomial; inertia of matrix; Bezout matrix; Hankel matrix; Hermite-Fujiwara theorem.

AMS subject classifications: 12D10, 15A57

1 Introduction

Let $f(z)$ be a complex polynomial of the form

$$f(z) = \sum_{i=0}^{n} f_i z^i \ (f_n \neq 0). \tag{1.1}$$

We denote by $\pi'(f)$, $\nu'(f)$, $\delta'(f)$ the number of roots of $f(z)$ (counting for multiplicities), lying in the upper half plane, the lower half plane of the complex plane and on the real axis $\mathbb{R}$, respectively. The triple $\mathrm{In}'(f) = (\pi'(f), \nu'(f), \delta'(f))$ is referred to as the inertia of $f(z)$ with respect to $\mathbb{R}$. For an interval I of $\mathbb{R}$, we use the symbol $\delta_I(f)$, $\tilde{\delta}_I(f)$, $\tilde{\delta}_I^{(k)}(f)$ to design the number of roots, different roots, and different roots with multiplicity k, respectively, of $f(z)$ lying in the interval I. Obviously, $\delta'(f) = \delta_{\mathbb{R}}(f)$. We also define the triple $\mathrm{In}(A) = (\pi(A), \nu(A), \delta(A))$ as the inertia of an $n \times n$ matrix A, where $\pi(A)$, $\nu(A)$ and $\delta(A)$ stand for the number of eigenvalues of A (counting for multiplicities) lying in the right half-plane, the left half-plane of the complex plane and on the imaginary axis $\mathrm{i}\mathbb{R}$, respectively.

For a pair of complex polynomials $g(z),\ h(z)$ with the maximal degree n, the Bezout matrix $B(g, h)$ is defined by the bilinear form

$$\frac{g(z)h(w) - g(w)h(z)}{z - w} = (1, z, \cdots, z^{n-1})B(g, h)(1, w, \cdots, w^{n-1})^{\mathrm{T}}. \tag{1.2}$$

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As is well-known, the classical Hermite-Fujiwara theorem (see, e.g., [1,7]) says that if the Bezout matrix $B(f, \bar{f})$ is nonsingular, then

$$\mathrm{In}'(f) = \mathrm{In}(-\mathrm{i}B(f, \bar{f})), \quad \delta'(f) = 0, \tag{1.3}$$

in which $\bar{f}(z)$ is the conjugate polynomial of $f(z)$, i.e.,

$$\bar{f}(z) = \sum_{i=0}^{n} \bar{f}_i z^i.$$

In this nonsingular case, $\delta_I(f) = 0$ for every interval I of the real axis. On the other hand, if the Bezout matrix $B(f, \bar{f})$ is singular, the polynomials $f(z)$ and $\bar{f}(z)$ are not prime and then $f(z)$ can be written as $f(z) = p(z)f_1(z)$, in which $p(z) = \gcd(f(z), \bar{f}(z))$ is a monic real polynomial of degree at least one, and $\gcd(f_1(z), \bar{f}_1(z)) = 1$ (see, e.g., [2,8]). Since $\mathrm{In}'(f) = \mathrm{In}'(p) + \mathrm{In}'(f_1)$, by (1.3) we have in turn

$$\mathrm{In}'(f) = \mathrm{In}'(p) + \mathrm{In}(-\mathrm{i}B(f_1, \bar{f}_1)), \quad \delta'(f) = \delta'(p). \tag{1.4}$$

In this singular case, $\delta_I(f) = \delta_I(p)$, $\tilde{\delta}_I(f) = \tilde{\delta}_I(p)$, and $\tilde{\delta}_I^{(k)}(f) = \tilde{\delta}_I^{(k)}(p)$ for every interval I of the real axis. Thus it is desirable to find the number $\delta_I(p)$, $\tilde{\delta}_I(p)$ and $\tilde{\delta}_I^{(k)}(p)$ for such interval I.

Throughout this paper, we always assume that $f(z)$ is given as in (1.1) such that $B(f(z), \bar{f}(z))$ is singular, and $p(z) = \gcd(f(z), \bar{f}(z))$ is of degree $m \geqslant 1$.

In what follows, we will concentrate on the intrinsic relations between the number $\delta_I(f)$ or $\tilde{\delta}_I(f)$ or even $\tilde{\delta}_I^{(k)}(f)$ for a given interval I of the real axis and the inertia of the related Bezout or Hankel matrices, rather than attempt to consider how to evaluate the number $\delta_I(f)$ or $\tilde{\delta}_I(f)$ or even $\tilde{\delta}_I^{(k)}(f)$ efficiently. Actually, there are many fast or even superfast algorithms for computing the inertia of Bezout matrices and Hankel matrices, which are available in the literature (see, e.g., [9–16] and the references therein). Based on the relations presented here together with these existing efficient algorithms, computationally we may obtain the most efficient procedures to evaluate the number $\delta_I(f)$ or $\tilde{\delta}_I(f)$ or even $\tilde{\delta}_I^{(k)}(f)$ with little effort. Moreover it is worth noting that the basic strategy adopted in this paper is much different from those given in [2–6].

