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RESEARCH ARTICLE

Limit theorems for flows of branching processes

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Abstract We construct two kinds of stochastic flows of discrete Galton-Watson branching processes. Some scaling limit theorems for the flows are proved, which lead to local and nonlocal branching superprocesses over the positive half line.

Keywords Stochastic flow, Galton-Watson branching process, continuous-state branching process, superprocess, nonlocal branching
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1 Introduction

Continuous-state branching processes (CB-processes) arose as weak limits of rescaled discrete Galton-Watson branching processes (see [12,14]). Continuous-state branching processes with immigration (CBI-processes) are generalizations of them describing the situation where immigrants may come from other sources of particles. Those processes can be obtained as the scaling limits of discrete branching processes with immigration (see [13,15]). A CBI-process was constructed in [7] as the strong solution of a stochastic equation driven by Brownian motions and Poisson random measures (see also [10]). A similar construction was given in [18] using a stochastic equation driven by time-space Gaussian white noises and Poisson random measures.

In the study of scaling limits of coalescent processes with multiple collisions, [6] constructed a flow of jump-type CB-processes as the weak solution flow of a system of stochastic equations driven by Poisson random measures (see also [4,5]. A more general flow of CBI-processes was constructed in [8] as strong solutions of stochastic equations driven by Gaussian white noise and Poisson random measures. The flows in [6,8] were also treated as path-valued processes

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with independent increments. Motivated by the work of [1,2] on tree-valued Markov processes, another flow of CBI-processes was introduced in [17], which was identified as a path-valued branching process. From the flows in [6,8,17], one can define some superprocesses or immigration superprocesses over the positive half line with local or nonlocal branching mechanisms. To study the genealogy trees for critical branching processes conditioned on non-extinction, Bakhtin [3] considered a flow of continuous CBI-processes driven by a time-space Gaussian white noise. He obtained the flow as a rescaling limit of systems of discrete Galton-Watson processes and also pointed out the connection of the model with a superprocess conditioned on non-extinction.

In this paper, we consider two flows of discrete Galton-Watson branching processes and show suitable rescaled sequences of the flows converge to the flows of [8] and [17], respectively. The main motivation of the work is to understand the connection between discrete and continuum tree-valued processes. Our results generalize those of [3] to flows of discontinuous CB-processes. To simplify the presentation, we only treat models without immigration, but the arguments given here carry over to those with immigration. We shall first prove limit theorems for the induced superprocesses, from which we derive the convergence of the finite-dimensional distributions of the path-valued branching processes.

In Section 2, we give a brief review of the flows of [8] and [17]. In Section 3, we consider flows consisting of independent branching processes and show their scaling limit gives a flow of the type of [8]. The formulation and convergence of interactive flows are discussed in Section 4, which lead to a flow in the class studied in [17].

Let

$$\mathbb{N} = \{0, 1, 2, \ldots\}, \quad \mathbb{N}_+ = \{1, 2, \ldots\}.$$

For any $a \geqslant 0$, let $M[0,a]$ be the set of finite Borel measures on $[0,a]$ endowed with the topology of weak convergence. We identify $M[0,a]$ with the set $F[0,a]$ of positive right continuous increasing functions on $[0,a]$. Let $B[0,a]$ be the Banach space of bounded Borel functions on $[0,a]$ endowed with the supremum norm $\|\cdot\|$. Let $C[0,a]$ denote its subspace of continuous functions. We use $B[0,a]^+$ and $C[0,a]^+$ to denote the subclasses of positive elements and $C[0,a]^{++}$ to denote the subset of $C[0,a]^+$ of functions bounded away from zero. For $\mu \in M[0,a]$ and $f \in B[0,a]$, write

$$\langle \mu, f \rangle = \int f \mathrm{d}\mu$$

if the integral exists.

2 Local and nonlocal branching flows

In this section, we recall some results on constructions and characterizations of the flow of CB-processes and the associated superprocess. It is well known that the law of a CB-process is determined by its *branching mechanism* ϕ, which is a function on $[0,\infty)$ and has the representation

$$\phi(z) = bz + \frac{1}{2}\sigma^2 z^2 + \int_0^\infty (\mathrm{e}^{-zu} - 1 + zu) m(\mathrm{d}u), \tag{1}$$

where $\sigma \geqslant 0$ and b are constants, and $(u \wedge u^2)m(\mathrm{d}u)$ is a finite measure on $(0,\infty)$. Let $W(ds, \mathrm{d}u)$ be a white noise on $(0,\infty)^2$ based on $\mathrm{d}s\mathrm{d}u$, and let $\widetilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u)$ be a compensated Poisson random measure on $(0,\infty)^3$ with intensity $\mathrm{d}sm(\mathrm{d}z)\mathrm{d}u$. By [8, Theorem 3.1], a CB-process with branching mechanism ϕ can be constructed as the pathwise unique strong solution $\{Y_t : t \geqslant 0\}$ to the stochastic equation:

$$\begin{aligned} Y_t = Y_0 + \sigma \int_0^t \int_0^{Y_{s-}} W(\mathrm{d}s, \mathrm{d}u) - \int_0^t bY_{s-}\mathrm{d}s \\ + \int_0^t \int_0^\infty \int_0^{Y_{s-}} z\widetilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}u). \end{aligned} \tag{2}$$

Let us fix a constant $a \geqslant 0$ and a function $\mu \in F[0,a]$. Let $\{Y_t(q) : t \geqslant 0\}$ denote the solution to (2) with $Y_0(q) = \mu(q)$. We can consider the solution flow $\{Y_t(q) : t \geqslant 0,\ q \in [0,a]\}$ of (2). As observed in [8], there is a version of the flow which is increasing in $q \in [0,a]$. Moreover, we can regard $\{(Y_t(q))_{t\geqslant 0} : q \in [0,a]\}$ as a path-valued stochastic process with independent increments. Let $\{Y_t : t \geqslant 0\}$ denote the $M[0,a]$-valued process so that $Y_t[0,q] = Y_t(q)$ for every $t \geqslant 0$ and $q \in [0,a]$. Then $\{Y_t : t \geqslant 0\}$ is a càdlàg superprocess with branching mechanism ϕ and trivial spatial motion (see [8, Theorems 3.9, 3.11]). For $\lambda \geqslant 0$, let $t \mapsto v(t,\lambda)$ be the unique locally bounded positive solution of

$$v(t,\lambda) = \lambda - \int_0^t \phi(v(s,\lambda))\mathrm{d}s, \quad t \geqslant 0. \tag{3}$$

For any $f \in B[0,a]^+$, define $x \mapsto v(t,f)(x)$ by

$$v(t,f)(x) = v(t,f(x)).$$

Then the superprocess $\{Y_t : t \geqslant 0\}$ has transition semigroup $(Q_t)_{t\geqslant 0}$ on $M[0,a]$ defined by

$$\int_{M[0,a]} \mathrm{e}^{-\langle \nu, f\rangle} Q_t(\mu, \nu) = \exp\{-\langle \mu, v(t,f)\rangle\}, \quad f \in B[0,a]^+. \tag{4}$$

By [16, Proposition 3.1], one can see that $v(t,f) \in C[0,a]^{++}$ for every $f \in C[0,a]^{++}$. Then it is easy to verify that $(Q_t)_{t\geqslant 0}$ is a Feller semigroup.

