Exact convergence rate in the central limit theorem for a branching random walk with a random environment in time*

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Abstract

In this paper, we obtain the exact convergence rate for the distribution of a supercritical branching random walk with an environment in time. This generalizes the results by Chen(2001) on the case with fixed environment. Moreover, in contrast with Chen(2001), the weaker moments condition on the underlying branching system is assumed here.

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1 Introduction

In this article, we will consider the convergence rate in the central limit theorem for *a branching random walk with a random environment in time.* In this model, the system is governed by the branching properties and random walks of the particles. More precisely, the offspring distribution and the motion law of each particle depend on its generation. Comparing with the classical branching random walks, the model presented here might be more approximate to the reality. In the classical branching random walk, the point processes indexed by the particles u, formulated by the number of its children and their displacements, have a fixed constant distribution for all particles u; here, these distributions may vary from generation to generation according to a random environment, just as in the case of a branching process in a random environment introduced in [2, 3, 24]. In other words, the distributions themselves may be realizations of a stochastic process, rather than being fixed. It is different to the usual *branching random walk in a random environment* (see e.g. [5, 13]), in which the authors considered the case where the offspring distribution of a particle situated at $z \in \mathbb{R}$ depends on a random environment indexed by z, while the moving mechanism is controlled by a fixed deterministic law.

For the model presented here, Biggins and Kyprianou (2004, [8]) showed the convergence of the natural martingale arising therein and Liu(2007,[20]) surveyed more limit theorems. Further, in [12], the authors showed the central limit theorems for the counting measure $Z_n(\cdot)$ which counts the number of particles of generation n situated in a given set for this model. Here we consider the problem on the convergence rate in the central limit theorems for the counting measure $Z_n(\cdot)$. For the classical branching random walk, Révész(1994,[23]) studied this problem and gave a conjecture on the exact convergence rate, which is proved by Chen(2001,[9]). In this article, we shall extend the work of Chen(2001,[9]) to the random environment case under the weaker moment condition for the underlying branching mechanism.

The main results will be stated in Section 2, while their proofs will be given in Sections 3.

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2 Description of the model and the main results

2.1 Description of the model

A random environment in time $\xi = (\xi_n)$ is formulated as a sequence of random variables independent and identically distributed with values in some measurable space $(\Theta, \mathcal{F})$. Each realization of ξ_n corresponds to two probability distributions $p(\xi_n)$ and G_{ξ_n}, where $p(\xi_n) = \{p_k(\xi_n) : k \in \mathbb{N}\}$ is a probability law on $\mathbb{N}$ and G_{ξ_n} a probability law on $\mathbb{R}$. Without loss of generality, we can take ξ_n as coordinate functions defined on the product space $(\Theta^{\mathbb{N}}, \mathcal{F}^{\otimes\mathbb{N}})$, equipped with a probability law τ which is invariant and ergodic under the usual shift transformation θ on $\Theta^{\mathbb{N}}$: $\theta(\xi_0, \xi_1, \cdots) = (\xi_1, \xi_2, \cdots)$.

When the environment $\xi = (\xi_n)$ is given, the process can be described as follows. The process begins at time 0 with one initial particle $\varnothing$ of generation 0 located at $S_\varnothing = 0 \in \mathbb{R}$; at time 1, it is replaced by $N = N_\varnothing$ new particles of generation 1, located at $L_i = L_{\varnothing i}, 1 \le i \le N$, where N is of distribution $p(\xi_0)$ and each L_i is an independent copy of the generic random variable H_{ξ_0} with the distribution G_{ξ_0}, given the environment ξ. In general, each particle $u = u_1...u_n$ of generation n is replaced at time $n+1$ by N_u new particles of generation $n+1$, with displacements $L_{u1}, L_{u2}, \cdots, L_{uN_u}$, so that the i-th child is located at

$$S_{ui} = S_u + L_{ui},$$

where N_u is of distribution $p(\xi_n)$ and each L_{ui} is an independent copy of the generic random variable $\widehat{L}_{\xi_n}$ with the distribution G_{ξ_n}, given the environment ξ. All the random variables N_u and L_u, indexed by all finite sequences u of positive integers, are independent of each other, given the environment ξ. We abbreviate $\widehat{L}_{\xi_n}$ (resp. G_{ξ_n}) as $\widehat{L}_n$ (resp. G_n) in the rest of this paper.

Let $(\Gamma, \mathbb{P}_\xi)$ be the probability space under which the process is defined when the environment ξ is fixed. As usual, $\mathbb{P}_\xi$ is called *quenched law*. The total probability space can be formulated as the product space $(\Gamma \times \Theta^{\mathbb{N}}, \mathbb{P})$, where $\mathbb{P} = \mathbb{P}_\xi \otimes \tau$ in the sense that for all measurable and positive function g, we have

$$\int g d\mathbb{P} = \iint g(y, \xi) d\mathbb{P}_\xi(y) d\tau(\xi),$$

(recall that τ is the law of the environment ξ). The probability $\mathbb{P}$ is called *annealed law*. The quenched law $\mathbb{P}_\xi$ may be viewed as the conditional probability of the annealed law $\mathbb{P}$ given ξ. We will use $\mathbb{E}_\xi$ to denote the expectation with respect to $\mathbb{P}_\xi$. Other expectations will be denoted simply $\mathbb{E}$ (there will be no confusion according to the context).

Let $\mathbb{T}$ be the genealogical tree with $\{N_u\}$ as defining elements. By definition, we have: (a) $\varnothing \in \mathbb{T}$; (b) $ui \in \mathbb{T}$ implies $u \in \mathbb{T}$; (c) if $u \in \mathbb{T}$, then $ui \in \mathbb{T}$ if and only if $1 \le i \le N_u$. Let $\mathbb{T}_n = \{u \in \mathbb{T} : |u| = n\}$ be the set of particles of generation n, where $|u|$ denotes the length of the sequence u and represents the number of generation to which u belongs.

2.2 The main results

Let $Z_n(\cdot)$ be the counting measure of particles of generation n: for $B \subset \mathbb{R}$,

$$Z_n(B) = \sum_{u \in \mathbb{T}_n} \mathbf{1}_B(S_u).$$

By convention, we will write $Z_n = Z_n(\mathbb{R})$. Then $\{Z_n\}$ constitutes a branching process in a random environment (see e.g. [2, 3, 24]). For $n \ge 0$, define

$$m_n = m(\xi_n) = \sum_{k=1}^{\infty} k p_k(\xi_n), \quad \Pi_n = m_0 \cdots m_{n-1}, \quad \Pi_0 = 1.$$

Throughout the paper, we will assume the following conditions

$$\mathbb{E} \ln m_0 > 0 \quad \text{and} \quad \mathbb{E} \left(\frac{1}{m_0} \sum_{k=2}^{\infty} k (\ln k)^{1+\lambda} p_k(\xi_0) \right) < \infty \quad \text{with } \lambda > 3. \tag{2.1}$$

Under these conditions, the underlying branching process is *supercritical* which means that the number of the particles tends to infinity with positive probability. Without loss of generality and for simplicity, we will always assume that

$$\mathbb{P}(N \geq 1) = 1. \tag{2.2}$$

Moreover, it is well known that in supercritical case, the normalized sequence

$$W_n = \Pi_n^{-1} Z_n, \quad n \geq 1$$

constitute a martingale with respect to the filtration $\mathscr{F}_n$ defined by: $\mathscr{F}_0 = \{\emptyset, \Omega\}, \quad \mathscr{F}_n = \sigma(\xi, N_u : |u| < n),$ for $n \geq 1$. Under (2.1), the limit

$$W = \lim_n W_n \tag{2.3}$$

exists a.s. with $\mathbb{E}W = 1$ (see for example [3]); by (2.2), $W > 0$ a.s.

