

L^p bounds for the commutators of singular integrals and maximal singular integrals with rough kernels ^{*}

(To appear in *Transactions of the American Mathematical Society*)

Yanping Chen ¹

Department of Applied Mathematics, School of Mathematics and Physics
University of Science and Technology Beijing
Beijing 100083, The People's Republic of China
E-mail: *yanpingch@126.com*

and

Yong Ding

School of Mathematical Sciences
Beijing Normal University
Laboratory of Mathematics and Complex Systems, Ministry of Education
Beijing 100875, The People's Republic of China
E-mail: *dingy@bnu.edu.cn*

ABSTRACT The commutator of convolution type Calderon-Zygmund singular integral operators with rough kernels $p.v. \frac{\Omega(x)}{|x|^n}$ are studied. The authors established the L^p ($1 < p < \infty$) boundedness of the commutators of singular integrals and maximal singular integrals with the kernel condition which is different from the condition $\Omega \in H^1(S^{n-1})$.

MR(2000) Subject Classification: 42B20, 42B25, 42B99

Keywords: Commutator, singular integral, Maximal singular integral, rough kernel, BMO, Bony paraproduct;

^{*}The research was supported by NSF of China (No. 10901017, 11371057), NCET of China (No. NCET-11-0574), the Fundamental Research Funds for the Central Universities (No. 2012CXQT09) and SRFDP of China (No. 20130003110003).

¹Corresponding author.

1 Introduction

The homogeneous singular integral operator T_Ω is defined by

$$T_\Omega f(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y) dy,$$

where $\Omega \in L^1(S^{n-1})$ satisfies the following conditions:

- (a) Ω is homogeneous function of degree zero on $\mathbb{R}^n \setminus \{0\}$, i.e.

$$\Omega(tx) = \Omega(x) \quad \text{for any } t > 0 \text{ and } x \in \mathbb{R}^n \setminus \{0\}. \quad (1.1)$$

- (b) Ω has mean zero on S^{n-1} , the unit sphere in \mathbb{R}^n , i.e.

$$\int_{S^{n-1}} \Omega(x') d\sigma(x') = 0. \quad (1.2)$$

For a function $b \in L_{loc}(\mathbb{R}^n)$, let A be a linear operator on some measurable function space. Then the commutator between A and b is defined by $[b, A]f(x) := b(x)Af(x) - A(bf)(x)$.

In 1965, Calderón [5] defined a commutator for the Hilbert transform H and a Lipschitz function b , which is connected closely the Cauchy integral along Lipschitz curves (see also [6]). Commutators have played an important role in harmonic analysis and PDE, for example in the theory of non-divergent elliptic equations with discontinuous coefficients (see [4, 11, 12, 18]). Moreover, there is also an interesting connection between the nonlinear commutator, considered by Rochberg and Weiss in [29], and Jacobian mapping of vector functions. They have been applied in the study of the nonlinear partial differential equations (see [13, 25]).

In 1976, Coifman, Rochberg and Weiss [14] obtained a characterization of L^p -boundedness of the commutators $[b, R_j]$ generated by the Riesz transforms R_j ($j = 1, \dots, n$) and a BMO function b . As an application of this characterization, a decomposition theorem of the real Hardy space is given in this paper. Moreover, the authors in [14] proved also that if $\Omega \in \text{Lip}(S^{n-1})$, then the commutator $[b, T_\Omega]$ for T_Ω and a BMO function b is bounded on L^p for $1 < p < \infty$, which is defined by

$$[b, T_\Omega]f(x) = p.v. \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} (b(x) - b(y))f(y)dy.$$

In the same paper, Coifman, Rochberg and Weiss [14] outlined a different approach, which is less direct but shows the close relationship between the weighted inequalities of the operator T and the weighted inequalities of the commutator $[b, T]$. In 1993, Alvarez, Bagby, Kurtz and Pérez [2] developed the idea of [14], and established a generalized boundedness criterion for the commutators of linear operators. The result of Alvarez, Bagby, Kurtz and Pérez (see [2, Theorem 2.13]) can be stated as follows.

Theorem A ([2]) *Let $1 < p < \infty$. If a linear operator T is bounded on $L^p(w)$ for all $w \in A_q$, ($1 < q < \infty$), where A_q denote the weight class of Muckenhoupt, then for $b \in BMO$, $\|[b, T]f\|_{L^p} \leq C\|b\|_{BMO}\|f\|_{L^p}$.*

Combining Theorem A with the well-known results by Duoandikoetxea [16] on the weighted L^p boundedness of the rough singular integral T_Ω , we know that if $\Omega \in L^q(S^{n-1})$ for some $q > 1$, then

$[b, T_\Omega]$ is bounded on L^p for $1 < p < \infty$. However, it is not clear up to now whether the operator T_Ω with $\Omega \in L^1 \setminus \bigcup_{q>1} L^q(S^{n-1})$ is bounded on $L^p(w)$ for $1 < p < \infty$ and all $w \in A_r$ ($1 < r < \infty$). Hence, if $\Omega \in L^1 \setminus \bigcup_{q>1} L^q(S^{n-1})$, the L^p boundedness of $[b, T_\Omega]$ can not be deduced from Theorem A.

The purpose of this paper is to give a sufficient condition which contains $\bigcup_{q>1} L^q(S^{n-1})$, such that the commutator of convolution operators are bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$, and this condition was introduced by Grafakos and Stefanov in [23], which is defined by

$$\sup_{\xi \in S^{n-1}} \int_{S^{n-1}} |\Omega(y)| \left(\frac{1}{|\xi \cdot y|} \right)^{1+\alpha} d\sigma(y) < \infty, \quad (1.3)$$

where $\alpha > 0$ is a fixed constant. It is well known that

$$\bigcup_{q>1} L^q(S^{n-1}) \subset L \log^+ L(S^{n-1}) \subset H^1(S^{n-1}).$$

Let $F_\alpha(S^{n-1})$ denote the space of all integrable functions Ω on S^{n-1} satisfying (1.3). The examples in [23] show that there is the following relationship between $F_\alpha(S^{n-1})$ and $H^1(S^{n-1})$ (the Hardy space on S^{n-1})

$$\bigcup_{q>1} L^q(S^{n-1}) \subset \bigcap_{\alpha>0} F_\alpha(S^{n-1}) \not\subset H^1(S^{n-1}) \not\subset \bigcup_{\alpha>0} F_\alpha(S^{n-1}).$$

The condition (1.3) above have been considered by many authors in the context of rough integral operators. One can consult [1, 7, 8, 9, 10, 17, 24] among numerous references, for its development and applications.