The rest of the paper is organized as follows. In Section 2, we give a factorization of $p(z)$ into a power product of its factors $p_k(z)$ $(k = 1, \cdots, s)$, in which each factor $p_k(z)$ is either the constant one or a monic real polynomial with only simple roots in the complex plane (Lemma 2.1 below), and then we formulate the inertia of each factor $p_k(z)$ with degree at least one in terms of the inertia of a certain Bezout or Hankel matrix. Consequently, we can formulate the inertia of $f(z)$ with respect to $\mathbb{R}$ and specially the number $\delta_{\mathbb{R}}(f)$ in terms of the inertia of a sequence of Bezout or Hankel matrices. The former result can be viewed as an improvement of the well-known Hermite-Fujiwara theorem (Theorem 2.3). In Section 3, we use the improvement result and a perturbation method of imposing a subtle rotation on the variable to deduce explicit formulas for the numbers $\delta_{\mathbb{R}^+}(f)$ and $\delta_{\mathbb{R}^-}(f)$, where $\mathbb{R}^+, \mathbb{R}^-$ stand for the positive real axis and the negative real axis, respectively (Theorem 3.2 and Corollary 3.3). Finally, in Section 4, we give some explicit formulas for the number $\delta_I(f)$ in the cases of $I = (a, +\infty)$, $I = (-\infty, a)$, and $I = (a, b)$, respectively (Theorem 4.2 and Corollary 4.3). Moreover, we can get more information about the roots of $f(z)$, such as the numbers $\tilde{\delta}_I^{(k)}(f)$ $(k = 1, 2, \cdots, s)$ of different roots with a given multiplicity k located in an interval I of the real axis (Corollaries 3.4, 4.4 and 4.5).

2 An improvement of Hermite-Fujiwara theorem for the inertia of polynomials

In this section, we give first a certain factorization of $p(z)$ into a power product of its factors with only simple roots in the complex plane, and then use it to formulate the inertia of $f(z)$ with respect to $\mathbb{R}$ in terms of the inertia of a sequence of Bezout or Hankel matrices. This later result can be viewed as an improvement of the well-known Hermite-Fujiwara theorem ([1]).

Starting from the real polynomial $p(z)$, by the help of Euclidean algorithm for polynomials, we compute easily a sequence of greatest common divisors $d_k(z)$ $(k = 1, \cdots, s)$:

$$\begin{aligned}
&d_1(z) = \gcd(p(z), p'(z)),\\
&d_2(z) = \gcd(d_1(z), d_1'(z)),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
&d_{s-1}(z) = \gcd(d_{s-2}(z), d_{s-2}'(z)) \neq 1,\\
&d_s(z) = \gcd(d_{s-1}(z), d_{s-1}'(z)) = 1.
\end{aligned} \tag{2.1}$$

Note that such s is a unique integer dependent on $p(z)$, and does coincide with the maximum of the multiplicities of the roots of $p(z)$. With these polynomials $d_k(z)$ $(k = 1, \cdots, s)$, we define in turn

$$q_1(z) = \frac{p(z)}{d_1(z)}, \quad q_k(z) = \frac{d_{k-1}(z)}{d_k(z)}, \quad k = 2, \cdots, s. \tag{2.2}$$

Obviously, such $q_k(z)$ $(k = 1, \cdots, s)$ are monic real polynomials with only simple roots in the complex plane and satisfy

$$p(z) = q_1(z)q_2(z)\cdots q_s(z). \tag{2.3}$$

Moreover, we check easily that the roots of $q_{k+1}(z)$ are also the roots of $q_k(z)$, and thus $q_{k+1}(z) \mid q_k(z)$ $(k = 1, \cdots, s-1)$. Thus we have that the number $\delta'(q_1)$ coincides with $\tilde{\delta}_{\mathbb{R}}(f)$. Now we define

$$p_k(z) = \frac{q_k(z)}{q_{k+1}(z)}, \quad k = 1, \cdots, s-1; \quad p_s(z) = q_s(z). \tag{2.4}$$

Then by (2.3) and (2.4) we obtain immediately a special factorization of $p(z)$ as follows. Here some of p_k $(k = 1, \cdots, s-1)$ are possibly equal to the constant one. For convenience, we introduce the symbol Δ:

$$\Delta = \{k \mid \deg p_k \geqslant 1,\ k = 1, \cdots, s\}.$$

Lemma 2.1. *Let $p(z) = \gcd(f(z), f'(z))$, and let $p_k(z)$ $(k = 1, \cdots, s)$ be monic real polynomials defined by* (2.1)–(2.4)*. Then $p(z)$ can be factorized as*

$$p(z) = \prod_{k\in\Delta} p_k^k(z), \tag{2.5}$$

in which $(p_i(z), p_j(z)) = 1$ for all $i, j \in \Delta$ $(i \neq j)$, and for $k \in \Delta$, $p_k(z)$ has only simple roots in the complex plane.

To determine the inertia $\mathrm{In}'(p)$ of $p(z)$ and the numbers $\delta_I(p)$, $\tilde{\delta}_I(p)$ and $\tilde{\delta}_I^{(k)}(p)$ for a given interval I of the real axis, it remains to determine the inertia $\mathrm{In}'(p_k)$ and the numbers $\delta_I(p_k)$,

$\tilde{\delta}_I(p_k)$, and, $\tilde{\delta}_I^{(k)}(p_k)$ for $k \in \Delta$, because

$$\begin{aligned}\mathrm{In}'(p) &= \sum_{k\in\Delta} k\mathrm{In}'(p_k),\\ \delta_I(p) &= \sum_{k\in\Delta} k\delta_I(p_k),\\ \tilde{\delta}_I(p) &= \sum_{k\in\Delta} \delta_I(p_k) = \delta_I(q_1),\\ \tilde{\delta}_I^{(k)}(p) &= \delta_I(p_k).\end{aligned} \tag{2.6}$$

Now we turn to consider generally the inertia of an arbitrary monic real polynomial $q(z)$ of degree at least one, with only simple roots in the complex plane. The following theorem shows that the inertia of $q(z)$ with respect to $\mathbb{R}$ can be formulated in terms of the inertia of the Bezout matrix $B(q, q')$ or the Hankel matrix $H(q, q') = (h_{i+j-1})_{i,j=1}^m$, in which $m = \deg q(z)$, h_1, h_2, $\cdots$, h_{2m-1} are the Markov parameters of the rational function $q'(z)/q(z)$.