For $n \geq 0$, define

$$l_n = l_n(\xi) = \mathbb{E}_\xi \widehat{L}_n, \quad \sigma_n^2 = \sigma_n^2(\xi) = \mathrm{Var}_\xi \widehat{L}_n, \quad \alpha_n = \mathbb{E}_\xi \big(\widehat{L}_n - l_n\big)^3, v_n(t) = \mathbb{E}_\xi e^{it(\widehat{L}_n - l_n)},$$
$$\ell_n = \ell_n(\xi) = \sum_{k=0}^{n-1} l_k, \quad s_n^2 = s_n^2(\xi) = \sum_{k=0}^{n-1} \sigma_k^2, \quad M_n^{(3)} = \sum_{k=0}^{n-1} \alpha_k.$$

We also need some conditions on the motion of the particles: for $\eta = \min\{2\lambda, \frac{3}{2}(1+\lambda)\}$,

$$\mathbb{E}\big(\widehat{L}_0 - l_0\big)^\eta < \infty \quad \text{and} \quad \limsup_{n\to\infty} \sup_{|t|\geq T} \left\{ \frac{1}{n} \sum_{k=0}^{n-1} |v_k(t)| \right\} < 1 \quad \text{a.s.} \tag{2.4}$$

Let $\{V_n\}$ be a sequence of random variables defined by

$$V_n = \frac{1}{\Pi_n} \sum_{u \in \mathbb{T}_n} (S_u - \ell_n).$$

Then we can state our main result as following:

Theorem 2.1. *Assume that the conditions* (2.1)∼(2.4) *hold. Then* $\lim_n V_n = V$ *a.s. for some random variable* V*, and for* $t \in \mathbb{R}$*,*

$$\sqrt{n}\Big[\Pi_n^{-1} Z_n(\ell_n + s_n t) - \Phi(t)W\Big] \xrightarrow{n\to\infty} \frac{1}{6}\mathbb{E}\alpha_0 (\mathbb{E}\sigma_0^2)^{-\frac{3}{2}}(1-t^2)\phi(t)W - (\mathbb{E}\sigma_0^2)^{-\frac{1}{2}}\phi(t)V \quad a.s., \tag{2.5}$$

where $Z_n(x) = Z_n((-\infty, x])$ *for* $x \in \mathbb{R}$ *and* $\phi(t) = \frac{1}{\sqrt{2\pi}} e^{-t^2/2}, \Phi(t) = \int_{-\infty}^t \phi(x)\mathrm{d}x.$

Remark 2.2. Compared with Chen(2001, [9]), the weaker moment condition on the underlying branching mechanism is assumed. Hence even under the constant environment, the result here considerably generalize that in[9].

We conjecture that it is possible to relax the moment condition on the reproduction law of the population.

3 Proof of Theorem 2.1

We first introduce some notation which will be used in the sequel.

For simplicity and without loss of generality, we will always assume that $l_n = 0$ hereafter (if not so, we only need to replace L_{ui} by $L_{ui} - l_n$).

The following σ-fields will be often used:

$$\begin{aligned} \mathscr{D}_0 = \{\emptyset, \Omega\}, \quad \mathscr{D}_n &= \sigma(\xi, N_u, L_{ui} : i \geq 1, |u| < n), \text{ for } n \geq 1 \\ \mathscr{I}_0 = \{\emptyset, \Omega\}, \quad \mathscr{I}_n &= \sigma(\xi_k, N_u, L_{ui} : k < n, i \geq 1, |u| < n) \text{ for } n \geq 1. \end{aligned}$$

We will use the following conditional probabilities and conditional expectations:

$$\mathbb{P}_{\xi,n}(\cdot) = \mathbb{P}_\xi(\cdot|\mathscr{D}_n); \quad \mathbb{E}_{\xi,n}(\cdot) = \mathbb{E}_\xi(\cdot|\mathscr{D}_n); \quad \mathbb{P}_n(\cdot) = \mathbb{P}(\cdot|\mathscr{I}_n), \quad \mathbb{E}_n(\cdot) = \mathbb{E}(\cdot|\mathscr{I}_n).$$

As usual, we write $\mathbb{N}^* = \{1, 2, 3, \cdots\}$ and denote by

$$U = \bigcup_{n=0}^{\infty} (\mathbb{N}^*)^n$$

the set of all finite sequences, where $(\mathbb{N}^*)^0 = \{\varnothing\}$ contains the null sequence $\varnothing$.

For all $u \in U$, let $\mathbb{T}(u)$ be the shifted tree of $\mathbb{T}$ at u with defining elements $\{N_{uv}\}$: we have 1) $\varnothing \in \mathbb{T}(u)$, 2) $vi \in \mathbb{T}(u) \Rightarrow v \in \mathbb{T}(u)$ and 3) if $v \in \mathbb{T}(u)$, then $vi \in \mathbb{T}(u)$ if and only if $1 \le i \le N_{uv}$. Define $\mathbb{T}_n(u) = \{v \in \mathbb{T}(u) : |v| = n\}$. Then $\mathbb{T} = \mathbb{T}(\varnothing)$ and $\mathbb{T}_n = \mathbb{T}_n(\varnothing)$.

3.1 A key decomposition

For $u \in (\mathbb{N}^*)^k (k \ge 0)$ and $n \ge 1$, let S_u be the position of u and write

$$Z_n(u, B) = \sum_{v \in \mathbb{T}_n(u)} \mathbf{1}_B(S_{uv} - S_u).$$

Then the law of $Z_n(u, B)$ under $\mathbb{P}_\xi$ is the same as that of $Z_n(B)$ under $P_{\theta^k\xi}$. Define

$$W_n(u, B) = Z_n(u, B)/\Pi_n(\theta^k\xi), \quad W_n(u, t) = W_n(u, (-\infty, t]),$$
$$W_n(B) = Z_n(B)/\Pi_n, \quad W_n(t) = W_n((-\infty, t]).$$

By definition, we see $\Pi_n(\theta^k\xi) = m_k \cdots m_{k+n-1}$, $Z_n(B) = Z_n(\varnothing, B)$, $W_n(B) = W_n(\varnothing, B)$, $W_n = W_n(\mathbb{R})$.

The following decomposition will play a key role in our approach: for $k \le n$,

$$Z_n(B) = \sum_{u \in \mathbb{T}_k} Z_{n-k}(u, B - S_u). \tag{3.1}$$

Remark that by our definition, for $u \in \mathbb{T}_k$,

$$Z_{n-k}(u, B - S_u) = \sum_{v_1 \cdots v_{n-k} \in \mathbb{T}_{n-k}(u)} \mathbf{1}_B(S_{uv_1 \cdots v_{n-k}})$$

represents number of the descendants of u at time n situated in B.