Now let us formulate our main results as follows.

Theorem 1. *Let Ω be a function in $L^1(S^{n-1})$ satisfying (1.1) and (1.2). If $\Omega \in F_\alpha(S^{n-1})$ for some $\alpha > 1$, then $[b, T_\Omega]$ extends to a bounded operator from L^p into itself for $\frac{\alpha+1}{\alpha} < p < \alpha + 1$.*

Corollary 1. *Let Ω be a function in $L^1(S^{n-1})$ satisfying (1.1) and (1.2). If $\Omega \in \bigcap_{\alpha>1} F_\alpha(S^{n-1})$, then $[b, T_\Omega]$ extends to a bounded operator from L^p into itself for $1 < p < \infty$.*

The proof of this result is in Section 4. In the proof of Theorem 1, we have used Littlewood-Paley decomposition and interpolation theorem argument to prove L^p ($1 < p < \infty$) norm inequalities for rough commutator $[b, T_\Omega]$. These techniques have been used to prove the L^p ($1 < p < \infty$) norm inequalities for rough singular integrals in [23] or [15]. They are very similar in spirit, though not in detail. In the following, we will point out the difference in the methods used to prove L^p ($1 < p < \infty$) norm inequalities for rough commutators and rough singular integrals.

Let T be a linear operator, we may decompose $T = \sum_{l \in \mathbb{Z}} T_l$ by using the properties of Littlewood-Paley functions and Fourier transform, reduce T to a sequence of composition operators $\{T_l\}_{l \in \mathbb{Z}}$. Hence, to get the L^p ($1 < p < \infty$) norm of T , it suffices to establish the delicate L^p ($1 < p < \infty$) norm of each T_l with a summation convergence factor, which can be obtained by interpolating between the delicate L^2 norm of T_l , which has a summation convergent factor, and the L^q ($1 < q < \infty$) norm of T_l , for each $l \in \mathbb{Z}$.

Let T be a rough singular integral. The delicate L^2 norm of each T_l can be obtained by using Fourier transform, the Plancherel theorem and the Littlewood-Paley theory. The L^q ($1 < q < \infty$) norm of each

T_l can be obtained by the method of rotations, the L^q ($1 < q < \infty$) bounds of the one dimensional case of Hardy-Littlewood operator and the Littlewood-Paley theory.

On the other hand, if T is a rough commutator of singular integral, the delicate L^2 norm of each T_l can be obtained by using the L^2 norm of the commutators of Littlewood-Paley operators (see Lemma 3.3) and Lemma 3.4 in Section 3. With these techniques and lemmas, G. Hu [26] obtained the result in Theorem 1 for $p = 2$. Therefore, it reduces the L^p ($1 < p < \infty$) norm of T to the L^q ($1 < q < \infty$) norm of T_l for each $l \in \mathbb{Z}$. Unfortunately, since each T_l is generated by a BMO function and a composition operator, the method of rotations, which deals with the same problem in rough singular integrals, fails to treat this problem directly. Hence we need to look for a new idea. We find the Bony paraproduct is the key technique to resolve the problem. In particular, it is worth to point out that main method used in this paper gives indeed a new application of Bony paraproduct. It is well known that the Bony paraproduct is an important tool in PDE. However, the idea presented in this paper shows that the Bony paraproduct is a powerful tool also for handling the integral operators with rough kernels in harmonic analysis.

It is well known that maximal singular integral operators T_Ω^* play a key role in studying the almost everywhere convergence of the singular integral operators. The mapping properties of the maximal singular integrals with convolution kernels have been extensively studied (see [15, 23, 30], for example). Therefore, another aim of this paper is to give the $L^p(\mathbb{R}^n)$ boundedness of the maximal commutator $[b, T_\Omega^*]$ associated to the singular integral T_Ω , which is defined by

$$[b, T_\Omega^*]f(x) = \sup_{j \in \mathbb{Z}} \left| \int_{|x-y| > 2^j} \frac{\Omega(x-y)}{|x-y|^n} (b(x) - b(y)) f(y) dy \right|.$$

The following theorem is another main result given in this paper:

Theorem 2. *Let Ω be a function in $L^1(S^{n-1})$ satisfying (1.1) and (1.2). If $\Omega \in F_\alpha(S^{n-1})$ for some $\alpha > 2$, then $[b, T_\Omega^*]$ extends to a bounded operator from L^p into itself for $\frac{\alpha}{\alpha-1} < p < \alpha$.*

Corollary 2. *Let Ω be a function in $L^1(S^{n-1})$ satisfying (1.1) and (1.2). If $\Omega \in \bigcap_{\alpha > 2} F_\alpha(S^{n-1})$, then $[b, T_\Omega^*]$ extends to a bounded operator from L^p into itself for $1 < p < \infty$.*

One will see that the maximal commutator $[b, T_\Omega^*]$ can be controlled pointwise by some composition operators of T_Ω , M , M_Ω and their commutators $[b, T_\Omega]$, $[b, M]$ and $[b, M_\Omega]$, where M is the standard Hardy-Littlewood maximal operator, M_Ω denotes the maximal operator with rough kernel, which is defined by

$$M_\Omega f(x) = \sup_{j \in \mathbb{Z}} \left| \int_{2^j < |x-y| \leq 2^{j+1}} \frac{\Omega(x-y)}{|x-y|^n} f(y) dy \right|.$$

The corresponding commutators $[b, M]$ and $[b, M_\Omega]$ are defined by

$$[b, M]f(x) = \sup_{r > 0} \frac{1}{r^n} \int_{|x-y| < r} |b(x) - b(y)| |f(y)| dy$$

and

$$[b, M_\Omega]f(x) = \sup_{j \in \mathbb{Z}} \left| \int_{2^j < |x-y| < 2^{j+1}} (b(x) - b(y)) \frac{\Omega(x-y)}{|x-y|^n} f(y) dy \right|.$$