Theorem 2.2. *Let $q(z)$ be a monic real polynomial of degree m ($m \geqslant 1$), which has only simple roots in the complex plane. Then*

$$\mathrm{In}'(q) = \{\nu(B(q, q')),\ \nu(B(q, q')),\ m - 2\nu(B(q, q'))\}, \tag{2.7}$$

or equivalently,

$$\mathrm{In}'(q) = \{\nu(H(q, q')),\ \nu(H(q, q')),\ m - 2\nu(H(q, q'))\}. \tag{2.8}$$

Proof. It is well known (see e.g. [2, Proposition 2.7]) that

$$B(q, q') = B(q, 1)H(q, q')B(q, 1). \tag{2.9}$$

We remark that (2.9) holds with $q'(z)$ replaced by an arbitrarily chosen polynomial $h(z)$ with $\deg h(z) \leqslant \deg q(z)$. Since $B(q, 1) = B(q, 1)^*$ is nonsingular, the Bezout matrix $B(q, q')$ and the Hankel matrix $H(q, q')$ have the same inertia. Now it suffices to prove (2.7) holds.

Let $q_\varepsilon(z) = q(z - \mathrm{i}\varepsilon)$, in which $\varepsilon > 0$ is sufficiently small. Then α is a root of $q(z)$ if and only if $\alpha + \mathrm{i}\varepsilon$ is a root of $q_\varepsilon(z)$, α lies in the open upper (lower) half-plane if and only if $\alpha + \mathrm{i}\varepsilon$ lies in the same half-plane for a sufficiently small $\varepsilon > 0$, and if α lies on the real axis then $\alpha + \mathrm{i}\varepsilon$ lies in the open upper half-plane. From above analysis we have $\nu'(q) = \nu'(q_\varepsilon)$. Note that the roots of $q(z)$ are symmetric with respect to the real axis, then

$$\mathrm{In}'(q) = \{\nu'(q_\varepsilon),\ \nu'(q_\varepsilon),\ m - 2\nu(q_\varepsilon)\}.$$

To describe the inertia of $q(z)$, it remains to compute $\nu(q_\varepsilon)$. First of all, we prove

$$\gcd(q_\varepsilon(z), \bar{q}_\varepsilon(z)) = 1. \tag{2.10}$$

We suppose that $q_\varepsilon(z) = q(z - \mathrm{i}\varepsilon)$ and $\bar{q}_\varepsilon(z) = q(z + \mathrm{i}\varepsilon)$ are not prime, or equivalently, $q(z - \mathrm{i}\varepsilon)$ and $q(z + \mathrm{i}\varepsilon)$ have a common root. If α is a common root of them, then $\alpha + \mathrm{i}\varepsilon, \alpha - \mathrm{i}\varepsilon$ are two different roots of $q(z)$ and the distance of them is equal to 2ε. Since $\varepsilon > 0$ is sufficiently small, it is impossible. So (2.10) holds and thus the Bezoutian matrix $B(q_\varepsilon, \bar{q}_\varepsilon)$ is nonsingular. By the well known Routh-Hurwitz-Fujiwara theorem, we have

$$\mathrm{In}'(q_\varepsilon) = \mathrm{In}(\frac{1}{\mathrm{i}}B(\varepsilon)), \tag{2.11}$$

in which $B(\varepsilon) = B(p_\varepsilon, \bar{p}_\varepsilon)$, $D = \mathrm{diag}(1, -1, \cdots, (-1)^{k-1})$.

We check easily that each entry of $B(\varepsilon)$ is a polynomial of ε and $B(0) = 0$, then $B(\varepsilon) = \varepsilon B'(0) + o(\varepsilon) = \varepsilon(B'(0) + o(1))$, in which $o(1) \to 0$ when ε tends to 0. This implies that

$$\mathrm{In}(\frac{1}{\mathrm{i}} B(\varepsilon)) = \mathrm{In}(\frac{1}{\mathrm{i}} B'(0) + o(1)). \tag{2.12}$$

Observe that

$$\begin{aligned}(1, z, \cdots, z^{m-1}) & B(\varepsilon)(1, \omega, \cdots, \omega^{m-1})^{\mathrm{T}} \\ &= \frac{q_\varepsilon(z)\bar{q}_\varepsilon(\omega) - q_\varepsilon(\omega)\bar{q}_\varepsilon(z)}{z - \omega} \\ &= \frac{q(z - \mathrm{i}\varepsilon)q(\omega + \mathrm{i}\varepsilon) - q(\omega - \mathrm{i}\varepsilon)q(z + \mathrm{i}\varepsilon)}{z - \omega},\end{aligned}$$

and thus

$$\begin{aligned}(1, z, \cdots, z^{m-1}) & B'(0)(1, \omega, \cdots, \omega^{m-1})^{\mathrm{T}} \\ &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\left(\frac{q(z - \mathrm{i}\varepsilon)q(\omega + \mathrm{i}\varepsilon) - q(\omega - \mathrm{i}\varepsilon)q(z + \mathrm{i}\varepsilon)}{z - \omega}\right)\Big|_{\varepsilon=0} \\ &= \mathrm{i}\left(-\frac{q'(z - \mathrm{i}\varepsilon)q(\omega + \mathrm{i}\varepsilon) - q'(\omega - \mathrm{i}\varepsilon)q(z + \mathrm{i}\varepsilon)}{z - \omega}\right. \\ &\qquad \left. + \frac{q(z - \mathrm{i}\varepsilon)q'(\omega + \mathrm{i}\varepsilon) - q(\omega - \mathrm{i}\varepsilon)q'(z + \mathrm{i}\varepsilon)}{z - \omega}\right)\Big|_{\varepsilon=0} \\ &= 2\mathrm{i}\left(\frac{q(z)q'(\omega) - q(\omega)q'(z)}{z - \omega}\right) \\ &= 2\mathrm{i}(1, z, \cdots, z^{m-1})B(q, q')(1, \omega, \cdots, \omega^{m-1})^{\mathrm{T}}.\end{aligned}$$