We can define another branching flow. For this purpose, let us consider an admissible family of branching mechanisms $\{\phi_q : q \in [0,a]\}$, where ϕ_q is given by (1) with parameters $(b,m) = (b_q, m_q)$ depending on $q \in [0,a]$. Here, by an *admissible family*, we mean that for each $z \geqslant 0$, the function $q \mapsto \phi_q(z)$ is decreasing and continuously differentiable with the derivative

$$\psi_\theta(z) = -\frac{\partial}{\partial\theta}\,\phi_\theta(z)$$

of the form

$$\psi_\theta(z) = h_\theta z + \int_0^\infty (1 - \mathrm{e}^{-zu}) n_\theta(\mathrm{d}u), \tag{5}$$

where $h_\theta \geqslant 0$ and $n_\theta(\mathrm{d}u)$ is a σ-finite kernel from $[0, a]$ to $(0, \infty)$ satisfying

$$\sup_{0\leqslant\theta\leqslant a}\left[h_\theta + \int_0^\infty u n_\theta(\mathrm{d}u)\right] < \infty.$$

Then we have

$$\phi_q(z) = \phi_0(z) - \int_0^q \psi_\theta(z)\mathrm{d}\theta, \quad z \geqslant 0. \tag{6}$$

Let $m(\mathrm{d}z, \mathrm{d}\theta)$ be the measure on $(0, \infty) \times [0, a]$ defined by

$$m([c, d] \times [0, q]) = m_q[c, d], \quad q \in [0, a],\ d > c > 0.$$

Let $W(\mathrm{d}s, \mathrm{d}u)$ be a white noise on $(0, \infty)^2$ based on $\mathrm{d}s\mathrm{d}u$ and $\widetilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}\theta, \mathrm{d}u)$ a compensated Poisson random measure on $(0, \infty)^2 \times [0, a] \times (0, \infty)$ with intensity $\mathrm{d}sm(\mathrm{d}z, \mathrm{d}\theta)\mathrm{d}u$. By the results in [17], for any $\mu \in F[0, a]$, the stochastic equation

$$\begin{aligned} Y_t(q) = \mu(q) - b_q \int_0^t Y_{s-}(q)\mathrm{d}s + \sigma \int_0^t \int_0^{Y_{s-}(q)} W(\mathrm{d}s, \mathrm{d}u) \\ + \int_0^t \int_0^\infty \int_{[0,q]} \int_0^{Y_{s-}(q)} z\widetilde{N}(\mathrm{d}s, \mathrm{d}z, \mathrm{d}\theta, \mathrm{d}u) \end{aligned} \tag{7}$$

has a unique solution flow $\{Y_t(q)\colon t \geqslant 0,\ q \in [0, a]\}$. For each $q \in [0, a]$, the one-dimensional process $\{Y_t(q)\colon t \geqslant 0\}$ is a CB-process with branching mechanism ϕ_q. It was proved in [17] that there is a version of the flow which is increasing in $q \in [0, a]$. Moreover, we can also regard $\{(Y_t(q))_{t\geqslant 0}\colon q \in [0, a]\}$ as a path-valued branching process. The solution flow of (7) also induces a càdlàg superprocess $\{Y_t\colon t \geqslant 0\}$ with state space $M[0, a]$. Let $f \mapsto \Psi(\cdot, f)$ be the operator on $C^+[0, a]$ defined by

$$\Psi(x, f) = \int_{[0,a]} f(x \vee \theta)h_\theta\mathrm{d}\theta + \int_{[0,a]} \mathrm{d}\theta \int_0^\infty (1 - \mathrm{e}^{-zf(x\vee\theta)})n_\theta(\mathrm{d}z). \tag{8}$$

The superprocess $\{Y_t\colon t \geqslant 0\}$ has local branching mechanism ϕ_0 and nonlocal branching mechanism given by (8) (see [17, Theorem 6.2]). Then the transition semigroup $(Q_t)_{t\geqslant 0}$ of $\{Y_t\colon t \geqslant 0\}$ is defined by

$$\int_{M[0,a]} \mathrm{e}^{-\langle \nu, f\rangle} Q_t(\mu, d\nu) = \exp\{-\langle \mu, V_t f\rangle\}, \quad f \in C^+[0, a], \tag{9}$$

where $t \mapsto V_t f$ is the unique locally bounded positive solution of

$$V_t f(x) = f(x) - \int_0^t [\phi_0(V_s f(x)) - \Psi(x, V_s f)]\mathrm{d}s, \quad t \geqslant 0, \ x \in [0, a]. \tag{10}$$

To study the scaling limit theorems of the discrete branching flows, we need to introduce a metric on $M[0,a]$. Let $\{h_0, h_1, h_2, \ldots\}$ be a countable dense subset of $\{h \in C[0,a]^+ \colon \|h\| \leqslant 1\}$ with $h_0 \equiv 1$. For convenience, we assume that each h_i is bounded away from zero. Then

$$\{h_0, h_1, h_2, \ldots\} \subset C[0,a]^{++}.$$

Now, we define a metric ρ on $M[0,a]$ by

$$\rho(\mu, \nu) = \sum_{i=0}^{\infty} \frac{1}{2^i} (1 \wedge |\langle \mu, h_i \rangle - \langle \nu, h_i \rangle|), \quad \mu, \nu \in M[0,a].$$

It is easy to see that the metric is compatible with the weak convergence topology of $M[0,a]$. In other words, we have $\mu_n \to \mu$ in $M[0,a]$ if and only if $\rho(\mu_n, \mu) \to 0$. For $\nu \in M[0,a]$, set

$$\mathrm{e}_{h_i}(\nu) = \mathrm{e}^{-\langle \nu, h_i \rangle}.$$

Theorem 1 *The metric space* $(M[0,a], \rho)$ *is a locally compact Polish* (*complete and separable*) *space, and* $\{\mathrm{e}_{h_i} \colon i = 0, 1, 2, \ldots\}$ *strongly separates the points of* $M[0,a]$, *that is, for every* $\nu \in M[0,a]$ *and* $\delta > 0$, *there exists a finite set* $\{\mathrm{e}_{h_{i_1}}, \mathrm{e}_{h_{i_2}}, \ldots, \mathrm{e}_{h_{i_k}}\} \subset \{\mathrm{e}_{h_i} \colon i = 0, 1, 2, \ldots\}$ *such that*

$$\inf_{\mu \colon \rho(\mu,\nu) \geqslant \delta} \max_{1 \leqslant j \leqslant k} |\mathrm{e}_{h_{i_j}}(\mu) - \mathrm{e}_{h_{i_j}}(\nu)| > 0.$$