We give the following $L^p(\mathbb{R}^n)$ boundedness of the commutators $[b, M_\Omega]$:

Theorem 3. *Let Ω be a function in $L^1(S^{n-1})$ satisfying (1.1). If $\Omega \in F_\alpha(S^{n-1})$ for some $\alpha > 1$, then $[b, M_\Omega]$ extends to a bounded operator from L^p into itself for $\frac{\alpha+1}{\alpha} < p < \alpha + 1$.*

Corollary 3. *Let Ω be a function in $L^1(S^{n-1})$ satisfying (1.1). If $\Omega \in \bigcap_{\alpha>1} F_\alpha(S^{n-1})$, then $[b, M_\Omega]$ extends to a bounded operator from L^p into itself for $1 < p < \infty$.*

Theorem 3 is actually a direct consequence of the $L^p(\mathbb{R}^n)$ boundedness of the commutator formed by a class of Littlewood-Paley square operator with rough kernel and a BMO function. In fact, if $\tilde{\Omega} = \Omega - \frac{A}{|S^{n-1}|}$ with $A = \int_{S^{n-1}} \Omega(x') d\sigma(x')$, then $\tilde{\Omega}$ satisfies (1.2). It is easy to check that

$$\begin{aligned} [b, M_\Omega]f(x) &\leq \sup_{j \in \mathbb{Z}} \left| \int_{2^j < |x-y| < 2^{j+1}} (b(x) - b(y)) \frac{\tilde{\Omega}(x-y)}{|x-y|^n} f(y) dy \right| + C[b, M]f(x) \\ &\leq C([b, g_\Omega]f(x) + [b, M]f(x)), \end{aligned} \quad (1.4)$$

where g_Ω and $[b, g_\Omega]$ denote the Littlewood-Paley square operator and its commutator, which are defined respectively by

$$g_\Omega f(x) = \left(\sum_{j \in \mathbb{Z}} \left| \int_{2^j < |x-y| \leq 2^{j+1}} \frac{\Omega(x-y)}{|x-y|^n} f(y) dy \right|^2 \right)^{1/2}$$

and

$$[b, g_\Omega]f(x) = \left(\sum_{j \in \mathbb{Z}} \left| \int_{2^j < |x-y| < 2^{j+1}} (b(x) - b(y)) \frac{\Omega(x-y)}{|x-y|^n} f(y) dy \right|^2 \right)^{1/2}.$$

Thus, (1.4) shows that Theorem 3 will follow from the $L^p(\mathbb{R}^n)$ boundedness of the commutators $[b, g_\Omega]$ and $[b, M]$. Since the $L^p(\mathbb{R}^n)$ boundedness of the later is well known (see [21]), hence, we need only give the $L^p(\mathbb{R}^n)$ boundedness of the commutator $[b, g_\Omega]$ which can be stated as follows.

Theorem 4. *Let Ω be a function in $L^1(S^{n-1})$ satisfying (1.1) and (1.2). If $\Omega \in F_\alpha(S^{n-1})$ for some $\alpha > 1$, then $[b, g_\Omega]$ extends to a bounded operator from L^p into itself for $\frac{\alpha+1}{\alpha} < p < \alpha + 1$.*

Corollary 4. *Let Ω be a function in $L^1(S^{n-1})$ satisfying (1.1) and (1.2). If $\Omega \in \bigcap_{\alpha>1} F_\alpha(S^{n-1})$, then $[b, g_\Omega]$ extends to a bounded operator from L^p into itself for $1 < p < \infty$.*

In fact, Theorem 4 is a corollary of Theorem 1. Write $T_\Omega f(x) = \sum_{j \in \mathbb{Z}} K_j * f(x)$, where $K_j(x) = \frac{\Omega(x)}{|x|^n} \chi_{\{2^j < |x| \leq 2^{j+1}\}}$. Define $T_j f(x) = K_j * f(x)$, then $[b, T_\Omega]f(x) = \sum_{j \in \mathbb{Z}} [b, T_j]f(x)$, and $[b, g_\Omega]f(x) = \left(\sum_{j \in \mathbb{Z}} |[b, T_j]f(x)|^2 \right)^{1/2}$. Then we get the L^p boundedness of $[b, g_\Omega]$ by using Theorem 1, Rademacher function and Khintchine's inequalities.

This paper is organized as follows. First, in Section 2, we give some important notations and tools, which will be used in the proofs of the main results. In Section 3, we give some lemmas which will be used in the proofs of the main results. In Section 4, we prove Theorem 1 by applying the lemmas in Section 3. Finally, we prove Theorem 2 by applying Theorem 3 and Theorem 4 in Section 5. Throughout this paper, the letter “ C ” will stand for a positive constant which is independent of the essential variables and not necessarily the same one in each occurrence.

2 Notations and preliminaries

Let us begin by giving some notations and important tools, which will be used in the proofs of our main results.

1. Schwartz class and Fourier transform. Denote by $\mathcal{S}(\mathbb{R}^n)$ and $\mathcal{S}'(\mathbb{R}^n)$ the Schwartz class and the space of tempered distributions, respectively. The notations “ $\widehat{\cdot}$ ” and “ \vee ” denote the Fourier transform and the inverse Fourier transform, respectively.

2. Smooth decomposition of identity and multipliers. Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ be a radial function satisfying $0 \leq \varphi \leq 1$ with its support is in the unit ball and $\varphi(\xi) = 1$ for $|\xi| \leq \frac{1}{2}$. The function $\psi(\xi) = \varphi(\frac{\xi}{2}) - \varphi(\xi) \in \mathcal{S}(\mathbb{R}^n)$ supported by $\{\frac{1}{2} \leq |\xi| \leq 2\}$ and satisfies the identity $\sum_{j \in \mathbb{Z}} \psi(2^{-j}\xi) = 1$, for $\xi \neq 0$.