For each n, we choose an integer $k_n < n$ as follows. Let β be a real number such that $\max\{\frac{3}{(2\lambda)}, \frac{2}{(1+\lambda)}\} < \beta < \frac{1}{2}$ and set $k_n = [n^\beta]$. Let $t_n = s_n t$ for $t \in \mathbb{R}$ and $n \ge 1$. Then on the basis of (3.1), the following decomposition will hold:

$$\Pi_n^{-1} Z_n(t_n) - \Phi(t)W = A_n + B_n + C_n, \tag{3.2}$$

where

$$\begin{aligned}
A_n &= \frac{1}{\Pi_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \left[W_{n-k_n}(u, t_n - S_u) - \mathbb{E}_{\xi,k_n}(W_{n-k_n}(u, t_n - S_u))\right], \\
B_n &= \frac{1}{\Pi_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \left[\mathbb{E}_{\xi,k_n}(W_{n-k_n}(u, t_n - S_u)) - \Phi(t)\right] \\
C_n &= (W_{k_n} - W)\Phi(t).
\end{aligned}$$

Here we remind that the random variables $W_{n-k_n}(u, t_n - S_u)$ are independent of each other under the conditional probability $\mathbb{P}_{\xi,k_n}$.

3.2 Convergence of the martingale $\{V_n\}$

In this subsection, we shall prove the following lemma, which ensures the convergence of the martingale $\{V_n\}$ defined before.

Lemma 3.1. *Under the conditions of Theorem 2.1, $\{V_n\}$ is a martingale with respect to $\{\mathscr{D}_n\}$ and converges a.s. to some random variables V.*

Proof. First we prove that $\{V_n, \mathscr{D}_n\}$ is a martingale. This fact follows from that

$$\begin{aligned}\mathbb{E}_{\xi,n} V_{n+1} &= \mathbb{E}_{\xi,n}\left(\frac{1}{\Pi_{n+1}} \sum_{u\in\mathbb{T}_{n+1}} S_u\right) = \frac{1}{\Pi_{n+1}}\mathbb{E}_{\xi,n}\left(\sum_{u\in\mathbb{T}_n}\sum_{i=1}^{N_u}(S_u + L_{ui})\right) \\ &= \frac{1}{\Pi_{n+1}}\sum_{u\in\mathbb{T}_n}\mathbb{E}_{\xi,n}\left(\sum_{i=1}^{N_u}(S_u + L_{ui})\right) \\ &= \frac{1}{\Pi_{n+1}}\sum_{u\in\mathbb{T}_n} m_n S_u = V_n.\end{aligned}$$

Now we go to prove the convergence of the martingale. We will do it by showing that $\sum_{n=1}^{\infty}(V_{n+1} - V_n) < +\infty$ a.s.

For convenience, we shall use the following notation:

$$X_u = S_u\left(\frac{N_u}{m_{|u|}} - 1\right) + \sum_{i=1}^{N_u}\frac{L_{ui}}{m_{|u|}}, \quad X'_u = X_u \mathbf{1}_{\{|X_u|\le \Pi_{|u|}\}};$$
$$I_n = V_{n+1} - V_n = \frac{1}{\Pi_n}\sum_{u\in\mathbb{T}_n} X_u, \quad I'_n = \frac{1}{\Pi_n}\sum_{u\in\mathbb{T}_n} X'_u.$$

For $u \in \mathbb{T}_n$, Let $\hat{X}_n$ and $\hat{N}_n$ be the generic random variables of X_u and N_u respectively, i.e. $\hat{X}_n$ ($\hat{N}_n$) has the same distribution with X_u (N_u).

We start the proof by proving the following inequality:

$$\frac{1}{(\ln \Pi_n)^{1+\lambda}}\mathbb{E}_\xi|\hat{X}_n|(\ln^+|\hat{X}_n|)^{1+\lambda} \le \frac{C_1}{(\ln \Pi_n)^{1+\lambda}} + \frac{C_2 n + C_3}{(\ln \Pi_n)^{1+\lambda}}\mathbb{E}_\xi\frac{\hat{N}_n}{m_n}(\ln \hat{N}_n)^{1+\lambda}, \tag{3.3}$$

where C_1, C_2, C_3 are constants depending on ξ.

For $u \in \mathbb{T}_n$, observe that

$$\begin{aligned}&|X_u| \le |S_u|\left(1 + \frac{N_u}{m_{|u|}}\right) + \frac{\left|\sum_{i=1}^{N_u} L_{ui}\right|}{m_{|u|}}, \\ &\ln^+|X_u| \le 2 + \ln^+|S_u| + \ln N_u + \ln^+\left|\frac{1}{m_{|u|}}\sum_{i=1}^{N_u} L_{ui}\right|, \\ &(\ln^+|X_u|)^{1+\lambda} \le 4^\lambda\left(2^{1+\lambda} + (\ln^+|S_u|)^{1+\lambda} + (\ln N_u)^{1+\lambda} + \left(\ln^+\left|\frac{1}{m_{|u|}}\sum_{i=1}^{N_u} L_{ui}\right|\right)^{1+\lambda}\right).\end{aligned}$$

(We have used the elementary inequalities : for $x, y > 0$, $\ln^+(x+y) \le 1 + \ln^+ x + \ln^+ y$, $\ln(1+x) \le 1 + \ln^+ x$ and the facts $N_u \ge 1$ and $m_{|u|} \ge 1$).

Then we get that

$$4^{-\lambda}|X_u|(\ln^+|X_u|)^{1+\lambda} \le \sum_{i=1}^{8}\mathbb{J}_i,$$

with
$$\mathbb{J}_1 = 2^{1+\lambda}|S_u|\left(1 + \frac{N_u}{m_{|u|}}\right), \qquad \mathbb{J}_2 = |S_u|(\ln^+|S_u|)^{1+\lambda}\left(1 + \frac{N_u}{m_{|u|}}\right),$$
$$\mathbb{J}_3 = |S_u|\left(1 + \frac{N_u}{m_{|u|}}\right)(\ln N_u)^{1+\lambda}, \qquad \mathbb{J}_4 = |S_u|\left(1 + \frac{N_u}{m_{|u|}}\right)\left(\ln^+\left|\frac{1}{m_{|u|}}\sum_{i=1}^{N_u} L_{ui}\right|\right)^{1+\lambda},$$

$$\mathbb{J}_5 = \frac{2^{1+\lambda}}{m_{|u|}}\left|\sum_{i=1}^{N_u} L_{ui}\right|, \quad \mathbb{J}_6 = \frac{(\ln^+|S_u|)^{1+\lambda}}{m_{|u|}}\left|\sum_{i=1}^{N_u} L_{ui}\right|, \quad \mathbb{J}_7 = \frac{(\ln N_u)^{1+\lambda}}{m_{|u|}}\left|\sum_{i=1}^{N_u} L_{ui}\right|,$$
$$\mathbb{J}_8 = \frac{1}{m_{|u|}}\left|\sum_{i=1}^{N_u} L_{ui}\right|\left(\ln^+\left|\frac{1}{m_{|u|}}\sum_{i=1}^{N_u} L_{ui}\right|\right)^{1+\lambda}.$$

There exists a constant K_ξ such that for $u \in \mathbb{T}_n$,

$$\mathbb{E}_\xi|S_u| \le \sum_{j=1}^{n}\mathbb{E}_\xi|\widehat{L}_j| \le K_\xi n, \mathbb{E}_\xi|S_u|^2 = \sum_{j=1}^{n}\mathbb{E}_\xi|\widehat{L}_j|^2 \le K_\xi n$$

because

$$\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\mathbb{E}_\xi|\widehat{L}_j|^q = \mathbb{E}|\widehat{L}_1|^q < \infty, \quad q = 1, 2.$$

Observe that for $\forall\varepsilon > 0$,

$$\mathbb{P}\left(n^{-1/2}\mathbb{E}_\xi|\widehat{L}_n| > \varepsilon\right) \le \frac{1}{n^2}\mathbb{E}(\mathbb{E}_\xi|\widehat{L}_n|)^4 \quad \le \quad \frac{1}{n^2}\mathbb{E}\big(\mathbb{E}_\xi|\widehat{L}_n|^4\big) = \frac{1}{n^2}\mathbb{E}|\widehat{L}_n|^4.$$

Then $n^{-1/2}\mathbb{E}_\xi|\widehat{L}_n| \to 0$ a.s. and hence $n^{-1/2}\mathbb{E}_\xi|\widehat{L}_n| \le K_\xi$.