Proof By [16, pp. 4, 7], we know that $M[0,a]$ is separable and locally compact, and thus, there is a complete metric on $M[0,a]$ compatible with the weak convergence topology. The following argument shows that the metric ρ defined above is complete. Suppose that $\{\mu_n\}_{n \geqslant 1} \subset M[0,a]$ is a Cauchy sequence under ρ. Then for every $m \geqslant 1$, $\{\langle \mu_n, h_m \rangle\}_{n \geqslant 1}$ is also a Cauchy sequence. We denote its limit by $\Phi(h_m)$. For $f \in C[0,a]^+$ satisfying $\|f\| \leqslant 1$, let $\{h_{i_k}\}_{k \geqslant 1} \subset \{h_0, h_1, h_2, \ldots\}$ be a sequence so that $\|h_{i_k} - f\| \to 0$ as $k \to \infty$. For $n \geqslant m \geqslant 1$, we have

$$\begin{aligned}
&\limsup_{m,n\to\infty} |\langle \nu_n, f \rangle - \langle \nu_m, f \rangle| \\
&\quad \leqslant \limsup_{m,n\to\infty} [|\langle \nu_n, f \rangle - \langle \nu_n, h_{i_k} \rangle| + |\langle \nu_n, h_{i_k} \rangle - \langle \nu_m, h_{i_k} \rangle| + |\langle \nu_m, h_{i_k} \rangle - \langle \nu_m, f \rangle|] \\
&\quad \leqslant 2\Phi(1)\|f - h_{i_k}\|.
\end{aligned}$$

Then letting $k \to \infty$, we have

$$\limsup_{m,n\to\infty} |\langle \nu_n, f \rangle - \langle \nu_m, f \rangle| = 0.$$

By the linearity, the above relation holds for all $f \in C[0,a]$. Therefore, the limit

$$\Phi(f) = \lim_{n\to\infty} \langle \mu_n, f\rangle$$

exists for each $f \in C[0,a]$. Clearly, $f \to \Phi(f)$ is a positive linear functional on $C[0,a]$. By the Riesz representation theorem, there exists $\mu \in M[0,a]$ such that $\langle \mu, f\rangle = \Phi(f)$ for every $f \in C[0,a]$. By the construction of Φ, we have $\mu_n \to \mu$. Therefore, $\rho(\mu_n,\mu) \to 0$. That proves the first assertion of the theorem.

For any $\nu \in M[0,a]$ and $\delta \geqslant 0$, there exists an $N_0 \in \mathbb{N}_+$ such that

$$\sum_{i=N_0+1}^{\infty} \frac{1}{2^i} < \frac{\delta}{2}.$$

Consider $\{h_0, h_1, \ldots, h_{N_0}\}$. For any $\mu \in M[0,a]$ satisfying $\rho(\mu,\nu) \geqslant \delta$, we have

$$\sum_{i=0}^{N_0} \frac{1}{2^i} (1 \wedge |\langle \mu, h_i\rangle - \langle \nu, h_i\rangle|) \geqslant \frac{\delta}{2},$$

and thus,

$$\sum_{i=0}^{N_0} (1 \wedge |\langle \mu, h_i\rangle - \langle \nu, h_i\rangle|) \geqslant \frac{\delta}{2}.$$

It follows that

$$|\langle \mu, h_j\rangle - \langle \nu, h_j\rangle| \geqslant \frac{\delta}{2N_0}$$

for some $j = 0, 1, \ldots, N_0$. Since

$$|\mathrm{e}^{-x} - \mathrm{e}^{-y}| = \mathrm{e}^{-y}|\mathrm{e}^{y-x} - 1| \geqslant \mathrm{e}^{-y}[(\mathrm{e}^{|y-x|} - 1) \wedge (1 - \mathrm{e}^{-|y-x|})], \quad x, y \in \mathbb{R},$$

we have

$$\begin{aligned} &\inf_{\mu:\, \rho(\mu,\nu)\geqslant\delta} \max_{0\leqslant i\leqslant N_0} |\mathrm{e}_{h_i}(\mu) - \mathrm{e}_{h_i}(\nu)| \\ &\geqslant \mathrm{e}^{-\max_{0\leqslant i\leqslant N_0}\langle \nu, h_i\rangle}[(\mathrm{e}^{\frac{\delta}{2N_0}} - 1) \wedge (1 - \mathrm{e}^{-\frac{\delta}{2N_0}})] \\ &> 0. \end{aligned}$$

That proves the second assertion. □

3 Flows of independent branching processes

In this section, we consider some flows of independent Galton-Watson branching processes. We shall study the scaling limit in the setting of superprocesses. Then we derive the convergence of the finite-dimensional distributions of the path-valued processes.

Let $\{g_i : i = 0, 1, 2, \ldots\}$ be a family of probability generating functions. For each $i \in \mathbb{N}$, suppose that there is a Galton-Watson branching process

(GW-process) $(X_n(i))_{n\geqslant 0}$ with offspring distribution given by g_i. In addition, we assume that $(X_n(i))_{n\geqslant 0}$, $i = 1, 2, \ldots$, are mutually independent. It is well known that for each $i \in \mathbb{N}$, $(X_n(i))_{n\geqslant 0}$ is a discrete-time $\mathbb{N}$-valued Markov chain with n-step transition matrix $P^n(j,k)$ defined by

$$\sum_{k=0}^{\infty} P^n(j,k)z^k = (g_i^n(z))^j, \quad |z| \leqslant 1, \tag{11}$$

where $g_i^n(z)$ is defined by $g_i^n(z) = g_i(g_i^{n-1}(z))$ successively with $g_i^0(z) = z$.

Suppose that for each integer $k \geqslant 1$, we have a sequence of GW-processes $\{(X_n^{(k)}(i))_{n\geqslant 0} \colon i \geqslant 0\}$ with offspring distribution given by $\{g_i^{(k)}\}$. Let γ_k be a positive real sequence so that $\gamma_k \to \infty$ increasingly as $k \to \infty$. For $m, n \in \mathbb{N}$, define

$$\overline{X}_n^{(k)}(m) = \sum_{i=0}^{m} X_n^{(k)}(i),$$

and

$$Y_t^{(k)}(x) = \frac{1}{k}\,\overline{X}_{[\gamma_k t]}^{(k)}([kx]), \quad k = 1, 2, \ldots,$$

where $[\cdot]$ denotes the integer part. Then the increasing function $x \mapsto Y_t^{(k)}(x)$ induces a random measure $Y_t^{(k)}(\mathrm{d}x)$ on $[0,\infty)$ so that $Y_t^{(k)}([0,x]) = Y_t^{(k)}(x)$ for $x \geqslant 0$. For convenience, we fix a constant $a \geqslant 0$ and consider the restriction of $\{Y_t^{(k)} \colon t \geqslant 0\}$ to $[0,a]$ without changing the notation. Clearly,

$$Y_0^{(k)} = \frac{1}{k}\sum_{i=0}^{[ka]} X_0^{(k)}(i)\delta_{i/k}$$

and

$$Y_t^{(k)} = \frac{1}{k}\sum_{i=0}^{[ka]} X_{[\gamma_k t]}^{(k)}(i)\delta_{i/k}.$$