For $j \in \mathbb{Z}$, denote by Δ_j and G_j the convolution operators whose the symbols are $\psi(2^{-j}\xi)$ and $\varphi(2^{-j}\xi)$, respectively. That is, Δ_j and G_j are defined by $\widehat{\Delta_j f}(\xi) = \psi(2^{-j}\xi)\widehat{f}(\xi)$ and $\widehat{G_j f}(\xi) = \varphi(2^{-j}\xi)\widehat{f}(\xi)$. By the Littlewood-Paley theory, for $1 < p < \infty$ and $\{f_j\} \in L^p(l^2)$, the following vector-value inequality holds (see [22, p.343])

$$\left\| \left(\sum_{j \in \mathbb{Z}} |\Delta_{j+k} f_j|^2 \right)^{1/2} \right\|_{L^p} \leq C \left\| \left(\sum_{j \in \mathbb{Z}} |f_j|^2 \right)^{1/2} \right\|_{L^p}, \quad \text{for } k \in [-10, 10]. \quad (2.1)$$

3. Homogeneous Triebel-Lizorkin space $\dot{F}_p^{s,q}(\mathbb{R}^n)$ and Besov space $\dot{B}_p^{s,q}(\mathbb{R}^n)$. For $0 < p, q \leq \infty$ ($p \neq \infty$) and $s \in \mathbb{R}$, the homogeneous Triebel-Lizorkin space $\dot{F}_p^{s,q}(\mathbb{R}^n)$ is defined by

$$\dot{F}_p^{s,q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{F}_p^{s,q}} = \left\| \left(\sum_{j \in \mathbb{Z}} 2^{-jsq} |\Delta_j f|^q \right)^{1/q} \right\|_{L^p} < \infty \right\}$$

and the homogeneous Besov space $\dot{B}_p^{s,q}(\mathbb{R}^n)$ is defined by

$$\dot{B}_p^{s,q}(\mathbb{R}^n) = \left\{ f \in \mathcal{S}'(\mathbb{R}^n) : \|f\|_{\dot{B}_p^{s,q}} = \left(\sum_{j \in \mathbb{Z}} 2^{-jsq} \|\Delta_j f\|_{L^p}^q \right)^{1/q} < \infty \right\},$$

where $\mathcal{S}'(\mathbb{R}^n)$ denotes the tempered distribution class on \mathbb{R}^n .

4. Sequence Carleson measures. A sequence of positive Borel measures $\{v_j\}_{j \in \mathbb{Z}}$ is called a sequence Carleson measures in $\mathbb{R}^n \times \mathbb{Z}$ if there exists a positive constant $C > 0$ such that $\sum_{j \geq k} v_j(B) \leq C|B|$ for all $k \in \mathbb{Z}$ and all Euclidean balls B with radius 2^{-k} , where $|B|$ is the Lebesgue measure of B . The norm of the sequence Carleson measures $v = \{v_j\}_{j \in \mathbb{Z}}$ is given by

$$\|v\| = \sup \left\{ \frac{1}{|B|} \sum_{j \geq k} v_j(B) \right\},$$

where the supremum is taken over all $k \in \mathbb{Z}$ and all balls B with radius 2^{-k} .

5. Homogeneous BMO-Triebel-Lizorkin space. For $s \in \mathbb{R}$ and $1 \leq q < +\infty$, the homogeneous BMO-Triebel-Lizorkin space $\dot{F}_\infty^{s,q}$ is the space of all distributions b for which the sequence $\{2^{sjq} |\Delta_j(b)(x)|^q dx\}_{j \in \mathbb{Z}}$ is a Carleson measure (see [19]). The norm of b in $\dot{F}_\infty^{s,q}$ is given by

$$\|b\|_{\dot{F}_\infty^{s,q}} = \sup \left[\frac{1}{|B|} \sum_{j \geq k} \int_B 2^{sjq} |\Delta_j(b)(x)|^q dx \right]^{\frac{1}{q}}$$

where the supremum is taken over all $k \in \mathbb{Z}$ and all balls B with radius 2^{-k} . For $q = +\infty$, we set $\dot{F}_\infty^{s,\infty} = \dot{B}_\infty^{s,\infty}$. Moreover, $\dot{F}_\infty^{0,2} = BMO$ (see [19, 20]).

6. Bony paraproduct and Bony decomposition. The paraproduct of Bony [3] between two functions f, g is defined by

$$\pi_f(g) = \sum_{j \in \mathbb{Z}} (\Delta_j f)(G_{j-3}g).$$

At least formally, we have the following Bony decomposition

$$fg = \pi_f(g) + \pi_g(f) + R(f, g) \quad \text{with} \quad R(f, g) = \sum_{i \in \mathbb{Z}} \sum_{|k-i| \leq 2} (\Delta_i f)(\Delta_k g). \quad (2.2)$$

3 Lemmas

We first give some lemmas, which will be used in the proof of Theorem 1 and Theorem 2.

Riesz potential and its inverse. For $0 < \tau < n$, the Riesz potential I_τ of order τ is defined on $\mathcal{S}'(\mathbb{R}^n)$ by setting $\widehat{I_\tau f}(\xi) = |\xi|^{-\tau} \widehat{f}(\xi)$. Another expression of I_τ is

$$I_\tau f(x) = \gamma(\tau) \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\tau}} dy,$$

where $\gamma(\tau) = 2^{-\tau} \pi^{-n/2} \Gamma(\frac{n-\tau}{2}) / \Gamma(\frac{\tau}{2})$. Moreover, for $0 < \tau < n$, the “inverse operator” I_τ^{-1} of I_τ is defined by $\widehat{I_\tau^{-1} f}(\xi) = |\xi|^\tau \widehat{f}(\xi)$, where \wedge denotes the Fourier transform.

With the notations above, we show the following two facts:

Lemma 3.1 For $0 < \tau < 1/2$, we have

$$\gamma(\tau) \leq C\tau, \quad (3.1)$$

where C is independent of τ .

Proof. Applying the Stirling’s formula, we have

$$\sqrt{2\pi} x^{x-1/2} e^{-x} \leq \Gamma(x) \leq 2\sqrt{2\pi} x^{x-1/2} e^{-x} \quad \text{for } x > 1.$$

Thus, by the equation $s\Gamma(s) = \Gamma(s+1)$ for $s > 0$, we get

$$\Gamma\left(\frac{n-\tau}{2}\right) = \frac{2}{n-\tau} \Gamma\left(\frac{n-\tau}{2} + 1\right) \leq 2\sqrt{2\pi} \left(\frac{n-\tau}{2} + 1\right)^{\left(\frac{n-\tau}{2} + \frac{1}{2}\right)} e^{-\left(\frac{n-\tau}{2}\right)-1} \cdot \frac{2}{n-\tau} \leq C \quad (3.2)$$

and

$$\Gamma\left(\frac{\tau}{2}\right) = \frac{2}{\tau} \Gamma\left(\frac{\tau}{2} + 1\right) \geq \sqrt{2\pi} \left(\frac{\tau}{2} + 1\right)^{\left(\frac{\tau}{2} + \frac{1}{2}\right)} e^{-\frac{\tau}{2}-1} \cdot \frac{2}{\tau} \geq C/\tau. \quad (3.3)$$

Hence, (3.1) follows from (3.2) and (3.3). Obviously, the constant C in (3.1) is independent of τ .