It follows from the last equation implies that $B'(0) = 2\mathrm{i}B(q, q')$. By (2.11) and (2.12), we have

$$\mathrm{In}'(q_\varepsilon) = \mathrm{In}(2B(q, q') + o(1)) = \mathrm{In}(B(q, q') + o(1)).$$

Since $q(z)$ has only simple roots in the complex plane,we have that the polynomials $q(z)$ and $q'(z)$ are prime. Then $B(q, q')$ is a nonsingular matrix. From the nonsingularity of $B(q, q')$ and the fact that the eigenvalues of a matrix are continuous functions of the entries of the matrix, we derive that $\mathrm{In}(B(q, q') + o(1)) = \mathrm{In}(B(q, q'))$. Hence,

$$\pi'(q) = \nu'(q) = \nu'(q_\varepsilon) = \nu(B(q, q')),$$

$$\delta'(q) = m - 2\nu'(q) = m - 2\nu(B(q, q')).$$

So (2.7) holds, as asserted. ■

Taking as basis Lemma 2.1 and Theorem 2.2 together with (1.4), (2.6), (2.7) and (2.8), we can formulate the inertia $\mathrm{In}'(f)$ in terms of the inertia of Bezout matrices $B(p_k, p_k')$ for $k \in \Delta$ or Hankel matrices $H(p_k, p_k')$ for $k \in \Delta$, which is a improvement to the well known Hermite-Fujiwara theorem for the inertia of polynomials.

Theorem 2.3. *Let $B(f, \bar{f})$ be singular, $p(z) = \gcd(f(z), \bar{f}(z))$ be of degree m $(m \geqslant 1)$ such* *that* (2.5) *holds and* $f(z) = p(z)f_1(z)$. *Then*

$$\begin{aligned}\mathrm{In}'(f) \;=\; \mathrm{In}\left(-\mathrm{i}B(f_1,\bar{f}_1)\right) + \Bigg\{ &\sum_{k\in\Delta} k\nu(B(p_k,p_k')),\ \sum_{k\in\Delta} k\nu(B(p_k,p_k')), \\ & m - 2\sum_{k\in\Delta} k\nu(B(p_k,p_k')) \Bigg\},\end{aligned}$$

or equivalently,

$$\begin{aligned}\mathrm{In}'(f) \;=\; \mathrm{In}\left(-\mathrm{i}B(f_1,\bar{f}_1)\right) + \Bigg\{ &\sum_{k\in\Delta} k\nu(H(p_k,p_k')),\ \sum_{k\in\Delta} k\nu(H(p_k,p_k')), \\ & m - 2\sum_{k\in\Delta} k\nu(H(p_k,p_k')) \Bigg\}.\end{aligned}$$

Since $\delta_{\mathbb{R}}(f) = \delta'(f)$, $\tilde{\delta}_{\mathbb{R}}(f) = \delta'(q_1) = \deg q_1(z) - 2\nu'(q_1)$ and $\tilde{\delta}^{(k)}_{\mathbb{R}}(f) = \delta'(p_k) = \deg p_k(z) - 2\nu'(p_k)$ for $k \in \Delta$, the following result is a direct consequence of Theorems 2.2 and 2.3.

Corollary 2.4. *Let* $B(f,\bar{f})$ *be singular, and let* $p(z)$ *be as in Theorem 2.3. Then*

$$\begin{aligned}\delta_{\mathbb{R}}(f) &= m - 2\sum_{k\in\Delta} k\nu(B(p_k,p_k')) = m - 2\sum_{k\in\Delta} k\nu(H(p_k,p_k')), \\ \tilde{\delta}_{\mathbb{R}}(f) &= \deg q_1(z) - 2\nu(B(q_1,q_1')) = \deg q_1(z) - 2\nu(H(q_1,q_1')),\end{aligned} \tag{2.13}$$

where $q_1(z)$ *is defined by* (2.2), *and*

$$\tilde{\delta}^{(k)}_{\mathbb{R}}(f) = \deg p_k(z) - 2\nu(B(p_k,p_k')) = \deg p_k(z) - 2\nu(H(p_k,p_k')) \tag{2.14}$$

for $k \in \Delta$.

3 Explicit formulas for the number $\delta_{\mathbb{R}^+}(f)$ and $\delta_{\mathbb{R}^-}(f)$ of positive and negative real roots of $f(z)$

In this section, we first evaluate the number $\delta_{\mathbb{R}^+}(q)$ of positive real roots of $q(z)$ in terms of the inertia of two associated Bezout matrices with $q(z)$, in which $q(z)$ is a monic real polynomial with only simple roots in the complex plane, then use this result to deduce explicit formulas for the numbers $\delta_{\mathbb{R}^+}(f)$ and $\delta_{\mathbb{R}^-}(f)$ of positive and negative real roots of $f(z)$, respectively.

By Theorem 2.2 and by use of a perturbation method of imposing a slight rotation on the variable, we can prove the following theorem on the rule of $\delta_{\mathbb{R}^+}(q)$.