In view of (11), for each $i \geqslant 0$, given $X_0^{(k)}(i) = x_i \in \mathbb{N}$, the conditional distribution $Q_{i,k}^{[\gamma_k t]}(x_i/k, \cdot)$ of $\{k^{-1}X_{[\gamma_k t]}^{(k)}(i) \colon t \geqslant 0\}$ on $E_k = \{0, 1/k, 2/k, \ldots\}$ is determined by

$$\int_{E_k} \mathrm{e}^{-\lambda y} Q_{i,k}^{[\gamma_k t]}\Big(\frac{x_i}{k}, \mathrm{d}y\Big) = \exp\Big\{-\frac{x_i}{k}\, v_i^{(k)}(t,\lambda)\Big\}, \tag{12}$$

where

$$v_i^{(k)}(t,\lambda) = -k\log(g_i^{(k)})^{[\gamma_k t]}(\mathrm{e}^{-\lambda/k}).$$

Let $Q_{\mu_k}^{(k)}$ denote the conditional law given

$$Y_0^{(k)} = \mu_k = k^{-1}\sum_{i=0}^{[ka]} x_i\delta_{i/k} \in M_k[0,a],$$

where

$$M_k[0,a] := \Big\{ k^{-1}\sum_{i=0}^{[ka]} x_i\delta_{i/k} \colon x_i \in \mathbb{N},\ k^{-1}\sum_{i=0}^{[ka]} x_i < \infty \Big\}.$$

For $f \in B[0,a]^+$, from (12), we have

$$\begin{aligned} Q^{(k)}_{\mu_k}\exp\{-\langle Y^{(k)}_t, f\rangle\} &= Q^{(k)}_{\mu_k}\exp\Big\{-\sum_{i=0}^{[ka]}\frac{1}{k}\,X^{(k)}_{[\gamma_k t]}(i) f\Big(\frac{i}{k}\Big)\Big\} \\ &= \prod_{i=1}^{[ka]}\int_{E_k} \mathrm{e}^{-f(i/k)y} Q^{[\gamma_k t]}_{i,k}\Big(\frac{x_i}{k}, \mathrm{d}y\Big) \\ &= \exp\Big\{-\sum_{i=0}^{[ka]}\frac{x_i}{k}\,v^{(k)}_i\Big(t, f\Big(\frac{i}{k}\Big)\Big)\Big\} \\ &= \exp\{-\langle \mu_k, v^{(k)}(t,f)\rangle\}, \end{aligned} \tag{13}$$

where $x \mapsto v^{(k)}(t,f)(x)$ is defined by

$$v^{(k)}(t,f)(x) = v^{(k)}_{[kx]}(t, f(x)).$$

For any $x, z \geqslant 0$, define

$$\phi_k(x,z) = k\gamma_k[g^{(k)}_{[kx]}(\mathrm{e}^{-z/k}) - \mathrm{e}^{-z/k}]. \tag{14}$$

For convenience of statement of the results, we formulate the following condition.

Condition (3.A) For each $a \geqslant 0$, the sequence $\{\phi_k(x,z)\}$ is Lipschitz with respect to z uniformly on $[0,\infty)\times[0,a]$ and there is a continuous function $(x,z) \mapsto \phi(x,z)$ such that $\phi_k(x,z) \to \phi(x,z)$ uniformly on $[0,\infty)\times[0,a]$ as $k \to \infty$.

Before giving the limit theorem for the sequence of the rescaled processes, we first introduce the limit process. By [16, Proposition 4.3], if Condition (3.A) is satisfied, then the limit function ϕ has the representation

$$\phi(x,z) = b(x)z + \frac{1}{2}\,c(x)z^2 + \int_0^\infty (\mathrm{e}^{-zu} - 1 + zu)m(x,\mathrm{d}u), \quad x, z \geqslant 0, \tag{15}$$

where b is a bounded function on $[0,\infty)$ and c is a positive bounded function on $[0,\infty)$. $(u\wedge u^2)m(x,\mathrm{d}u)$ is a bounded kernel from $[0,\infty)$ to $(0,\infty)$. Conversely, for any continuous function $(x,z) \mapsto \phi(x,z)$ given by (15), we can construct a family of probability generating functions $\{g^{(k)}_i \colon i = 0,1,2,\ldots\}$ such that sequence (14) satisfies Condition (3.A) (see [16, p.93]).

For any $l \geqslant 0$, let $B_l[0,\infty)^+$ be the set of positive bounded functions on $[0,\infty)$ satisfying $\|f\| \leqslant l$. By a modification of the proof of [16, Theorem 3.42], it is not hard to show that for each $T \geqslant 0$ and $l \geqslant 0$, $v^{(k)}(t, f)(x)$ converges uniformly on the set $[0, T] \times [0, \infty) \times B_l[0, \infty)^+$ of (t, x, f) to the unique locally bounded positive solution $(t, x) \mapsto v(t, f)(x)$ of the evolution equation

$$v(t, f)(x) = f(x) - \int_0^t \phi(x, v(s, f)(x))\mathrm{d}s. \tag{16}$$

Let $\{Y_t : t \geqslant 0\}$ be the superprocess with state space $M[0, a]$ and transition semigroup $(Q_t)_{t \geqslant 0}$ defined by

$$\int_{M[0,a]} \mathrm{e}^{-\langle \nu, f\rangle} Q_t(\mu, \nu) = \exp\left\{-\langle \mu, v(t, f)\rangle\right\}, \quad f \in B[0, a]^+. \tag{17}$$

Using (16) and Gronwall's inequality, one can see that $x \mapsto v(t, f)(x)$ is continuous on $[0, a]$ for every $f \in C[0, a]^+$. Then by [16, Proposition 3.1], it is easy to see that $v(t, f) \in C[0, a]^{++}$ for every $f \in C[0, a]^{++}$. From this and (17), it follows that $(Q_t)_{t \geqslant 0}$ is a Feller semigroup. Note that if $\phi(x, z) = \phi(z)$ independent of $x \geqslant 0$, then $(Q_t)_{t \geqslant 0}$ is the same transition semigroup as that defined by (3) and (4). In this case, the corresponding superprocess can be defined by the stochastic integral equation (2).

Let $D([0, \infty), M[0, a])$ denote the space of càdlàg paths from $[0, \infty)$ to $M[0, a]$ furnished with the Skorokhod topology. The proof of the next theorem is a modification of that of [16, Theorem 3.43].

Theorem 2 *Suppose that Condition* (3.A) *is satisfied. Let* $\{Y_t : t \geqslant 0\}$ *be a càdlàg superprocess with transition semigroup* $(Q_t)_{t \geqslant 0}$ *defined by* (16) *and* (17). *If* $Y_0^{(k)}$ *converges to* Y_0 *in distribution on* $M[0, a]$, *then* $\{Y_t^{(k)} : t \geqslant 0\}$ *converges to* $\{Y_t : t \geqslant 0\}$ *in distribution on* $D([0, \infty), M[0, a])$.