Lemma 3.2 For the multiplier G_k ($k \in \mathbb{Z}$), $b \in BMO(\mathbb{R}^n)$, and any fixed $0 < \tau < 1/2$, we have

$$|G_k b(x) - G_k b(y)| \leq C \frac{2^{k\tau}}{\tau} |x-y|^\tau \|b\|_{BMO}, \quad (3.4)$$

where C is independent of k and τ .

Proof. Note that $I_\tau(I_\tau^{-1}f) = f$, we have

$$G_k b(x) = \gamma(\tau) \int_{\mathbb{R}^n} \frac{I_\tau^{-1}(G_k b)(z)}{|x-z|^{n-\tau}} dz.$$

Hence

$$\begin{aligned} |G_k b(x) - G_k b(y)| &= \left| \gamma(\tau) \int_{\mathbb{R}^n} I_\tau^{-1}(G_k b)(z) \left(\frac{1}{|x-z|^{n-\tau}} - \frac{1}{|y-z|^{n-\tau}} \right) dz \right| \\ &\leq \gamma(\tau) \|I_\tau^{-1}(G_k b)\|_{L^\infty} \int_{\mathbb{R}^n} \left| \frac{1}{|x-z|^{n-\tau}} - \frac{1}{|y-z|^{n-\tau}} \right| dz \\ &= \gamma(\tau) \|I_\tau^{-1}(G_k b)\|_{L^\infty} \int_{\mathbb{R}^n} \left| \frac{1}{|x-y+z|^{n-\tau}} - \frac{1}{|z|^{n-\tau}} \right| dz. \end{aligned} \quad (3.5)$$

We first show that

$$\left\| \frac{1}{|x-y+z|^{n-\tau}} - \frac{1}{|z|^{n-\tau}} \right\|_{L^1} \leq C\tau^{-1}|x-y|^\tau. \quad (3.6)$$

In fact,

$$\begin{aligned} &\int_{\mathbb{R}^n} \left| \frac{1}{|x-y+z|^{n-\tau}} - \frac{1}{|z|^{n-\tau}} \right| dz \\ &= \int_{|z| \leq 2|x-y|} \left| \frac{1}{|x-y+z|^{n-\tau}} - \frac{1}{|z|^{n-\tau}} \right| dz + \int_{|z| > 2|x-y|} \left| \frac{1}{|x-y+z|^{n-\tau}} - \frac{1}{|z|^{n-\tau}} \right| dz \\ &\leq \int_{|z| \leq 3|x-y|} \frac{1}{|z|^{n-\tau}} dz + \int_{|z| \leq 2|x-y|} \frac{1}{|z|^{n-\tau}} dz + C \int_{|z| > 2|x-y|} \frac{|x-y|}{|z|^{n-\tau+1}} dz \\ &\leq C \frac{|x-y|^\tau}{\tau}, \end{aligned}$$

where C is independent of τ . By (3.5), (3.6) and (3.1), we get

$$|G_k b(x) - G_k b(y)| \leq C|x-y|^\tau \|I_\tau^{-1}(G_k b)\|_{L^\infty}, \quad (3.7)$$

where C is independent of τ . We now estimate $\|I_\tau^{-1}(G_k b)\|_{L^\infty}$. Since $G_k \Delta_u b = 0$ for $u \geq k+1$, we have

$$\|I_\tau^{-1}(G_k b)\|_{L^\infty} = \left\| I_\tau^{-1} G_k \left(\sum_{u \in \mathbb{Z}} \Delta_u b \right) \right\|_{L^\infty} \leq \sum_{u \leq k+1} \|G_k(I_\tau^{-1} \Delta_u b)\|_{L^\infty} \leq \sum_{u \leq k+1} \|I_\tau^{-1} \Delta_u b\|_{L^\infty}. \quad (3.8)$$

Taking a radial function $\tilde{\psi} \in \mathcal{S}(\mathbb{R}^n)$ such that $\text{supp}(\tilde{\psi}) \subset \{1/4 \leq |x| \leq 4\}$ and $\tilde{\psi} = 1$ in $\{1/2 \leq |x| \leq 2\}$.

Then we have

$$\widehat{I_\tau^{-1} \Delta_u b}(\xi) = 2^{u\tau} \tilde{\psi}(2^{-u}\xi) |2^{-u}\xi|^\tau \widehat{\Delta_u b}(\xi).$$

Set a function h by $\widehat{h}(\xi) = \tilde{\psi}(\xi) |\xi|^\tau$. Then

$$I_\tau^{-1} \Delta_u b(x) = 2^{u\tau} \int_{\mathbb{R}^n} 2^{un} h(2^u(x-y)) \Delta_u b(y) dy.$$

So we have

$$\|I_\tau^{-1} \Delta_u b\|_{L^\infty} \leq 2^{u\tau} \|2^{un} h(2^u \cdot)\|_{L^1} \|\Delta_u b\|_{L^\infty} = 2^{u\tau} \|h\|_{L^1} \|\Delta_u b\|_{L^\infty}.$$

Thus, if there exists a constant $C > 0$, independent of τ , such that

$$\|h\|_{L^1} \leq C, \quad (3.9)$$

then by (3.7)-(3.8), we have

$$\begin{aligned} |G_k b(x) - G_k b(y)| &\leq C|x-y|^\tau \sum_{u \leq k+1} 2^{u\tau} \|\Delta_u b\|_{L^\infty} \\ &\leq C|x-y|^\tau 2^{k\tau} \sup_{u \in \mathbb{Z}} \|\Delta_u b\|_{L^\infty} \sum_{u \leq k+1} 2^{(u-k)\tau}. \end{aligned}$$