By our assumptions on environment, we see that S_u, N_u and L_{ui} are mutually independent under $\mathbb{P}_\xi$. On the basis of the above estimates, we have the following inequalities: for $u \in \mathbb{T}_n$,

$$\begin{aligned}
&\mathbb{E}_\xi\mathbb{J}_1 = 2^{1+\lambda}\mathbb{E}_\xi|S_u|\mathbb{E}_\xi\left(1 + \frac{N_u}{m_{|u|}}\right) \le K_\xi n;\\
&\mathbb{E}_\xi\mathbb{J}_2 \le \mathbb{E}_\xi(|S_u|^2 + |S_u|) \le K_\xi n;\\
&\mathbb{E}_\xi\mathbb{J}_3 \le \mathbb{E}_\xi|S_u|\mathbb{E}_\xi\left(1 + \frac{N_u}{m_{|u|}}\right)(\ln N_u)^{1+\lambda} \le K_\xi n\left(1 + \mathbb{E}_\xi\frac{\hat{N}_n}{m_n}(\ln\hat{N}_n)^{1+\lambda}\right);\\
&\mathbb{E}_\xi\mathbb{J}_4 \le \mathbb{E}_\xi|S_u|\mathbb{E}_\xi\left(1 + \frac{N_u}{m_{|u|}}\right)\left(\ln^+\left|\frac{1}{m_{|u|}}\sum_{i=1}^{N_u} L_{ui}\right|\right)^{1+\lambda} \text{(then by the concavity of } (\ln x)^{1+\lambda})\\
&\quad \le (K_\xi n)\mathbb{E}_\xi\left(1 + \frac{N_u}{m_{|u|}}\right)\left(\ln^+\frac{1}{m_{|u|}}\sum_{i=1}^{N_u}\mathbb{E}_\xi\{|L_{ui}|\,\big|N_u\}\right)^{1+\lambda}\\
&\quad \le K_\xi n\left(K_\xi n^{1/2} + \mathbb{E}_\xi\frac{N_u}{m_{|u|}}(\ln N_u)^{1+\lambda}\right)\\
&\quad = K_\xi n^{3/2} + K_\xi n\mathbb{E}_\xi\frac{\hat{N}_n}{m_n}(\ln\hat{N}_n)^{1+\lambda};\\
&\mathbb{E}_\xi\mathbb{J}_5 \le 2^{1+\lambda}\mathbb{E}_\xi|\widehat{L}_n| \le K_\xi n^{1/2};\\
&\mathbb{E}_\xi\mathbb{J}_6 = \mathbb{E}_\xi(\ln^+|S_u|)^{1+\lambda}\mathbb{E}_\xi\frac{1}{m_{|u|}}\left|\sum_{i=1}^{N_u} L_{ui}\right|\\
&\quad \le (\ln^+\mathbb{E}_\xi|S_u|)^{1+\lambda}\mathbb{E}_\xi|\widehat{L}_n| \le (\ln^+(K_\xi n))^{1+\lambda}K_\xi n \le K_\xi n(\ln n)^{1+\lambda};\\
&\mathbb{E}_\xi\mathbb{J}_7 \le \mathbb{E}_\xi\widehat{L}_n\mathbb{E}_\xi\frac{N_u}{m_{|u|}}(\ln N_u)^{1+\lambda} = K_\xi n^{1/2}\mathbb{E}_\xi\frac{\hat{N}_n}{m_n}(\ln\hat{N}_n)^{1+\lambda};\\
&\mathbb{E}_\xi\mathbb{J}_8 \le \mathbb{E}_\xi\frac{\left(\sum_{i=1}^{N_u} L_{ui}\right)^2}{m_{|u|}} \le \mathbb{E}_\xi|\widehat{L}_n|^2 \le K_\xi n.
\end{aligned}$$

Hence we get that for $u \in \mathbb{T}_n$,

$$\mathbb{E}_\xi|\hat{X}_n|(\ln^+|\hat{X}_n|)^{1+\lambda} = \mathbb{E}_\xi|X_u|(\ln^+|X_u|)^{1+\lambda} \le K_\xi n^{3/2}\left(1 + \mathbb{E}_\xi\frac{\hat{N}_n}{m_n}(\ln\hat{N}_n)^{1+\lambda}\right). \tag{3.4}$$

Now we go to prove the $\sum_{n=0}^{\infty} I_n < \infty$ a.s. Observe that

$$\sum_{n=0}^{\infty} I_n = \sum_{n=0}^{\infty}(I_n - I_n') + \sum_{n=0}^{\infty}(I_n' - \mathbb{E}_{\xi,n} I_n') + \sum_{n=0}^{\infty} \mathbb{E}_{\xi,n} I_n'.$$

We shall prove that each of the three series on the right hand side converges.

For the first series, we see that

$$\begin{aligned}
\mathbb{E}_\xi |I_n - I_n'| &= \mathbb{E}_\xi \left| \frac{1}{\Pi_n} \sum_{u\in\mathbb{T}_n} X_u \mathbf{1}_{\{|X_u|>\Pi_n\}} \right| \\
&\le \mathbb{E}_\xi \left\{ \frac{1}{\Pi_n} \sum_{u\in\mathbb{T}_n} \mathbb{E}_{\xi,n}(|X_u| \mathbf{1}_{\{|X_u|>\Pi_n\}}) \right\} \\
&= \mathbb{E}_\xi \big(|\hat{X}_n| \mathbf{1}_{\{|\hat{X}_n|>\Pi_n\}} \big) \\
&\le \frac{1}{(\ln \Pi_n)^{1+\lambda}} \mathbb{E}_\xi |\hat{X}_n| (\ln^+ |\hat{X}_n|)^{1+\lambda} \\
&\le \frac{K_\xi n^{1/2}}{(\ln \Pi_n)^{1+\lambda}} \left(1 + \mathbb{E}_\xi \frac{\hat{N}_n}{m_n} (\ln \hat{N}_n)^{1+\lambda} \right).
\end{aligned}$$

As $\lim_{n\to\infty} \frac{\ln \Pi_n}{n} = \mathbb{E} \ln m_0 > 0$ a.s., for a given constant δ_1 satisfying $0 < \delta_1 < \mathbb{E} \ln m_0$ and for n large, $\ln \Pi_n > \delta_1 n$. Hence for n large,

$$\mathbb{E}_\xi |I_n - I_n'| \le \frac{K_\xi}{(\delta_1)^{1+\lambda} n^{\lambda - \frac{1}{2}}} \left(1 + \mathbb{E}_\xi \frac{\hat{N}_n}{m_n} (\ln \hat{N}_n)^{1+\lambda} \right).$$