Proof For $f \in C[0, a]^{++}$ and $\nu \in M[0, a]$, set

$$\mathrm{e}_f(\nu) = \mathrm{e}^{-\langle \nu, f\rangle}.$$

Clearly, the function $\nu \mapsto \mathrm{e}_f(\nu)$ is continuous in ρ. Denote by D_1 the linear span of $\{\mathrm{e}_f : f \in C[0, a]^{++}\}$. By Theorem 1, we have D_1 is an algebra which strongly separates the points of $M[0, a]$. Let $C_0(M[0, a])$ be the space of continuous functions F on $M[0, a]$ such that $F(\nu_n) \to 0$ as $\nu_n([0, a]) \to \infty$. Then D_1 is uniformly dense in $C_0(M[0, a])$ by the Stone-Weierstrass theorem (see [11, pp. 98, 99]). On the other hand, for any $f \in C[0, a]^{++}$, since $v(t, f)$ is bounded away from zero and $v_k(t, f)(x) \to v(t, f)(x)$ uniformly on $[0, \infty)$ for every $t \geqslant 0$, we have $v_k(t, f)$ is also bounded away from zero for k sufficiently large. Without loss of generality, we may assume $v_k(t, f) \geqslant c$ and $v(t, f) \geqslant c$ for some $c > 0$. Let $Q_t^{(k)}$ denote the transition semigroup of $Y_t^{(k)}$. We get from (13) and (17) that, for any $M \geqslant 0$,

$$\begin{aligned}
&\sup_{\nu\in M_k[0,a]} |Q_t^{(k)}\mathrm{e}_f(\nu) - Q_t\mathrm{e}_f(\nu)| \\
&= \sup_{\nu\in M_k[0,a]} |\exp\{-\langle\nu, v_k(t,f)\rangle\} - \exp\{-\langle\nu, v(t,f)\rangle\}| \\
&\leqslant \sup_{\langle\nu,1\rangle\leqslant M,\, \nu\in M_k[0,a]} |\exp\{-\langle\nu, v_k(t,f)\rangle\} - \exp\{-\langle\nu, v(t,f)\rangle\}| \\
&\quad + \sup_{\langle\nu,1\rangle> M,\, \nu\in M_k[0,a]} |\exp\{-\langle\nu, v_k(t,f)\rangle\} - \exp\{-\langle\nu, v(t,f)\rangle\}| \\
&\leqslant \sup_{\langle\nu,1\rangle\leqslant M,\, \nu\in M_k[0,a]} |\langle\nu, v_k(t,f)\rangle - \langle\nu, v(t,f)\rangle| + \sup_{\langle\nu,1\rangle> M,\, \nu\in M_k[0,a]} 2\mathrm{e}^{-\langle\nu,c\rangle} \\
&\leqslant M\|v_k(t,f) - v(t,f)\| + 2\mathrm{e}^{-Mc}.
\end{aligned}$$

Since $M \geqslant 0$ was arbitrary, we have

$$\lim_{k\to\infty} \sup_{\nu\in M_k[0,a]} |Q_t^{(k)}\mathrm{e}_f(\nu) - Q_t\mathrm{e}_f(\nu)| = 0$$

for every $t \geqslant 0$. Thus,

$$\lim_{k\to\infty} \sup_{\nu\in M_k[0,a]} |Q_t^{(k)}F(\nu) - Q_tF(\nu)| = 0$$

for every $t \geqslant 0$ and $F \in C_0(M[0,a])$. By [9, pp. 226, 233, 234], we conclude that $\{Y_t^{(k)}: t \geqslant 0\}$ converges to $\{Y_t: t \geqslant 0\}$ in distribution on $D([0,\infty), M[0,a])$. □

Let $\{0 \leqslant a_1 < a_2 < \cdots < a_n = a\}$ be an ordered set of constants. Denote by $\{Y_{t,a_i}: t \geqslant 0\}$ and $\{Y_{t,a_i}^{(k)}: t \geqslant 0\}$ the restriction of $\{Y_t: t \geqslant 0\}$ and $\{Y_t^{(k)}: t \geqslant 0\}$ to $[0, a_i]$, $i = 1, 2, \ldots, n$, respectively. The following theorem is an extension of Theorem 2.

Theorem 3 *Suppose that Condition* (3.A) *is satisfied. If $Y_{0,a}^{(k)}$ converges to $Y_{0,a}$ in distribution on $M[0,a]$, then $\{(Y_{t,a_1}^{(k)}, \ldots, Y_{t,a_n}^{(k)}): t \geqslant 0\}$ converges to $\{(Y_{t,a_1}, \ldots, Y_{t,a_n}): t \geqslant 0\}$ in distribution on $D([0,\infty), M[0,a_1] \times \cdots \times M[0,a_n])$.*

Proof Let $f_i \in C[0,a_i]$ for $i = 1, \ldots, n$. By Theorem 2, we see that for every $1 \leqslant i \leqslant n$, $\{\langle Y_{t,a_i}^{(k)}, f_i\rangle: t \geqslant 0\}$ is tight in $D([0,\infty), \mathbb{R})$. Thus, $\{\sum_{i=1}^n \langle Y_{t,a_i}^{(k)}, f_i\rangle: t \geqslant 0\}$ is tight in $D([0,\infty), \mathbb{R})$. Then the tightness criterion of [19] implies that $\{(Y_{t,a_1}^{(k)}, \ldots, Y_{t,a_n}^{(k)}): t \geqslant 0\}$ is tight in $D([0,\infty), M[0,a_1] \times \cdots \times M[0,a_n])$. Let $\{(Z_{t,a_1}, \ldots, Z_{t,a_n}): t \geqslant 0\}$ be a weak limit point of $\{(Y_{t,a_1}^{(k)}, \ldots, Y_{t,a_n}^{(k)}): t \geqslant 0\}$. By an argument similar to the proof of [8, Theorem 5.8], one can show that $\{(Z_{t,a_1}, \ldots, Z_{t,a_n}): t \geqslant 0\}$ and $\{(Y_{t,a_1}, \ldots, Y_{t,a_n}): t \geqslant 0\}$ have the same distributions on $D([0,\infty), M[0,a_1] \times \cdots \times M[0,a_n])$. That gives the desired result. □

Corollary 1 *Suppose that Condition* (3.A) *holds. Let $\{0 \leqslant a_1 < a_2 < \cdots < a_n = a\}$ be an ordered set of constants. Let $Y_t(a_i) := Y_t[0,a_i]$ and $Y_t^{(k)}(a_i) :=$*

$Y_t^{(k)}[0,a_i]$ *for every* $t \geqslant 0$, $i = 1, \dots, n$, *respectively. If* $(Y_0^{(k)}(a_1), \dots, Y_0^{(k)}(a_n))$ *converges to* $(Y_0(a_1), \dots, Y_0(a_n))$ *in distribution on* $\mathbb{R}_+^n$, *then* $\{(Y_t^{(k)}(a_1), \dots, Y_t^{(k)}(a_n))\colon t \geqslant 0\}$ *converges to* $\{(Y_t(a_1), \dots, Y_t(a_n))\colon t \geqslant 0\}$ *in distribution on* $D([0,\infty), \mathbb{R}_+^n)$.

4 Flows of interactive branching processes

In this section, we prove some limit theorems for a sequence of flows of interactive branching processes, which leads to a superprocesses with local branching and nonlocal branching. From those limit theorems, we derive the convergence of the finite-dimensional distributions of the path-valued branching processes.