Since for some $0 < \tau < 1$,

$$\sum_{u \leq k+1} 2^{(u-k)\tau} = \sum_{j=-1}^{\infty} 2^{-j\tau} = \frac{2^\tau}{1-2^{-\tau}} = \frac{2^{2\tau}}{2^\tau-1} = \frac{2^{2\tau}}{\tau 2^{\theta\tau}} < \frac{C}{\tau}, \quad \text{for } 0 < \tau < 1/2,$$

where C is independent of τ . Using the fact (see [22, p.615])

$$\sup_{u \in \mathbb{Z}} \|\Delta_u b\|_{L^\infty} \leq C_n \|b\|_{BMO}, \quad (3.10)$$

we have

$$|G_k b(x) - G_k b(y)| \leq C \frac{|x-y|^\tau 2^{k\tau}}{\tau} \|b\|_{BMO},$$

where C is independent of k and τ . Thus, to finish the proof of Lemma 3.2, it remains to show (3.9). In fact,

$$\|h\|_{L^1} = \int_{|x| < 1} |h(x)| dx + \int_{|x| \geq 1} |h(x)| dx \leq C_n (\|h\|_{L^2} + \| |\cdot|^n h(\cdot) \|_{L^2}) := C_n (I_1 + I_2).$$

Since $\text{supp}(\tilde{\psi}) \subset \{1/4 \leq |\xi| \leq 4\}$ and $0 < \delta < 1/2$, we get

$$I_1 = \|\tilde{\psi}(\xi)|\xi|^\tau\|_{L^2} \leq C,$$

where C is independent of τ . Thus, to get (3.9), we need only verify that $I_2 \leq C$. To do this, let us recall some notations about the multi-index. For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$, denote $\partial^\alpha f = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n} f$, $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ for $x \in \mathbb{R}^n$. By [22, p.425], we know that

$$(1 + |\xi|^2)^{n/2} = \sum_{|\alpha| \leq n} \frac{n!}{\alpha_1! \dots \alpha_n!} \xi^\alpha \frac{\xi^\alpha}{(1 + |\xi|^2)^{n/2}}$$

and the function $m_\alpha(\xi) = \frac{\xi^\alpha}{(1 + |\xi|^2)^{n/2}}$ is an L^p ($1 < p < \infty$) multiplier whenever $|\alpha| \leq n$. Hence

$$((1 + |\xi|^2)^{n/2} h(\xi))^\vee = \sum_{|\alpha| \leq n} C_{\alpha,n} (m_\alpha(\xi) \xi^\alpha h(\xi))^\vee = C \sum_{|\alpha| \leq n} C_{\alpha,n} (m_\alpha(\xi) \widehat{\partial^\alpha \tilde{h}}(\xi))^\vee,$$

where \vee denote the inverse Fourier transform. Applying the equation above, we get

$$\begin{aligned} I_2 &\leq C \|(1 + |\xi|^2)^{n/2} h(\xi)\|_{L^2} = \|((1 + |\xi|^2)^{n/2} h(\xi))^\vee\|_{L^2} \\ &\leq C \sum_{|\alpha| \leq n} C_{\alpha,n} \|\widehat{\partial^\alpha \tilde{h}}\|_{L^2} \\ &= C \sum_{|\alpha| \leq n} C_{\alpha,n} \|\partial^\alpha \tilde{h}\|_{L^2} = C \sum_{|\alpha| \leq n} C_{\alpha,n} \|\partial^\alpha (\tilde{\psi}(\xi)|\xi|^\tau)\|_{L^2}. \end{aligned}$$

Notice that

$$\partial^\alpha (\tilde{\psi}(\xi)|\xi|^\tau) = \sum_{\beta \leq \alpha} C_{\alpha_1}^{\beta_1} \dots C_{\alpha_n}^{\beta_n} (\partial^\beta \tilde{\psi}(\xi)) (\partial^{\alpha-\beta} (|\xi|^\tau)), \quad (3.11)$$

where the sum in (3.11) is taken over all multi-indices β with $0 \leq \beta_j \leq \alpha_j$ for all $1 \leq j \leq n$. Trivial computations show that there exists $C > 0$, independent of τ , such that $|\partial^{\alpha-\beta} (|\xi|^\tau)| \leq C$ for $1/4 < |\xi| < 4$

and $0 < \tau < 1/2$. Further, by $\tilde{\psi} \in C_0^\infty(\mathbb{R}^n)$, then $|\partial^\beta \tilde{\psi}(\xi)| \leq C$. So we get $|\partial^\alpha(\psi(\xi)|\xi|^\tau)| \leq C$. From this we get

$$I_2 \leq \sum_{|\alpha| \leq n} C_{\alpha,n} \|\partial^\alpha(\psi(\xi)|\xi|^\tau)\|_{L^\infty} \left(\int_{1/4 \leq |\xi| \leq 4} d\xi \right)^{1/2} \leq C,$$

where C is dependent only on n , but independent of τ . This completes the estimate of (3.9) and Lemma 3.2 follows.

Lemma 3.3 (see [27]). *Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be a radial function such that $\text{supp } \phi \subset \{1/2 \leq |\xi| \leq 2\}$ and $\sum_{l \in \mathbb{Z}} \phi^3(2^{-l}\xi) = 1$ for $|\xi| \neq 0$. Define the multiplier operator S_l by $\widehat{S_l f}(\xi) = \phi(2^{-l}\xi)\widehat{f}(\xi)$, S_l^2 by $S_l^2 f = S_l(S_l f)$. For $b \in BMO(\mathbb{R}^n)$, denote by $[b, S_l]$ (respectively, $[b, S_l^2]$) the commutator of S_l (respectively, S_l^2). Then for $1 < p < \infty$ and $f \in L^p(\mathbb{R}^n)$, we have*

- (i) $\left\| \left(\sum_{l \in \mathbb{Z}} |[b, S_l](f)|^2 \right)^{1/2} \right\|_{L^p} \leq C(n, p) \|b\|_{BMO} \|f\|_{L^p};$
- (ii) $\left\| \left(\sum_{l \in \mathbb{Z}} |[b, S_l^2](f)|^2 \right)^{1/2} \right\|_{L^p} \leq C(n, p) \|b\|_{BMO} \|f\|_{L^p};$
- (iii) $\left\| \sum_{l \in \mathbb{Z}} [b, S_l](f_l) \right\|_{L^p} \leq C(n, p) \|b\|_{BMO} \left\| \left(\sum_{l \in \mathbb{Z}} |f_l|^2 \right)^{1/2} \right\|_{L^p}, \{f_l\} \in L^p(l^2).$