Theorem 3.1. *Let* $q(z)$ *be a monic real polynomial of degree at least one, with only simple roots in the complex plane and* $q(0) \neq 0$. *Let* C_q *be the first companion matrix of* $q(z)$. *Then*

$$\delta_{\mathbb{R}^+}(q) = \pi(B(q,zq')) - \nu(B(q,q')) = \pi(B(q,q')C_q) - \nu(B(q,q')), \tag{3.1}$$

or equivalently,

$$\delta_{\mathbb{R}^+}(q) = \pi(H(q,zq')) - \nu(H(q,q')) = \pi(H(q,q')C_q^T) - \nu(H(q,q')). \tag{3.2}$$

Proof. Observe that the set of roots of the real polynomial $q(z)$ is symmetric with respect to $\mathbb{R}$, and a small clockwise rotation of the set via the origin $z = 0$ will destroy its symmetry property. We define $q_\theta(z) = q(z\mathrm{e}^{\mathrm{i}\theta})$ for $\theta \geqslant 0$, and choose sufficiently small positive θ such that $q_\theta(z)$ has neither roots located on $\mathbb{R}$ nor a pair of complex conjugate roots symmetric with respect to $\mathbb{R}$. In that case, the polynomials $q_\theta(z)$ and $\bar{q}_\theta(z)$ have no common roots in the complex plane, and thus $B(q_\theta, \bar{q}_\theta)$ is nonsingular. From the classical Hermite-Fujiwara theorem (see (1.3) above), we have

$$\nu'(q_\theta) = \nu(-\mathrm{i}B(q_\theta, \bar{q}_\theta)). \tag{3.3}$$

By the construction, for those sufficiently small $\theta > 0$, the roots of $q(z)$ lying in the upper (lower, resp.) half plane are changed to the roots of $q_\theta(z)$ in the same region, but the roots of $q(z)$ lying on the positive (negative, resp.) real axis are changed to the roots of $q_\theta(z)$ in the lower (upper, resp.) half plane. Then, by Theorem 2.2 we have in turn

$$\nu'(q_\theta) = \nu'(q) + \delta_{\mathbb{R}^+}(q) = \nu(B(q, q')) + \delta_{\mathbb{R}^+}(q). \tag{3.4}$$

Combining (3.3) and (3.4), we get

$$\delta_{\mathbb{R}^+}(q) = \nu(-\mathrm{i}B(q_\theta, \bar{q}_\theta)) - \nu(B(q, q')). \tag{3.5}$$

On the other hand, each entry of the Bezout matrix $B(p_\theta, \bar{q}_\theta)$ being a differential function of θ and $B(q_0, \bar{q}_0) = B(q, \bar{q}) = 0$, we have the expanded form

$$B(q_\theta, \bar{q}_\theta) = \theta B'(q_0, \bar{q}_0) + o(\theta). \tag{3.6}$$

By the definition of Bezout matrix (see (1.2)), we have

$$\begin{aligned}
&(1, z, \cdots, z^{n-1})B'(q_0, \bar{q}_0)(1, w, \cdots, w^{n-1})^{\mathrm{T}} \\
&= \left(\frac{q_\theta(z)\bar{q}_\theta(w) - q_\theta(z)\bar{q}_\theta(w)}{z - w}\right)'_{\theta=0} \\
&= \left(\frac{q(z\mathrm{e}^{\mathrm{i}\theta})q(w\mathrm{e}^{-\mathrm{i}\theta}) - q(w\mathrm{e}^{\mathrm{i}\theta})q(z\mathrm{e}^{-\mathrm{i}\theta})}{z - w}\right)'_{\theta=0} \\
&= 2\mathrm{i}\left(\frac{zq'(z)q(w) - q(z)wq'(w)}{z - w}\right) \\
&= (1, z, \cdots, z^{n-1})(2\mathrm{i}B(zq', q))(1, w, \cdots, w^{n-1})^{\mathrm{T}}.
\end{aligned}$$

This implies that $B'(q_0, \bar{q}_0) = 2\mathrm{i}B(zq', q)$. Since $q(0) \neq 0$, and thus the real polynomials $zq'(z)$ and $q(z)$ have no common zeros, $-\mathrm{i}B'(q_0, \bar{q}_0) = 2B(zq', q)$ is real symmetric and nonsingular. Therefore, from (3.6) it follows that

$$-\mathrm{i}B(q_\theta, \bar{q}_\theta) = \theta(-\mathrm{i}B'(q_0, \bar{q}_0) + o(1)) = \theta(2B(zq', q) + o(1)),$$

and thus

$$\begin{aligned}
\nu(-\mathrm{i}B(q_\theta, \bar{q}_\theta)) &= \nu(2B(zq', q) + o(1)) = \nu(2B(zq', q)) \\
&= \nu(B(zq', q)) = \pi(B(q, zq')).
\end{aligned} \tag{3.7}$$

Inserting (3.7) into (3.5), we obtain the first equality of (3.1) immediately.