Let g_0 be a probability generating function, and let $\{h_i\colon i = 1, 2, \dots\}$ be a family of probability generating functions. For each $i \geqslant 1$, define $g_i := g_0 h_1 \cdots h_i$ and suppose that

$$\{\xi_{n,j}(i)\colon n = 0, 1, 2, \dots;\, j = 1, 2, \dots\},$$
$$\{\eta_{n,j}(i)\colon n = 0, 1, 2, \dots;\, j = 1, 2, \dots\}$$

are two independent families of positive integer-valued i.i.d. random variables with distributions given by g_i and h_i, respectively. Given another family of positive integer-valued random variables $\{z_i\colon i = 1, 2, \dots\}$ independent of $\{\xi_{n,j}(i)\colon i = 1, 2, \dots\}$ and $\{\eta_{n,j}(i)\colon i = 1, 2, \dots\}$, we define inductively $X_0(0) = z_0$ and

$$X_{n+1}(0) = \sum_{j=1}^{X_n(0)} \xi_{n,j}(0), \quad n = 0, 1, 2, \dots. \tag{18}$$

Suppose that $\{X_n(i)\colon n = 0, 1, 2, \dots\}$ has been constructed for $i = 0, 1, \dots, m-1$, we define $\{X_n(m)\colon n = 0, 1, 2, \dots\}$ by $X_0(m) = z_m$ and

$$X_{n+1}(m) = \sum_{j=1}^{X_n(m)} \xi_{n,j}(m) + \sum_{j=1}^{\overline{X}_n(m-1)} \eta_{n,j}(m), \quad n = 0, 1, 2, \dots, \tag{19}$$

where

$$\overline{X}_n(m-1) = \sum_{i=0}^{m-1} X_n(i), \quad n = 0, 1, 2, \dots.$$

Intuitively, $\{X_n(m)\colon n = 0, 1, 2, \dots\}$ is a GW-process with immigration, whose offspring distribution is given by g_m, immigration distribution is given by h_m, and immigration rate is $\{\overline{X}_n(m-1)\colon n = 0, 1, 2, \dots\}$.

It is easy to show that for any $m \in \mathbb{N}$, $\{(X_n(0), X_n(1), \dots, X_n(m))\colon n = 0, 1, 2, \dots\}$ is a discrete-time $\mathbb{N}^{m+1}$-valued Markov chain with one-step transition probability $Q(x, \mathrm{d}y)$ determined by

$$\int_{\mathbb{N}^{m+1}} \mathrm{e}^{-\langle \lambda, y\rangle} Q(x, \mathrm{d}y) = \prod_{i=0}^{m} [g_i(\mathrm{e}^{-\lambda_i})]^{x_i} [h_i(\mathrm{e}^{-\lambda_i})]^{\sum_{j=0}^{i-1} x_j}, \quad \lambda, x \in \mathbb{N}^{m+1}, \tag{20}$$

where x_i and λ_i denote the i-th component of x and λ, respectively.

Suppose that for each integer $k \geqslant 1$, we have two sequence of processes $\{(X_n^{(k)}(i))_{n\geqslant 0} \colon i \geqslant 0\}$ and $\{(\overline{X}_n^{(k)}(i))_{n\geqslant 0} \colon i \geqslant 0\}$ with parameters $g_0^{(k)}$ and $\{h_i^{(k)} \colon i = 1, 2, \ldots\}$. Suppose that γ_k is a positive real sequence such that $\gamma_k \to \infty$ increasingly as $k \to \infty$. Let $[\gamma_k t]$ denote the integer part of $\gamma_k t \geqslant 0$. Define

$$Y_t^{(k)}(x) := \frac{1}{k} \overline{X}_{[\gamma_k t]}^{(k)}([kx]) = \frac{1}{k} \sum_{i=0}^{[kx]} X_{[\gamma_k t]}^{(k)}(i), \quad k = 1, 2, \ldots. \tag{21}$$

Let $Y_t^{(k)}(\mathrm{d}x)$ denote the random measure on $[0, \infty)$ induced by the random function $Y_t^{(k)}(x)$. We are interested in the asymptotic behavior of the continuous-time process $\{Y_t^{(k)}(\mathrm{d}x) \colon t \geqslant 0\}$ as $k \to \infty$. Let $h_0^{(k)} \equiv 1$. For any $z \geqslant 0$ and $\theta \geqslant 0$, set

$$\phi_\theta^{(k)}(z) = k\gamma_k [g_{[k\theta]}^{(k)}(\mathrm{e}^{-z/k}) - \mathrm{e}^{-z/k}] \tag{22}$$

and

$$\psi_\theta^{(k)}(z) = k^2 \gamma_k [1 - h_{[k\theta]}^{(k)}(\mathrm{e}^{-z/k})]. \tag{23}$$

Let us consider the following set of conditions.

Condition (4.A) For every $l \geqslant 0$, the sequence $\{\phi_0^{(k)}\}$ is uniformly Lipschitz on $[0, l]$ and there is a function ϕ_0 on $[0, \infty)$ such that $\phi_0^{(k)}(z) \to \phi_0(z)$ uniformly on $[0, l]$ as $k \to \infty$.

Condition (4.B) There is a function ψ on $[0, \infty)^2$ such that, for every $l \geqslant 0$, $\psi_\theta^{(k)}(z) \to \psi_\theta(z)$ uniformly on $[0, l]^2$ as $k \to \infty$ and

$$\sup_{\theta \in [0, a]} \frac{\mathrm{d}}{\mathrm{d}z} \psi_\theta(z)|_{z=0^+} < \infty.$$

Proposition 1 *If Conditions* (4.A) *and* (4.B) *hold, then for every* $q \geqslant 0$, *there is a branching mechanism* ϕ_q *such that* $\phi_q^{(k)}(z) \to \phi_q(z)$ *uniformly on* $[0, l]$ *for every* $l \geqslant 0$ *as* $k \to \infty$. *Moreover, the family of branching mechanisms* $\{\phi_q \colon q \geqslant 0\}$ *is admissible with*

$$\frac{\partial}{\partial \theta} \phi_\theta(z) = -\psi_\theta(z).$$

Proof If Conditions (4.A) and (4.B) hold, then the limit function ϕ_0 has the representation (1) with $(b, m) = (b_0, m_0)$ and ψ_θ has representation (5) (see [16,

p. 76]). By the definition of $g_i^{(k)}$, it is simple to check that, for every $q \geqslant 0$,

$$\phi_q^{(k)}(z) = k\gamma_k[g_0^{(k)}(\mathrm{e}^{-z/k}) - \mathrm{e}^{-z/k}]\prod_{i=1}^{[kq]} h_i^{(k)}(\mathrm{e}^{-z/k})$$

$$-\sum_{i=1}^{[kq]} k\gamma_k[1 - h_i^{(k)}(\mathrm{e}^{-z/k})]\mathrm{e}^{-z/k}\prod_{j=i+1}^{[kq]} h_j^{(k)}(\mathrm{e}^{-z/k}). \tag{24}$$