Lemma 3.4 (see [26]). *Let $m_\sigma \in C_0^\infty(\mathbb{R}^n)$ ($0 < \sigma < \infty$) be a family of multipliers such that $\text{supp}(m_\sigma) \subset \{|\xi| \leq 2\sigma\}$, and for some constants C , $0 < A \leq 1/2$, and $\alpha > 0$,*

$$\|m_\sigma\|_{L^\infty} \leq C \min\{A\sigma, \log^{-\alpha-1}(2 + \sigma)\}, \quad \|\nabla m_\sigma\|_{L^\infty} \leq C.$$

Let T_σ be the multiplier operator defined by

$$\widehat{T_\sigma f}(\xi) = m_\sigma(\xi)\widehat{f}(\xi).$$

For $b \in BMO$, denote by $[b, T_\sigma]$ the commutator of T_σ . Then for any fixed $0 < v < 1$, there exists a positive constant $C = C(n, v)$ such that

$$\begin{aligned} \|[b, T_\sigma]f\|_{L^2} &\leq C(A\sigma)^v \log(1/A) \|b\|_{BMO} \|f\|_{L^2}, \text{ if } \sigma < 10/\sqrt{A}; \\ \|[b, T_\sigma]f\|_{L^2} &\leq C \log^{-(\alpha+1)v+1}(2 + \sigma) \|b\|_{BMO} \|f\|_{L^2}, \text{ if } \sigma \geq 10/\sqrt{A}; \end{aligned}$$

Similar to the proof of Lemma 3.4, it is easy to get

Lemma 3.5 *Let $m_\sigma \in C_0^\infty(\mathbb{R}^n)$ ($0 < \sigma < \infty$) be a family of multipliers such that $\text{supp}(m_\sigma) \subset \{|\xi| \leq 2\sigma\}$, and for some constants C , $0 < A \leq 1/2$, and $\alpha > 0$, $j \in \mathbb{N}$*

$$\|m_\sigma\|_{L^\infty} \leq C \min\{A2^{-j}\sigma, \log^{-\alpha-1}(2 + 2^j\sigma)\}, \quad \|\nabla m_\sigma\|_{L^\infty} \leq C2^j.$$

Let T_σ be the multiplier operator defined by

$$\widehat{T_\sigma f}(\xi) = m_\sigma(\xi)\widehat{f}(\xi).$$

For $b \in BMO$, denote by $[b, T_\sigma]$ the commutator of T_σ . Then for any fixed $0 < v < 1$, there exists a positive constant $C = C(n, v)$, $0 < \beta < 1$ such that

$$\|[b, T_\sigma]f\|_{L^2} \leq C2^{-\beta j} (A\sigma)^v \log(1/A) \|b\|_{BMO} \|f\|_{L^2}, \text{ if } \sigma < 10/\sqrt{A};$$

$$\|[b, T_\sigma]f\|_{L^2} \leq C \log^{-(\alpha+1)v+1}(2+2^j\sigma) \|b\|_{BMO} \|f\|_{L^2}, \text{ if } \sigma \geq 10/\sqrt{A};$$

Lemma 3.6. For any $j \in \mathbb{Z}$, let $K_j(x) = \frac{\Omega(x)}{|x|^n} \chi_{\{2^j < |x| \leq 2^{j+1}\}}(x)$. Suppose $\Omega \in L^1(S^{n-1})$ satisfying (1.1). Then for $1 < p < \infty$, the following vector valued inequality

$$\left\| \left(\sum_{j \in \mathbb{Z}} \|K_j * |f_j|^2\| \right)^{1/2} \right\|_{L^p} \leq C_p \|\Omega\|_{L^1} \left\| \left(\sum_{j \in \mathbb{Z}} |f_j|^2 \right)^{1/2} \right\|_{L^p}$$

holds for any $\{f_j\}$ in $L^p(l^2)$.

Proof. Note that for $\Omega \in L^1(S^{n-1})$ and any local integrable function f on \mathbb{R}^n , we have

$$\sigma^*(f)(x) := \sup_{j \in \mathbb{Z}} \|K_j * f(x)\| \leq CM_\Omega f(x) \quad \text{for any } x \in \mathbb{R}^n,$$

where

$$M_\Omega f(x) = \sup_{r>0} \frac{1}{r^n} \int_{|x-y|<r} |\Omega(x-y)| |f(y)| dy. \quad (3.12)$$

By the L^q boundedness of M_Ω for all $q > 1$ with $\Omega \in L^1(S^{n-1})$, σ^* is also a bounded operator on $L^q(\mathbb{R}^n)$ for all $q > 1$ with $\Omega \in L^1(S^{n-1})$. Thus, by applying Lemma in [15, p.544], we know that, for $1 < p < \infty$, the vector valued inequality (3.12) holds.

Lemma 3.7. For any $j \in \mathbb{Z}$, define the operator T_j by $T_j f = K_j * f$, where $K_j(x) = \frac{\Omega(x)}{|x|^n} \chi_{\{2^j < |x| \leq 2^{j+1}\}}(x)$. Denote by $[b, S_{l-j} T_j S_{l-j}^2]$ the commutator of $S_{l-j} T_j S_{l-j}^2$. Suppose $\Omega \in L^1(S^{n-1})$ satisfying (1.1). Then for any fixed $0 < \tau < 1/2$, $b \in BMO(\mathbb{R}^n)$, $1 < p < \infty$,

$$\left\| \sum_{j \in \mathbb{Z}} [b, S_{l-j} T_j S_{l-j}^2] f \right\|_{L^p} \leq C \|b\|_{BMO} \max\left\{ \frac{2^{\tau l}}{\tau}, 2 \right\} \|\Omega\|_{L^1} \|f\|_{L^p}, \quad (3.13)$$

where C is independent of τ and l .