Furthermore, the Barnett factorization formula for Bezout matrices (see, e.g., [2, Proposition 2.8]) says that $B(q, zq') = B(q, q')C_q$, in which C_q is the first companion matrix of $q(z)$. Then the rule of $\delta_{\mathbb{R}^+}(q)$ can be rewritten as

$$\delta_{\mathbb{R}^+}(q) = \pi(B(q, q')C_q) - \nu(B(q, q')). \tag{3.8}$$

On the other hand, the rule

$$\delta_{\mathbb{R}^+}(q) = \pi(H(q, zq')) - \nu(H(q, q'))$$

follows from (3.1) and (2.7) (and its extended form). The second rule of $\delta_{\mathbb{R}^+}(q)$ in (3.2) holds as well, since by a direct computation we can check that the relation $H(q, zq') = H(q, q')C_q^{\mathrm{T}}$ is valid. ■

As stated in the introduction part, the number $\delta_{\mathbb{R}^\pm}(f)$ coincide with the numbers $\delta_{\mathbb{R}^\pm}(p)$, in which $p(z) = \gcd(f(z), \bar{f}(z))$. Then by (2.5) we have

$$\delta_{\mathbb{R}^\pm}(f) = \sum_{k\in\Delta} k\delta_{\mathbb{R}^\pm}(p_k). \tag{3.9}$$

Thus to count $\delta_{\mathbb{R}^\pm}(f)$ it remains only to determine the numbers $\delta_{\mathbb{R}^\pm}(p_k)$ for each factor $p_k(z)$, $k \in \Delta$.

Applying Theorem 3.1, Corollary 3.2, (3.8) and (3.9) to each factor $p_k(z)$ of $p(z)$, $k \in \Delta$, we obtain two explicit formulas for the number $\delta_{\mathbb{R}^+}(f)$ in terms of the inertia of a sequence of Bezout or Hankel matrices.

Theorem 3.2. *Let $f(z)$ be given as in* (1.1) *and let $p(z) = \gcd(f(z), \bar{f}(z))$ have a factorization as in* (2.5). *If $p(0) \neq 0$, then*

$$\begin{aligned}\delta_{\mathbb{R}^+}(f) &= \sum_{k\in\Delta} k[\pi(B(p_k, zp_k')) - \nu(B(p_k, p_k'))] \\ &= \sum_{k\in\Delta} k[\pi(B(p_k, p_k')C_{p_k}) - \nu(B(p_k, p_k'))],\end{aligned}$$

or equivalently,

$$\begin{aligned}\delta_{\mathbb{R}^+}(f) &= \sum_{k\in\Delta} k[\pi(H(p_k, zp_k')) - \nu(H(p_k, p_k'))] \\ &= \sum_{k\in\Delta} k[\pi(H(p_k, p_k')C_{p_k}^{\mathrm{T}}) - \nu(H(p_k, p_k'))].\end{aligned}$$

Remark that if $p(0) = 0$, then $f(0) = 0$, so that there exists a positive integer t found easily such that $f(z) = z^t\tilde{f}(z)$, $\tilde{f}(0) \neq 0$. In this case, we use $\tilde{f}(z)$ instead of the original $f(z)$.

By Corollary 2.4 and Theorem 3.2, we can give explicit formulas for the number $\delta_{\mathbb{R}^-}(f)$ in terms of the inertia of Bezout or Hankel matrices as well.

Corollary 3.3. *Let $f(z)$ and $p(z)$ be as in Theorem 3.2. Then*

$$\begin{aligned}\delta_{\mathbb{R}^-}(f) &= \sum_{k\in\Delta} k[\nu(B(p_k, zp_k')) - \nu(B(p_k, p_k'))] \\ &= \sum_{k\in\Delta} k[\nu(B(p_k, p_k')C_{p_k}) - \nu(B(p_k, p_k'))],\end{aligned} \tag{3.10}$$

or equivalently,

$$\begin{aligned}\delta_{\mathbb{R}^-}(f) &= \sum_{k\in\Delta} k[\nu(H(p_k, zp_k')) - \nu(H(p_k, p_k'))] \\ &= \sum_{k\in\Delta} k[\nu(H(p_k, p_k')C_{p_k}^{\mathrm{T}}) - \nu(H(p_k, p_k'))].\end{aligned}$$

Proof. In view of (2.9) and its extended form, and the relations $B(p_k, zp_k') = B(p_k, p_k')C_{p_k}$ and $H(p_k, zp_k') = H(p_k, p_k')C_{p_k}^{\mathrm{T}}$ for $k \in \Delta$, we need only to verify the first equality in (3.10). It follows from Corollary 2.4 and Theorem 3.2 that

$$\delta_{\mathbb{R}^-}(f) = \delta_{\mathbb{R}^-}(p) = \deg p(z) - \sum_{k\in\Delta} k[\pi(B(p_k, zp_k')) + \nu(B(p_k, p_k'))], \tag{3.11}$$

moreover,

$$\deg p(z) = \sum_{k\in\Delta} k \deg p_k(z).$$

Note that $p_k(0) \neq 0$ and thus each $B(p_k, zp_k')$ $(k \in \Delta)$ is a nonsingular real symmetric matrix, so that

$$\pi(B(p_k, zp_k')) + \nu(B(p_k, zp_k')) = \deg p_k(z), \quad k \in \Delta, \tag{3.12}$$

Thus (3.11) can be rewritten as

$$\begin{aligned}\delta_{\mathbb{R}^-}(f) &= \sum_{k\in\Delta} k \deg p_k(z) - \sum_{k\in\Delta} k[\pi(B(p_k, zp_k')) + \nu(B(p_k, p_k'))] \\ &= \sum_{k\in\Delta} k[(\deg p_k(z) - \pi(B(p_k, zp_k'))) - \nu(B(p_k, p_k'))] \\ &= \sum_{k\in\Delta} k[\nu(B(p_k, zp_k')) - \nu(B(p_k, p_k'))],\end{aligned}$$

as required. ■

At the end of this section, we point out that similar results to (2.10)–(2.11) are valid for $\tilde{\delta}_{\mathbb{R}^\pm}(f)$ in the case of $p(0) \neq 0$, where $\tilde{\delta}_{\mathbb{R}^\pm}(f)$ are the numbers of different roots of f in $\mathbb{R}^\pm$, analogously for $\tilde{\delta}_{\mathbb{R}^\pm}^{(k)}(f)$ $(k \in \Delta)$.