By elementary calculations, we have

$$\prod_{i=1}^{[kq]} h_i^{(k)}(\mathrm{e}^{-z/k}) = \exp\bigg\{-\sum_{i=1}^{[kq]}\frac{1}{k^2\gamma_k\zeta_i^{(k)}}\,\psi_{i/k}^{(k)}(z)\bigg\},$$

where $\zeta_i^{(k)} \in [h_i^{(k)}(\mathrm{e}^{-z/k}), 1]$. It is easy to show that $\prod_{i=1}^{[kq]} h_i^{(k)}(\mathrm{e}^{-z/k})$ converges to 1 uniformly on $[0, l]$ for every $l \geqslant 0$ if Condition (4.B) holds, and hence, for each $1 \leqslant i \leqslant [kq]$, $\prod_{j=i+1}^{[kq]} h_j^{(k)}(\mathrm{e}^{-z/k})$ converges to 1 uniformly on $[0, l]$ for every $l \geqslant 0$. By letting $k \to \infty$ in (24), we see that $\phi_q^{(k)}(z)$ uniformly converge to a function $\phi_q(z)$ on $[0, l]$ for every $l \geqslant 0$ and (6) holds. Then the desired result follows readily. □

Proposition 2 *To each admissible family of branching mechanisms* $\{\phi_q : q \geqslant 0\}$ *with* $(\partial/\partial\theta)\phi_\theta(z) = -\psi_\theta(z)$, *there correspond two sequences* $\{\phi_0^{(k)}\}$ *and* $\{\psi_\theta^{(k)}\}$ *in form of* (22) *and* (23), *respectively, such that Conditions* (4.A) *and* (4.B) *are satisfied.*

Proof By [16, p. 93], there is a sequence $\{\phi_0^{(k)}\}$ in form of (22) satisfying Condition (4.A). By [16, p. 102], there is a family of probability generating functions $\{\overline{h}_\theta^{(k)}\}$ such that

$$k[1 - \overline{h}_\theta^{(k)}(\mathrm{e}^{-z/k})] \to \psi_\theta(z)$$

uniformly on $[0, l]^2$ for every $a \geqslant 0$ as $k \to \infty$. Let

$$\widetilde{h}_\theta^{(k)}(z) = 1 + \frac{1}{k\gamma_k}\,[\overline{h}_\theta^{(k)}(z) - 1], \quad \theta \geqslant 0,\ |z| \leqslant 1.$$

Clearly, $\{\widetilde{h}_\theta^{(k)} : \theta \geqslant 0\}$ is a family of probability generating functions and

$$k^2\gamma_k[1 - \widetilde{h}_\theta^{(k)}(\mathrm{e}^{-z/k})] \to \psi_\theta(z)$$

uniformly on $[0, l]^2$ for every $l \geqslant 0$ as $k \to \infty$. For each $k \geqslant 1$, define

$$h_i^{(k)} = \widetilde{h}_{i/k}^{(k)}, \quad i = 1, 2, \ldots.$$

Then by the continuity of $(\theta, z) \mapsto \psi_\theta(z)$, we get the result. $\square$

Given a constant $a \geqslant 0$, denote by $\{Y_{t,a}^{(k)} : t \geqslant 0\}$ the restriction of $\{Y_t^{(k)} : t \geqslant 0\}$ to $[0, a]$. Then it is easy to see

$$Y_{0,a}^{(k)} = \frac{1}{k} \sum_{i=0}^{[ka]} X_0^{(k)}(i)\delta_{i/k}, \quad Y_{t,a}^{(k)} = \frac{1}{k} \sum_{i=0}^{[ka]} X_{[\gamma_k t]}^{(k)}(i)\delta_{i/k}.$$

Then $\{Y_{t,a}^{(k)} : t \geqslant 0\}$ is a measure-valued Markov process with state space $M_k[0, a]$. From (20), for

$$\nu = k^{-1} \sum_{i=0}^{[ka]} x_i^{(k)} \delta_{i/k} \in M_k[0, a]$$

and $f \in C[0, a]^{++}$, one can see that the (discrete) generator L_k of $\{Y_{t,a}^{(k)} : t \geqslant 0\}$ is given by

$$\begin{aligned} L_k \mathrm{e}^{-\langle \nu, f\rangle} &= \gamma_k \Big[\prod_{i=0}^{[ka]} g_i^{(k)}(\mathrm{e}^{-f(i/k)/k})^{x_i} h_i^{(k)}(\mathrm{e}^{-f(i/k)/k})^{\sum_{j=0}^{i-1} x_j} - \mathrm{e}^{-\langle \nu, f\rangle} \Big] \\ &= \mathrm{e}^{-\langle \nu, f\rangle} \gamma_k \Big[\exp\Big\{ \sum_{i=0}^{[ka]} \log \big(g_i^{(k)}(\mathrm{e}^{-f(i/k)/k})^{x_i} h_i^{(k)}(\mathrm{e}^{-f(i/k)/k})^{\sum_{j=0}^{i-1} x_j} \big) \\ &\qquad + \langle \nu, f\rangle \Big\} - 1 \Big] \\ &= \mathrm{e}^{-\langle \nu, f\rangle} \gamma_k [\exp\{\alpha_k + \beta_k\} - 1], \end{aligned} \tag{25}$$

where

$$\begin{aligned} \alpha_k &= \sum_{i=0}^{[ka]} x_i \Big[\log g_i^{(k)}(\mathrm{e}^{-f(i/k)/k}) + \frac{f(i/k)}{k} \Big], \\ \beta_k &= \sum_{i=0}^{[ka]} \sum_{j=0}^{i-1} x_j \log h_i^{(k)}(\mathrm{e}^{-f(i/k)/k}). \end{aligned}$$

By the definition of $g_i^{(k)}$, we have

$$\begin{aligned} \alpha_k &= \sum_{i=0}^{[ka]} x_i \Big[\log g_0^{(k)}(\mathrm{e}^{-f(i/k)/k}) + \sum_{j=0}^{i} \log h_j^{(k)}(\mathrm{e}^{-f(i/k)/k}) + \frac{f(i/k)}{k} \Big] \\ &= \sum_{i=0}^{[ka]} x_i \Big[\log g_0^{(k)}(\mathrm{e}^{-f(i/k)/k}) + \frac{f(i/k)}{k} \Big] + \sum_{i=0}^{[ka]} \sum_{j=0}^{i} x_i \log h_j^{(k)}(\mathrm{e}^{-f(i/k)/k}). \end{aligned}$$

It follows that

$$\begin{aligned}\alpha_k+\beta_k = &\sum_{i=0}^{[ka]} x_i\Big[\log g_0^{(k)}(\mathrm{e}^{-f(i/k)/k}) + \frac{f(i/k)}{k}\Big] + \sum_{i=0}^{[ka]}\sum_{j=0}^{i} x_i \log h_j^{(k)}(\mathrm{e}^{-f(i/k)/k}) \\ &+\sum_{i=0}^{[ka]}\sum_{j=0}^{i-1} x_j \log h_i^{(k)}(\mathrm{e}^{-f(i/k)/k}) \\ =&\sum_{i=0}^{[ka]} x_i\Big[\log g_0^{(k)}(\mathrm{e}^{-f(i/k)/k}) + \frac{f(i/k)}{k}\Big] + \sum_{i=0}^{[ka]}\sum_{j=0}^{i} x_i \log h_j^{(k)}(\mathrm{e}^{-f(i/k)/k}) \\ &+\sum_{i=0}^{[ka]}\sum_{j=i+1}^{[ka]-1} x_i \log h_j^{(k)}(\mathrm{e}^{-f(j/k)/k}) \\ =&\sum_{i=0}^{[ka]} x_i\Big[\log g_0^{(k)}(\mathrm{e}^{-f(i/k)/k}) + \frac{f(i/k)}{k}\Big] + \sum_{i=0}^{[ka]}\sum_{j=0}^{[ka]} x_i \log h_j^{(k)}(\mathrm{e}^{-f(\frac{i\vee j}{k})/k}) \\ =&\frac{1}{\gamma_k}\Big[\sum_{i=0}^{[ka]}\frac{x_i}{k\zeta_i^{(k)}}\,\phi_0^{(k)}\Big(f\Big(\frac{i}{k}\Big)\Big) - \sum_{i=0}^{[ka]}\frac{x_i}{k}\Big(\sum_{j=0}^{[ka]}\frac{1}{k\zeta_{i,j}^{(k)}}\,\psi_{j/k}^{(k)}\Big(f\Big(\frac{i\vee j}{k}\Big)\Big)\Big)\Big],\end{aligned}$$