Observe that for $\lambda > 3/2$,

$$\mathbb{E} \sum_{n=1}^{\infty} \frac{1}{n^{\lambda-\frac{1}{2}}} \mathbb{E}_\xi \frac{\hat{N}_n}{m_n} (\ln \hat{N}_n)^{1+\lambda} = \sum_{n=1}^{\infty} \frac{1}{n^{\lambda-\frac{1}{2}}} \mathbb{E} \frac{N}{m_0} (\ln N)^{1+\lambda} < \infty.$$

Then it follows that

$$\sum_{n=1}^{\infty} \frac{1}{n^{\lambda-\frac{1}{2}}} \mathbb{E}_\xi \frac{\hat{N}_n}{m_n} (\ln \hat{N}_n)^{1+\lambda} < \infty \quad \text{a.s.}$$

Hence a.s.

$$\mathbb{E}_\xi \left| \sum_{n=0}^{\infty} (I_n - I_n') \right| \le \sum_{n=0}^{\infty} \mathbb{E}_\xi |I_n - I_n'| < \infty,$$

and it follows that $\sum_{n=0}^{\infty}(I_n - I_n')$ is convergent a.s.

For the series $\sum_{n=0}^{\infty} \mathbb{E}_{\xi,n} I_n'$, we use the fact $\mathbb{E}_{\xi,n} I_n = 0$. Then we have

$$\mathbb{E}_\xi \left| \sum_{n=0}^{\infty} \mathbb{E}_{\xi,n} I_n' \right| = E_\xi \left| \sum_{n=0}^{\infty} \mathbb{E}_{\xi,n}(I_n - I_n') \right| \le \sum_{n=0}^{\infty} \mathbb{E}_\xi |I_n - I_n'| < \infty.$$

And the a.s. convergence of the series $\sum_{n=0}^{\infty} \mathbb{E}_{\xi,n} I_n'$ follow.

Now we go to prove that

$$\sum_{n=0}^{\infty} (I_n' - \mathbb{E}_{\xi,n} I_n') < \infty \text{ a.s.} \tag{3.5}$$

By the convergence of L^2 bounded martingale (see e.g. [10, P. 251, Ex. 4.9]), we only need to show the convergence of the series: $\sum_{n=0}^{\infty} \mathbb{E}_\xi (I_n' - \mathbb{E}_{\xi,n} I_n')^2$.

Notice that for a positive constant δ_2,

$$\mathbb{E}_\xi (I_n' - \mathbb{E}_{\xi,n} I_n')^2 = \mathbb{E}_\xi \left(\frac{1}{\Pi_n} \sum_{u\in\mathbb{T}_n} (X_u - \mathbb{E}_{\xi,n} X_u) \right)^2$$

$$
\begin{aligned}
&= \mathbb{E}_\xi\left(\frac{1}{\Pi_n^2}\sum_{u\in\mathbb{T}_n}\mathbb{E}_{\xi,n}(X'_u-\mathbb{E}_{\xi,n}X'_u)^2\right)\\
&\le \mathbb{E}_\xi\frac{1}{\Pi_n^2}\sum_{u\in\mathbb{T}_n}\mathbb{E}_{\xi,n}X_u'^2=\frac{1}{\Pi_n}\mathbb{E}_\xi(\hat{X}_n^2\mathbf{1}_{\{|\hat{X}_n|\le\Pi_n\}})\\
&= \frac{1}{\Pi_n}\mathbb{E}_\xi\left(\hat{X}_n^2\mathbf{1}_{\{|\hat{X}_n|\le\Pi_n\}}\mathbf{1}_{\{|\hat{X}_n|\le\delta_2\}}+\hat{X}_n^2\mathbf{1}_{\{|\hat{X}_n|\le\Pi_n\}}\mathbf{1}_{\{|\hat{X}_n|>\delta_2\}}\right)\\
&\le \frac{\delta_2^2}{\Pi_n}+\frac{1}{\Pi_n}\mathbb{E}_\xi\frac{\hat{X}_n^2\Pi_n(\ln\Pi_n)^{-(1+\lambda)}}{|\hat{X}_n|(\ln^+|\hat{X}_n|)^{-(1+\lambda)}}\\
&\le \frac{\delta_2^2}{\Pi_n}+\frac{\mathbb{E}_\xi|\hat{X}_n|(\ln^+|\hat{X}_n|)^{1+\lambda}}{(\ln\Pi_n)^{1+\lambda}}
\end{aligned}
$$

Then by the above arguments, we see that the series $\sum_{n=0}^\infty\mathbb{E}_\xi(I'_n-\mathbb{E}_{\xi,n}I'_n)^2<\infty$ and the desired convergence of (3.5) follows.

Combining the above arguments, we see that the martingale V_n converges to the limit $V=\sum_{n=1}^\infty(V_{n+1}-V_n)$. □

3.3 Proof of the main theorem

By virtue of the decomposition (3.2), We shall divide the proof of the main theorem into the following lemmas.

Lemma 3.2. *Under the hypothesis of Theorem 2.1, a.s.*

$$\sqrt{n}A_n\xrightarrow{n\to\infty}0. \tag{3.6}$$

Lemma 3.3. *Under the hypothesis of Theorem 2.1, a.s.*

$$\sqrt{n}B_n\xrightarrow{n\to\infty}\frac{1}{6}\mathbb{E}\alpha_0(\mathbb{E}\sigma_0^2)^{-\frac{3}{2}}(1-t^2)\phi(t)W-(\mathbb{E}\sigma_0^2)^{-\frac{1}{2}}\phi(t)V \tag{3.7}$$

Lemma 3.4. *Under the hypothesis of Theorem 2.1, a.s.*

$$\sqrt{n}C_n\xrightarrow{n\to\infty}0\ \ a.s. \tag{3.8}$$

Now we go to prove the lemmas subsequently.

Proof of Lemma 3.2. For ease of the notations, we will use the following notations. For $|u|=n$, we denote $X_{n,u}=W_{n-k_n}(u,t_n-S_u)-\mathbb{E}_{\xi,k_n}W_{n-k_n}(u,t_n-S_u)$. Then we see that $|X_{n,u}|\le W_{n-k_n}+1$.

We introduce the following notations:

$$
\begin{aligned}
Y_n &= \frac{1}{Z_{k_n}}\sum_{u\in\mathbb{T}_{k_n}}X_{n,u},\\
\bar{X}_{n,u} &= X_{n,u}\mathbf{1}_{\{|X_{n,u}|<Z_{k_n}\}},\\
\bar{Y}_n &= \frac{1}{Z_{k_n}}\sum_{u\in\mathbb{T}_{k_n}}\bar{X}_{n,u}.
\end{aligned}
$$

Using the fact that W_n converges to $W>0$ a.s., then to prove Lemma 3.2, we only need to prove that

$$\sqrt{n}Y_n\xrightarrow{n\to\infty}0\text{ a.s.} \tag{3.9}$$

We will use the extended Borel-Cantelli Lemma. We can obtain the required result once we prove that $\forall\varepsilon>0$,

$$\sum_{n=1}^\infty\mathbb{P}_{k_n}(|\sqrt{n}Y_n|>\varepsilon)<\infty. \tag{3.10}$$