Corollary 3.4. *Let $f(z)$ and $p(z)$ be as in Theorem 3.2. Then*

$$\begin{aligned}\tilde{\delta}_{\mathbb{R}^+}(f) &= \pi(B(q_1, zq_1')) - \nu(B(q_1, q_1')) \\ &= \pi(B(q_1, q_1')C_{q_1}) - \nu(B(q_1, q_1')) \\ &= \pi(H(q_1, zq_1')) - \nu(H(q_1, q_1')) \\ &= \pi(H(q_1, q_1')C_{q_1}^{\mathrm{T}}) - \nu(H(q_1, q_1')), \\ \tilde{\delta}_{\mathbb{R}^-}(f) &= \nu(B(q_1, zq_1')) - \nu(B(q_1, q_1')) \\ &= \nu(B(q_1, q_1')C_{q_1}) - \nu(B(q_1, q_1')) \\ &= \nu(H(q_1, zq_1')) - \nu(H(q_1, q_1')) \\ &= \nu(H(q_1, q_1')C_{q_1}^{\mathrm{T}}) - \nu(H(q_1, q_1')),\end{aligned}$$

and for $k \in \Delta$,

$$\begin{aligned}
\tilde{\delta}^{(k)}_{\mathbb{R}^+}(f) &= \pi(B(p_k, zp'_k)) - \nu(B(p_k, p'_k)) \\
&= \pi(B(p_k, p'_k)C_{p_k}) - \nu(B(p_k, p'_k)) \\
&= \pi(H(p_k, zp'_k)) - \nu(H(p_k, p'_k)) \\
&= \pi(H(p_k, p'_k)C^{\mathrm{T}}_{p_k}) - \nu(H(p_k, p'_k)),
\end{aligned}$$

and

$$\begin{aligned}
\tilde{\delta}^{(k)}_{\mathbb{R}^-}(f) &= \nu(B(p_k, zp'_k)) - \nu(B(p_k, p'_k)) \\
&= \nu(B(p_k, p'_k)C_{q_k}) - \nu(B(p_k, p'_k)) \\
&= \nu(H(p_k, zp'_k)) - \nu(H(p_k, p'_k)) \\
&= \nu(H(p_k, p'_k)C^{\mathrm{T}}_{q_k}) - \nu(H(p_k, p'_k)).
\end{aligned}$$

4 Explicit formulas for the number $\delta_I(f)$ in the cases of $I = (a, +\infty)$, $I = (-\infty, a)$, and $I = (a, b)$

By using the results established in Section 3, we present in this section some explicit formulas for the number $\delta_I(f)$ in the cases of $I = (a, +\infty)$, $I = (-\infty, a)$, and $I = (a, b)$, respectively, where a and b are real numbers.

We first consider the number $\delta_I(p)$ for the case of $I = (a, +\infty)$. We need the following property of Bezout matrix, which can be easily verified by replacing the variables z and w in (1.2) with $z + z_0$ and $w + z_0$, respectively (see, e.g., [2, Proposition 4.1]).

Lemma 4.1. *Let z_0 be a complex number and $g(z), h(z)$ be a pair of polynomials with maximal degree n, and let $\tilde{g}(z) = g(z + z_0)$, $\tilde{h}(z) = h(z + z_0)$. Then*

$$B(\tilde{g}, \tilde{h}) = V_n(z_0)B(g, h)V_n(z_0)^{\mathrm{T}},$$

in which

$$V_n(z_0) = \left(\begin{pmatrix} i \\ j \end{pmatrix} z_0^{j-i} \right)_{i,j=0}^{n-1}$$

is the nonsingular $n \times n$ generalized Vandermonde matrix associated with z_0 ($V_n(0) = I_n$).

If $g(z)$, $h(z)$ are both real polynomials and z_0 is a real number, then $B(\tilde{g}, \tilde{h})$ and $B(g, h)$ are real symmetric matrices such that, by Lemma 4.1, $\mathrm{In}(B(\tilde{g}, \tilde{h})) = \mathrm{In}(B(g, h))$. Let now a be a real number and $q(z)$ be a monic real polynomial with only simple roots in the complex plane. Without loss of generality, we assume that $q(a) \neq 0$ (otherwise, $q(z)$ can be factorized as $q(z) = (z - a)^t \hat{q}(z)$ for some integer t with $\hat{q}(a) \neq 0$. Since $\delta_I(q) = \delta_I(\hat{q})$, $I = (a, +\infty)$, we turn to consider the number $\delta_I(\hat{q})$). Now we define a monic real polynomial $\tilde{q}(z)$ by setting $\tilde{q}(z) = q(z + a)$. It is obvious that $\tilde{q}(z)$ is also a monic real polynomial with only simple roots in the complex plane, $\tilde{q}(0) \neq 0$ and satisfies $\delta_{\mathbb{R}^+}(\tilde{q}) = \delta_I(q)$ for $I = (a, +\infty)$. By Lemma 4.1 and

Theorem 3.1, we have that if $q(a) \neq 0$,

$$\begin{aligned} \delta_I(q) &= \delta_{\mathbb{R}^+}(\tilde{q}) \\ &= \pi(B(\tilde{q}, z\tilde{q}')) - \nu(B(\tilde{q}, \tilde{q}')) \\ &= \pi(B(q(z+a), zq'(z+a))) - \nu(B(q(z+a), q'(z+a)) \\ &= \pi(B(q, (z-a)q')) - \nu(B(q, q')). \end{aligned} \tag{4.1}$$

For the case of $I = (-\infty, a)$, by Theorem 2.2 and (4.1) we have that if $q(a) \neq 0$,

$$\begin{aligned} \delta_I(q) &= \deg q(z) - \pi(B(q, (z-a)q')) - \nu(B(q, q')) \\ &= \nu(B(q, (z-a)q')) - \nu(B(q, q')). \end{aligned} \tag{4.2}$$

Applying (4.1) and (4.2) to each factor $p_k(z)$ $(k \in \Delta)$, we obtain the following theorem.