where $\zeta_i^{(k)}$ is between $\mathrm{e}^{-f(i/k)/k}$ and $g_0^{(k)}(\mathrm{e}^{-f(i/k)/k})$, $\zeta_{i,j}^{(k)} \in [h_j^{(k)}(\mathrm{e}^{-f(\frac{i\vee j}{k})/k}), 1]$. Clearly, both $\zeta_i^{(k)}$ and $\zeta_{i,j}^{(k)}$ converge to 1 uniformly as $k\to\infty$ if Conditions (4.A) and (4.B) hold. Then the above equality implies

$$\alpha_k+\beta_k = \frac{1}{\gamma_k}\,[\langle \nu, \phi_0^{(k)}(f(\cdot))\rangle - \langle \nu, \Psi^{(k)}(\cdot, f)\rangle + o(1)], \tag{26}$$

where

$$\Psi^{(k)}(\cdot, f) = \sum_{j=0}^{[ka]}\frac{1}{k}\,\psi_{j/k}^{(k)}\Big(f\Big(\cdot\vee\frac{j}{k}\Big)\Big).$$

Let $\{Y_{t,a}\colon t\geqslant 0\}$ be the càdlàg superprocess with transition semigroup $(Q_t)_{t\geqslant 0}$ defined by (9) and (10).

Theorem 4 *Suppose that Conditions* (4.A) *and* (4.B) *are satisfied. If $Y_{0,a}^{(k)}$ converges to $Y_{0,a}$ in distribution on $M[0,a]$, then $\{Y_{t,a}^{(k)}\colon t\geqslant 0\}$ converges to $\{Y_{t,a}\colon t\geqslant 0\}$ in distribution on $D([0,\infty), M[0,a])$.*

Proof As in the proof of [15, Theorem 2.1], we shall prove the convergence of the generators. Let D_1 be the algebra as defined in Theorem 2. For $f\in C[0,a]^{++}$, let

$$L\mathrm{e}^{-\langle \nu, f\rangle} = \mathrm{e}^{-\langle \nu, f\rangle}[\langle \nu, \phi_0(f(\cdot))\rangle - \langle \nu, \Psi(\cdot, f)\rangle], \quad \nu\in M[0,a],$$

and extend the definition of L to D_1 by linearity. By (9), one can check that L is a restriction of strong generator of $(Q_t)_{t\geqslant 0}$; see (1.10) of [9, p. 8]. Note also that $L := \{(f, Lf)\colon f \in D_1\}$ is a linear space of $C_0(M[0,a]) \times C_0(M[0,a])$. On the other hand, letting $f(x) = \lambda$ in (9) and (10), we have the function $\lambda \mapsto V_t(\lambda)$ is strictly increasing on $[0,\infty)$ for every $t \geqslant 0$ (see [16, p. 58]). Therefore, $V_t(\lambda) > 0$ for every $\lambda > 0$ and $t \geqslant 0$. In view of (9), for any $f \in C[0,a]^{++}$, we have $V_t f \in C[0,a]^{++}$ for every $t \geqslant 0$. Then D_1 is invariant under $(Q_t)_{t\geqslant 0}$, which is a core of the strong generator of $(Q_t)_{t\geqslant 0}$ (see [9, p. 17]). In other words, the closure of L generates $(Q_t)_{t\geqslant 0}$ uniquely (see [9, pp. 15, 17]). Based on (25) and (26), one can see

$$\lim_{k\to\infty} \sup_{\nu\in M_k[0,a]} |L_k \mathrm{e}^{-\langle \nu, f\rangle} - L\mathrm{e}^{-\langle \nu, f\rangle}| = 0$$

for every $f \in C[0,a]^{++}$, which implies

$$\lim_{k\to\infty} \sup_{\nu\in M_k[0,a]} |L_k F(\nu) - LF(\nu)| = 0$$

for every $F \in D_1$. By [9, pp. 226, 233, 234], we conclude that $\{Y^{(k)}_{t,a} : t \geqslant 0\}$ converges to the immigration superprocess $\{Y_{t,a}\colon t \geqslant 0\}$ in distribution on $D([0,\infty), M[0,a])$. $\square$

Let $\{0 \leqslant a_1 < a_2 < \cdots < a_n = a\}$ be an ordered set of constants. Denote by $\{Y_{t,a_i} : t \geqslant 0\}$ and $\{Y^{(k)}_{t,a_i} : t \geqslant 0\}$ the restriction of $\{Y_t \colon t \geqslant 0\}$ and $\{Y^{(k)}_t : t \geqslant 0\}$ to $[0,a_i]$, respectively. Let

$$Y_t(a_i) := Y_t[0,a_i], \quad Y^{(k)}_t(a_i) := Y^{(k)}_t[0,a_i]$$

for every $t \geqslant 0$, $i = 1,2,\ldots,n$. By arguments similar to those in Section 3, we have the following result.

Theorem 5 *Suppose that Conditions* (4.A) *and* (4.B) *are satisfied. If* $Y^{(k)}_{0,a}$ *converges to* $Y_{0,a}$ *in distribution on* $M[0,a]$, *then* $\{(Y^{(k)}_{t,a_1},\ldots,Y^{(k)}_{t,a_n})\colon t \geqslant 0\}$ *converges to* $\{(Y_{t,a_1},\ldots,Y_{t,a_n})\colon t \geqslant 0\}$ *in distribution on* $D([0,\infty), M[0,a_1] \times \cdots \times M[0,a_n])$.

Corollary 2 *Suppose that Conditions* (4.A) *and* (4.B) *are satisfied. If* $(Y^{(k)}_0(a_1),\ldots,Y^{(k)}_0(a_n))$ *converges to* $(Y_0(a_1),\ldots,Y_0(a_n))$ *in distribution on* $\mathbb{R}^n_+$, *then* $\{(Y^{(k)}_t(a_1),\ldots,Y^{(k)}_t(a_n))\colon t \geqslant 0\}$ *converges to* $\{(Y_t(a_1),\ldots,Y_t(a_n))\colon t \geqslant 0\}$ *in distribution on* $D([0,\infty),\mathbb{R}^n_+)$.

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