Notice that

$$\mathbb{P}_{k_n}(|Y_n|>\frac{\varepsilon}{\sqrt{n}})$$

$$\leq \quad \mathbb{P}_{k_n}(Y_n \neq \bar{Y}_n) + \mathbb{P}_{k_n}(|\bar{Y}_n - \mathbb{E}_{\xi,k_n}\bar{Y}_n| \geq \frac{\varepsilon}{\sqrt{n}}) + \mathbb{P}_{k_n}(|\mathbb{E}_{\xi,k_n}| > \frac{\varepsilon}{\sqrt{n}}).$$

We will proceed the proof in four steps.
Step 1 We first prove that

$$\sum_{n=1}^{\infty} \mathbb{P}_{k_n}(Y_n \neq \overline{Y}_n) < \infty. \tag{3.11}$$

We need the following result:

Lemma 3.5. *([19]) Assume that* (2.1) *and* (2.2) *hold, then*

$$\mathbb{E}(W^* + 1)(\ln(W^* + 1))^{\lambda} < \infty. \tag{3.12}$$

Furthermore, for $\beta > \frac{1}{\lambda}$ and $\{r_n\}$ with $\liminf_{n\to\infty} \frac{\ln r_n}{n^\beta} > 0$,

$$\sum_{n=1}^{\infty} \mathbb{E}\Big[(W^* + 1)\mathbf{1}_{\{W^*+1\geq r_n\}}\Big] < +\infty. \tag{3.13}$$

We observe that

$$\mathbb{P}_{k_n}(Y_n \neq \overline{Y}_n) \leq \Big[\mathbb{E}\big((W_{n-k_n} + 1)\mathbf{1}_{\{W_{n-k_n}+1\geq r_n\}}\big)\Big]_{r_n=Z_{k_n}} \leq \Big[\mathbb{E}\big((W^* + 1)\mathbf{1}_{\{W^*+1\geq r_n\}}\big)\Big]_{r_n=Z_{k_n}}.$$

Notice that $r_n = Z_{k_n}$ satisfies the condition $\liminf_{n\to\infty} \frac{\ln r_n}{n^\beta} > 0$, because $Z_{k_n}/\pi_{k_n} \to W > 0$ a.s., $(\pi_n)^{\frac{1}{n}} \to \exp\{\mathbb{E}\ln m_0\}$ a.s. and $k_n \sim n^\beta$. From this and Lemma 3.5, we obtain that

$$\sum_{n=1}^{\infty} \Big[\mathbb{E}\big((W^* + 1)\mathbf{1}_{\{W^*+1\geq r_n\}}\big)\Big]_{r_n=Z_{k_n}} < +\infty. \tag{3.14}$$

Then (3.11) follows.
Step 2. We next prove that $\forall \varepsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{P}_{k_n}(|\overline{Y}_n - \mathbb{E}_\xi \overline{Y}_n| > \frac{\varepsilon}{\sqrt{n}}) < \infty. \tag{3.15}$$

Observe that $\forall u \in \mathbb{T}_{k_n}, n \geq 1$,

$$\begin{aligned}
\mathbb{E}_{k_n}\bar{X}_{n,u}^2 &= \int_0^\infty 2x\mathbb{P}_{k_n}(|\bar{X}_{n,u}| > x)dx = 2\int_0^\infty x\mathbb{P}_{k_n}(|X_{n,u}\mathbf{1}_{\{|X_{n,u}|<Z_{k_n}\}} > x)dx \\
&\leq 2\int_0^{Z_{k_n}} x\mathbb{P}_{k_n}(|W_{n-k_n}(u) + 1| > x)dx = 2\int_0^{Z_{k_n}} x\mathbb{P}(|W_{n-k_n} + 1| > x)dx \\
&\leq 2\int_0^{Z_{k_n}} x\mathbb{P}(W^* + 1 > x)dx.
\end{aligned}$$

Then we have that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \mathbb{P}_{k_n}(|\overline{Y}_n - \mathbb{E}_\xi \overline{Y}_n| > \frac{\varepsilon}{\sqrt{n}}) \\
&= \sum_{n=1}^{\infty} \mathbb{E}_{k_n}\mathbb{P}_{\xi,k_n}(|\overline{Y}_n - \mathbb{E}_{\xi,k_n}\overline{Y}_n| > \frac{\varepsilon}{\sqrt{n}}) \\
&\leq \varepsilon^{-2}\sum_{n=1}^{\infty} n\mathbb{E}_{k_n}\left(Z_{k_n}^{-2}\sum_{u\in\mathbb{T}_{k_n}} \mathbb{E}_{\xi,k_n}\overline{X}_{n,u}^2\right) = \varepsilon^{-2}\sum_{n=1}^{\infty} n\left(Z_{k_n}^{-2}\sum_{u\in\mathbb{T}_{k_n}} \mathbb{E}_{k_n}\overline{X}_{n,u}^2\right) \\
&\leq \varepsilon^{-2}\sum_{n=1}^{\infty} \frac{n}{Z_{k_n}}\int_0^{Z_{k_n}} 2y\mathbb{P}(|W^* + 1| > y)dy
\end{aligned}$$

$$
\begin{aligned}
&= \varepsilon^{-2}\left[c_2 + \int_1^\infty \left(\sum_{n=1}^\infty \frac{n}{Z_{k_n}}\mathbf{1}_{\{y<Z_{k_n}\}}\right)\cdot 2y\mathbb{P}(|W^*+1|>y)dy\right] \\
&\le \varepsilon^{-2}\left[c_2 + \int_1^\infty \left(c_3 + c_4(\ln y)^{2/\beta-1}\right)\mathbb{P}(|W^*+1|>y)dy\right] \\
&\le \varepsilon^{-2}\big(c_1 + c_4\mathbb{E}(W^*+1)(\ln(W^*+1))^{2/\beta-1}\big) \\
&\le \varepsilon^{-2}\big(c_1 + c_4\mathbb{E}(W^*+1)(\ln(W^*+1))^{\lambda}\big) < \infty.
\end{aligned}
$$

Step 3. First we see

$$
\begin{aligned}
&\mathbb{P}_{k_n}\left(|\mathbb{E}_{\xi,k_n}\bar{Y}_n| > \frac{\varepsilon}{\sqrt{n}}\right) \\
\le\ & \frac{\sqrt{n}}{\varepsilon}\mathbb{E}_{k_n}|\mathbb{E}_{\xi,k_n}\bar{Y}_n| = \frac{\sqrt{n}}{\varepsilon}\mathbb{E}_{k_n}\Big|\frac{1}{Z_{k_n}}\sum_{u\in\mathbb{T}_{k_n}}\mathbb{E}_{\xi,k_n}\bar{X}_{n,u}\Big| \\
=\ & \frac{\sqrt{n}}{\varepsilon}\mathbb{E}_{k_n}\Big|\frac{1}{Z_{k_n}}\sum_{u\in\mathbb{T}_{k_n}}(-\mathbb{E}_{\xi,k_n}X_{n,u}\mathbf{1}_{\{|X_{n,u}|\ge Z_{k_n}\}})\Big| \\
\le\ & \frac{\sqrt{n}}{\varepsilon}\frac{1}{Z_{k_n}}\sum_{u\in\mathbb{T}_{k_n}}\mathbb{E}_{k_n}(W_{n-k_n}(u)+1)\mathbf{1}_{\{W_{n-k_n}(u)+1\ge Z_{k_n}\}} \\
=\ & \frac{\sqrt{n}}{\varepsilon}\Big[\mathbb{E}(W_{n-k_n}+1)\mathbf{1}_{\{W_{n-k_n}+1\ge r_n\}}\Big]_{r_n=Z_{k_n}} \\
\le\ & \frac{\sqrt{n}}{\varepsilon}\Big[\mathbb{E}(W^*+1)\mathbf{1}_{\{W^*+1\ge r_n\}}\Big]_{r_n=Z_{k_n}}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
&\sum_{n=1}^\infty\Big[\mathbb{E}\sqrt{n}(W^*+1)\mathbf{1}_{\{W^*+1\ge r_n\}}\Big]_{r_n=Z_{k_n}} \\
\le\ & \sum_{n=1}^\infty\Big[\mathbb{E}(W^*+1)\ln^{1/2\beta}(W^*+1)\mathbf{1}_{\{W^*+1\ge r_n\}}\Big]_{r_n=Z_{k_n}} \\
=\ & \mathbb{E}(W^*+1)\ln^{1/2\beta}(W^*+1)\sum_{n=1}^\infty\Big[\mathbf{1}_{\{W^*+1\ge r_n\}}\Big]_{r_n=Z_{k_n}} \\
\le\ & c_6\mathbb{E}(W^*+1)\ln^{3/2\beta}(W^*+1) \le c_6\mathbb{E}(W^*+1)\ln^{\lambda}(W^*+1) < \infty.
\end{aligned}
$$