Theorem 4.2. *Let a be a real number and let $p(z) = \gcd(f(z), \bar{f}(z))$ have a factorization as (2.5). If $f(a) \neq 0$, then*

$$\delta_I(f) = \begin{cases} \sum\limits_{k\in\Delta} k[\pi(B(p_k, (z-a)p_k')) - \nu(B(p_k, p_k'))], & I = (a, +\infty); \\ \sum\limits_{k\in\Delta} k[\nu(B(p_k, (z-a)p_k')) - \nu(B(p_k, p_k'))], & I = (-\infty, a). \end{cases} \tag{4.3}$$

Let now $I = (a, b)$ be a finite interval of the real axis. It is obvious that $\delta_I(f) = \delta_{I_1}(f) - \delta_{I_2}(f)$, where $I_1 = (a, +\infty)$ and $I_2 = (b, +\infty)$. By Theorem 4.2, we obtain immediately an explicit formula for the number $\delta_I(f)$ under the assumption that $f(a)f(b) \neq 0$.

Corollary 4.3. *Let $f(z)$, $p(z)$ and $p_k(z)$ $(k \in \Delta)$ be the same as in Theorem 4.2, and $I = (a, b)$ be a finite interval of the real axis. If $f(a)f(b) \neq 0$, then*

$$\delta_I(f) = \sum_{k\in\Delta} k[\pi(B(p_k, (z-a)p_k')) - \pi(B(p_k, (z-b)p_k'))]. \tag{4.4}$$

Similarly, formulas (4.3) and (4.4) can also be formulated in terms of the inertia of a sequence of Hankel matrices $H(p_k, (z-a)p_k')$, $H(p_k, (z-b)p_k')$ and $H(p_k, p_k')$ $(k \in \Delta)$. Moreover, the similar rules to (2.13)–(2.14) hold as well in the following:

Corollary 4.4. *Let $f(z)$, $p(z)$, $p_k(z)$ $(k \in \Delta)$ and I be the same as in Theorem 4.2. If $f(a) \neq 0$, then*

$$\tilde{\delta}_I(f) = \begin{cases} \pi(B(q_1, (z-a)q_1')) - \nu(B(q_1, q_1')), & I = (a, +\infty); \\ \nu(B(q_1, (z-a)q_1')) - \nu(B(q_1, q_1')), & I = (-\infty, a), \end{cases}$$

and for $k \in \Delta$,

$$\tilde{\delta}_I^{(k)}(f) = \begin{cases} \pi(B(p_k, (z-a)p_k')) - \nu(B(p_k, p_k')), & I = (a, +\infty); \\ \nu(B(p_k, (z-a)p_k')) - \nu(B(p_k, p_k')), & I = (-\infty, a). \end{cases}$$

Corollary 4.5. *Let $f(z)$, $p(z)$, $p_k(z)$ $(k \in \Delta)$ and I be the same as in Corollary 4.3. If $f(a)f(b) \neq 0$, then*

$$\begin{aligned}\tilde{\delta}_I(f) &= \nu(B(q_1,(z-a)q_1')) - \nu(B(q_1,(z-b)q_1'))\\ &= \nu(H(q_1,(z-a)q_1')) - \nu(H(q_1,(z-b)q_1')),\end{aligned}$$

and for $k \in \Delta$,

$$\begin{aligned}\tilde{\delta}_I^{(k)}(f) &= \nu(B(p_k,(z-a)p_k')) - \nu(B(p_k,(z-b)p_k'))\\ &= \nu(H(p_k,(z-a)p_k')) - \nu(H(p_k,(z-b)p_k')).\end{aligned}$$

Finally, we point out that the results established in this paper can be used to determine the number of roots of a complex polynomial located on a given straight line, a ray or a line segment of the complex plane. Let $f(z)$ be a complex polynomial given as in (1.1), and let $z_1,\ z_2$ be two distinct points in the complex plane. Let

$$S(I,z_1,z_2) = \{(1-t)z_1 + tz_2 \,|\, t \in I\},$$

in which I is an interval of the real axis. Then $S(I,z_1,z_2)$ stands for a straight line, a ray and a line segment in the complex plane in the cases of $I=\mathbb{R}$, $I=\mathbb{R}^+$ and $I=(a,b)$, respectively. Moreover,

$$\delta_{S(I,z_1,z_2)}(f) = \delta_I(g), \quad \tilde{\delta}_{S(I,z_1,z_2)}(f) = \tilde{\delta}_I(g), \quad \tilde{\delta}^{(k)}_{S(I,z_1,z_2)}(f) = \tilde{\delta}_I^{(k)}(g),\ k \in \Delta,$$

in which $g(z) = f(z_1 + z(z_2 - z_1))$, $\delta_{S(I,z_1,z_2)}(f)$, $\tilde{\delta}_{S(I,z_1,z_2)}(f)$ and $\tilde{\delta}^{(k)}_{S(I,z_1,z_2)}(f)$ for $k \in \Delta$ stand for the number of roots, different roots, and different roots with multiplicity k of $f(z)$ lying on $S(I,z_1,z_2)$, respectively.

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