Combining step 1-3, we obtain (3.10) and hence the lemma is proved. □

Proof of Lemma 3.3 . For ease of notation, set $H(t) = (1-t^2)\phi(t)$.

Observe that

$$
B_n = B_{n1} + B_{n2} + B_{n3} + B_{n4}, \tag{3.16}
$$

where

$$
\begin{aligned}
B_{n1} &= \frac{1}{\Pi_n}\sum_{u\in\mathbb{T}_{k_n}}\left(\mathbb{E}_{\xi,k_n}W_{n-k_n}(u,t_n-S_u) - \Phi\left(\frac{t_n-S_u}{\sqrt{s_n^2-s_{k_n}^2}}\right) - \frac{M_n^{(3)}-M_{k_n}^{(3)}}{6(s_n^2-s_{k_n}^2)^{3/2}}H\left(\frac{t_n-S_u}{\sqrt{s_n^2-s_{k_n}^2}}\right)\right); \\
B_{n2} &= \frac{1}{\Pi_n}\sum_{u\in\mathbb{T}_{k_n}}\left(\Phi\left(\frac{t_n-S_u}{\sqrt{s_n^2-s_{k_n}^2}}\right) - \Phi(t)\right); \\
B_{n3} &= \frac{M_n^{(3)}-M_{k_n}^{(3)}}{6(s_n^2-s_{k_n}^2)^{3/2}}\frac{1}{\Pi_n}\sum_{u\in\mathbb{T}_{k_n}}\left(H\left(\frac{t_n-S_u}{\sqrt{s_n^2-s_{k_n}^2}}\right) - H(t)\right); \\
B_{n4} &= \frac{M_n^{(3)}-M_{k_n}^{(3)}}{6(s_n^2-s_{k_n}^2)^{3/2}}H(t)W_{k_n}.
\end{aligned}
$$

Then the lemma will be proved once we show that

$$\sqrt{n}B_{n1} \xrightarrow{n\to\infty} 0; \tag{3.17}$$

$$\sqrt{n}B_{n2} \xrightarrow{n\to\infty} -(\mathbb{E}\sigma_0^2)^{-\frac{1}{2}}\phi(t)V; \tag{3.18}$$

$$\sqrt{n}B_{n3} \xrightarrow{n\to\infty} 0; \tag{3.19}$$

$$\sqrt{n}B_{n4} \xrightarrow{n\to\infty} \frac{1}{6}\mathbb{E}\alpha_0(\mathbb{E}\sigma_0^2)^{-\frac{3}{2}}H(t)W. \tag{3.20}$$

Next we prove these results subsequently.

First, We go to prove (3.17). Here a key role will be played by the following result on the asymptotic expansion of the distribution of the sum of independent random variables:

Proposition 3.6. *Under the hypothesis of Theorem 2.1, for a.e. ξ,*

$$\epsilon_n = n^{1/2}\sup_{x\in\mathbb{R}}\left|\mathbb{P}_{\xi,k_n}\left(\frac{\sum_{k=k_n}^{n-1}\widehat{L}_k}{\sqrt{s_n^2-s_{k_n}^2}}\le x\right)-\Phi(x)-\frac{M_n^{(3)}-M_{k_n}^{(3)}}{6(s_n^2-s_{k_n}^2)^{3/2}}H(x)\right| \xrightarrow{n\to\infty} 0.$$

Proof. Let $X_k=0$ for $0\le k\le k_n-1$ and $X_k=\widehat{L}_k$ for $k_n\le k\le n-1$. Then the random variables $\{X_k\}$ are independent under P_ξ. By virtue of the Markov inequality and Theorem 1 of Bai and Zhao(1986,[4]), we obtain the following result:

$$\begin{aligned}&\sup_{x\in\mathbb{R}}\left|\mathbb{P}_{\xi,k_n}\left(\frac{\sum_{k=0}^{n-1}\widehat{L}_k}{\sqrt{s_n^2-s_{k_n}^2}}\le x\right)-\Phi(x)-\frac{M_n^{(3)}-M_{k_n}^{(3)}}{6(s_n^2-s_{k_n}^2)^{3/2}}H(x)\right|\\ \le& K_\xi\left\{(s_n^2-s_{k_n}^2)^{-2}\sum_{j=k_n}^{n-1}\mathbb{E}_\xi|L_j|^4+n^6\left(\sup_{\delta>T}\frac{1}{n}\left(k_n+\sum_{j=k_n}^{n-1}\mathbb{E}_\xi|v_j(t)|\right)+\frac{1}{2n}\right)^n\right\}.\end{aligned}$$

By our conditions on the environment, we know that

$$\lim_{n\to\infty} n(s_n^2-s_{k_n}^2)^{-2}\sum_{j=k_n}^{n-1}\mathbb{E}_\xi|\widehat{L}_k|^4=\mathbb{E}|\widehat{L}_0|^4/\mathbb{E}\sigma_0^2. \tag{3.21}$$

$$\lim_{n\to\infty}\delta_n=\frac{11}{128}\frac{\mathbb{E}\sigma_0^2}{\mathbb{E}\alpha_0}. \tag{3.22}$$

Then for n large enough, $\delta_n>\frac{1}{12}\frac{\mathbb{E}\sigma_0^2}{\mathbb{E}\alpha_0}$ and hence by use of (2.4),

$$n^6\left(\sup_{\delta>T}\frac{1}{n}\left(k_n+\sum_{j=k_n}^{n-1}\mathbb{E}_\xi|v_j(t)|\right)+\frac{1}{2n}\right)^n=o(n^{-\frac{1}{2}}). \tag{3.23}$$

The proposition comes from (3.21) and (3.23). □

From this proposition, it is easy to see that

$$\sqrt{n}|B_{n1}|\le W_{k_n}\epsilon_n \xrightarrow{n\to\infty} 0.$$

Hence (3.17) is proved.

The next thing is to prove (3.18).

Observe that

$$|\sqrt{n}B_{n2}+(\mathbb{E}\sigma_0^2)^{-\frac{1}{2}}\phi(t)V|\le B_{n21}+B_{n22}+B_{n23}+B_{n24},$$

$$\text{with}\ \ B_{n21}=\frac{1}{\Pi_{k_n}}\sum_{u\in\mathbb{T}_{k_n}}\left|\sqrt{n}\left[\Phi\left(\frac{t_n-S_u}{\sqrt{s_n^2-s_{k_n}^2}}\right)-\Phi(t)\right]+(\mathbb{E}\sigma_0^2)^{-\frac{1}{2}}\phi(t)S_u\right|\mathbf{1}_{\{|S_u|\le k_n\}},$$

$$\begin{aligned}
B_{n22} &= \frac{\sqrt{n}}{\Pi_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \left| \Phi\left(\frac{t_n - S_u}{\sqrt{s_n^2 - s_{k_n}^2}} \right) - \Phi(t) \right| \mathbf{1}_{\{|S_u| > k_n\}}, \\
B_{n23} &= (\mathbb{E}\sigma_0^2)^{-\frac{1}{2}} \phi(t) |V_{k_n} - V|, \\
B_{n24} &= (\mathbb{E}\sigma_0^2)^{-\frac{1}{2}} \phi(t) \frac{1}{\Pi_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} |S_u| \mathbf{1}_{\{|S_u| > k_n\}}.
\end{aligned}$$

By the choice of $\beta < 1/2$ and the conditions on the environment, we have that

$$\widetilde{\epsilon}_n = \sup_{|y| \le k_n} \left| \sqrt{n} \left[\Phi\Big((s_{n-1}^2 - s_{k_n}^2)^{-\frac{1}{2}} (t_n - y) \Big) - \Phi(t) \right] + (\mathbb{E}\sigma_0^2)^{-\frac{1}{2}} \phi(t) y \right| \xrightarrow{n \to \infty} 0. \tag{3.24}$$

Hence

$$B_{n21} \le V_{k_n} \widetilde{\epsilon}_n \xrightarrow{n \to \infty} 0. \tag{3.25}$$

Next we go to prove that

$$B_{n23} \xrightarrow{n \to \infty} 0; \qquad B_{n24} \xrightarrow{n \to \infty} 0. \tag{3.26}$$

This will follow from the facts:

$$\frac{1}{\Pi_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} |S_u| \mathbf{1}_{\{|S_u| > k_n\}} \xrightarrow{n \to \infty} 0 \ \text{ a.s.}; \ \ \sqrt{n} \frac{1}{\Pi_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \mathbf{1}_{\{S_u| > k_n\}} \xrightarrow{n \to \infty} 0 \text{ a.s.} \tag{3.27}$$

In order to prove (3.27), we firstly observe that

$$\begin{aligned}
&\mathbb{E} \left(\sum_{n=1}^{\infty} \frac{1}{\Pi_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} |S_u| \mathbf{1}_{\{|S_u| > k_n\}} \right) \\
= &\sum_{n=1}^{\infty} \mathbb{E} |\hat{S}_{k_n}| \mathbf{1}_{\{|\hat{S}_{k_n}| > k_n\}} \le \sum_{n=1}^{\infty} k_n^{1-\eta} \mathbb{E} |\hat{S}_{k_n}|^{\eta} \le \sum_{n=1}^{\infty} k_n^{-\frac{\eta}{2}} \sum_{j=0}^{k_n - 1} \mathbb{E} |\widehat{L}_j|^{\eta} = \sum_{n=1}^{\infty} k_n^{1-\frac{\eta}{2}} \mathbb{E} |\widehat{L}_0|^{\eta}, \\
&\mathbb{E} \left(\sum_{n=1}^{\infty} \sqrt{n} \frac{1}{\Pi_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \mathbf{1}_{\{|S_u| > k_n\}} \right) \\
= &\sum_{n=1}^{\infty} \sqrt{n} \mathbb{E} \mathbf{1}_{\{|\hat{S}_{k_n}| > k_n\}} \le \sum_{n=1}^{\infty} \sqrt{n} k_n^{-\eta} \mathbb{E} |\hat{S}_{k_n}|^{\eta} \le \sum_{n=1}^{\infty} \sqrt{n} k_n^{-\frac{\eta}{2} - 1} \sum_{j=0}^{k_n - 1} \mathbb{E} |\widehat{L}_j|^{\eta} = \sum_{n=1}^{\infty} n^{\frac{1}{2}} k_n^{-\frac{\eta}{2}} \mathbb{E} |\widehat{L}_0|^{\eta}.
\end{aligned}$$

The assumptions on β, k_n and η ensure that the series in the right hand side of the above two expressions are convergent. Then

$$\sum_{n=1}^{\infty} \frac{1}{\Pi_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} |S_u| \mathbf{1}_{\{|S_u| > k_n\}} < \infty, \quad \sum_{n=1}^{\infty} \sqrt{n} \frac{1}{\Pi_{k_n}} \sum_{u \in \mathbb{T}_{k_n}} \mathbf{1}_{\{|S_u| > k_n\}} < \infty \quad \text{a.s.},$$

which deduce (3.27), and consequently, (3.26) is proved.

Due to Lemma 3.1, it is immediate that

$$B_{n24} \xrightarrow{n \to \infty} 0 \quad \text{a.s.} \tag{3.28}$$

From (3.25) (3.26) and (3.28), we derive (3.18).

Now we turn to the proof of (3.19).

According to the hypothesis of Theorem 2.1, it follows from the Birkoff ergodic theorem that

$$\lim_{n \to \infty} \sqrt{n} \frac{M_n^{(3)} - M_{k_n}^{(3)}}{6(s_n^2 - s_{k_n}^2)^{3/2}} = \frac{1}{6} (\mathbb{E}\sigma_0^2)^{-3/2} \mathbb{E}\alpha_0. \tag{3.29}$$

Notice that

$$
\begin{aligned}
&\left|\frac{1}{\Pi_n}\sum_{u\in\mathbb{T}_{k_n}}\left(H\left(\frac{t_n-S_u}{\sqrt{s_n^2-s_{k_n}^2}}\right)-H(t)\right)\right| \\
\le\ &\frac{2}{\Pi_n}\sum_{u\in\mathbb{T}_{k_n}}\mathbf{1}_{\{|S_u|>k_n\}}+\frac{1}{\Pi_n}\sum_{u\in\mathbb{T}_{k_n}}\left|H\left(\frac{t_n-S_u}{\sqrt{s_n^2-s_{k_n}^2}}\right)-H(t)\right|\mathbf{1}_{\{|S_u|\le k_n\}}.
\end{aligned}
$$

The first term in the last expression above tends to 0 a.s. by (3.27), and the second one tends to 0 a.s. because

$$\sup_{|y|\le k_n}\left|H\left(\frac{t_n-y}{\sqrt{s_n^2-s_{k_n}^2}}\right)-H(t)\right|\xrightarrow{n\to\infty}0.$$

Then combining the above facts, (3.19) is proved.

It remains to prove (3.20), which is an immediate consequence of (2.3) and (3.29).

Therefore, the proof of Lemma 3.3 is completed. □

Proof of Lemma 3.4. This lemma follows from the Borel-Cantelli Lemma and the following result given by [15].

Proposition 3.7. *Assume the condition* (2.1). *Then*

$$W-W_n=o(n^{-\lambda})\qquad a.s.$$

□

Now the main theorem follows from the decomposition (3.2) and Lemmas 3.2∼ 3.4.

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