Proof. For any $j, l \in \mathbb{Z}$, we may write

$$[b, S_{l-j} T_j S_{l-j}^2] f = [b, S_{l-j}](T_j S_{l-j}^2 f) + S_{l-j} [b, T_j](S_{l-j}^2 f) + S_{l-j} T_j ([b, S_{l-j}^2] f).$$

Thus,

$$\begin{aligned} \left\| \sum_{j \in \mathbb{Z}} [b, S_{l-j} T_j S_{l-j}^2] f \right\|_{L^p} &\leq \left\| \sum_{j \in \mathbb{Z}} [b, S_{l-j}](T_j S_{l-j}^2 f) \right\|_{L^p} + \left\| \sum_{j \in \mathbb{Z}} S_{l-j} T_j ([b, S_{l-j}^2] f) \right\|_{L^p} \\ &\quad + \left\| \sum_{j \in \mathbb{Z}} S_{l-j} [b, T_j](S_{l-j}^2 f) \right\|_{L^p} \\ &:= L_1 + L_2 + L_3. \end{aligned} \quad (3.14)$$

Below we shall estimate L_i for $i = 1, 2, 3$, respectively. For L_1 , by Lemma 3.3 (iii), Lemma 3.6 and the Littlewood-Paley theory, we have

$$\begin{aligned} L_1 &\leq C \|b\|_{BMO} \left\| \left(\sum_{j \in \mathbb{Z}} |T_j S_{l-j}^2 f|^2 \right)^{1/2} \right\|_{L^p} \\ &\leq C \|\Omega\|_{L^1} \|b\|_{BMO} \left\| \left(\sum_{j \in \mathbb{Z}} |S_j^2 f|^2 \right)^{1/2} \right\|_{L^p} \leq C \|\Omega\|_{L^1} \|b\|_{BMO} \|f\|_{L^p}. \end{aligned}$$

Similarly, we have $L_2 \leq C \|\Omega\|_{L^1} \|b\|_{BMO} \|f\|_{L^p}$.

Hence, by (3.14), to show (3.12) it remains to give the estimate of L_3 . We will apply the Bony paraproduct to do this. By (2.2), we have

$$\begin{aligned} [b, T_j]S_{l-j}^2 f(x) &= b(x)(T_j S_{l-j}^2 f)(x) - T_j(b S_{l-j}^2 f)(x) \\ &= [\pi_{(T_j S_{l-j}^2 f)}(b)(x) - T_j(\pi_{(S_{l-j}^2 f)}(b))(x)] + [R(b, T_j S_{l-j}^2 f)(x) - T_j(R(b, S_{l-j}^2 f))(x)] \\ &\quad + [\pi_b(T_j S_{l-j}^2 f)(x) - T_j(\pi_b(S_{l-j}^2 f))(x)]. \end{aligned}$$

Thus

$$\begin{aligned} L_3 &\leq \left\| \sum_{j \in \mathbb{Z}} S_{l-j} [\pi_{(T_j S_{l-j}^2 f)}(b) - T_j(\pi_{(S_{l-j}^2 f)}(b))] \right\|_{L^p} \\ &\quad + \left\| \sum_{j \in \mathbb{Z}} S_{l-j} [R(b, T_j S_{l-j}^2 f) - T_j(R(b, S_{l-j}^2 f))] \right\|_{L^p} \\ &\quad + \left\| \sum_{j \in \mathbb{Z}} S_{l-j} [\pi_b(T_j S_{l-j}^2 f) - T_j(\pi_b(S_{l-j}^2 f))] \right\|_{L^p} \\ &:= M_1 + M_2 + M_3. \end{aligned} \tag{3.15}$$

(a) *The estimate of M_1 .* For M_1 , by $\Delta_i S_{l-j} g = 0$ for $g \in \mathcal{S}'(\mathbb{R}^n)$ when $|i - (l - j)| \geq 3$, we get

$$\begin{aligned} &\pi_{(T_j S_{l-j}^2 f)}(b)(x) - T_j(\pi_{(S_{l-j}^2 f)}(b))(x) \\ &= \sum_{|i-(l-j)| \leq 2} \{(T_j \Delta_i S_{l-j}^2 f)(x)(G_{i-3} b)(x) - T_j[(\Delta_i S_{l-j}^2 f)(G_{i-3} b)](x)\} \\ &= \sum_{|i-(l-j)| \leq 2} [G_{i-3} b, T_j](\Delta_i S_{l-j}^2 f)(x). \end{aligned} \tag{3.16}$$

Note that

$$\begin{aligned} |[G_{i-3} b, T_j](\Delta_i S_{l-j}^2 f)(x)| &= \left| \int_{2^j \leq |x-y| < 2^{j+1}} \frac{\Omega(x-y)}{|x-y|^n} (G_{i-3} b(x) - G_{i-3} b(y)) \Delta_i S_{l-j}^2 f(y) dy \right| \\ &\leq C \int_{2^j \leq |x-y| < 2^{j+1}} \frac{|\Omega(x-y)|}{|x-y|^n} |G_{i-3} b(x) - G_{i-3} b(y)| |\Delta_i S_{l-j}^2 f(y)| dy. \end{aligned} \tag{3.17}$$

By Lemma 3.2, we have

$$\begin{aligned} |[G_{i-3} b, T_j] \Delta_i S_{l-j}^2 f(x)| &\leq C \frac{2^{i\tau}}{\tau} \|b\|_{BMO} \int_{2^j \leq |x-y| < 2^{j+1}} \frac{|\Omega(x-y)|}{|x-y|^n} |x-y|^\tau |\Delta_i S_{l-j}^2 f(y)| dy \\ &\leq C \frac{2^{(i+j)\tau}}{\tau} \|b\|_{BMO} \int_{2^j \leq |x-y| < 2^{j+1}} \frac{|\Omega(x-y)|}{|x-y|^n} |\Delta_i S_{l-j}^2 f(y)| dy \\ &= C \frac{2^{(i+j)\tau}}{\tau} \|b\|_{BMO} T_{|\Omega|,j}(|\Delta_i S_{l-j}^2 f|)(x), \end{aligned} \tag{3.18}$$

where

$$T_{|\Omega|,j} f(x) = \int_{2^j \leq |x-y| < 2^{j+1}} \frac{|\Omega(x-y)|}{|x-y|^n} f(y) dy.$$

Then, by (3.16), (3.18) and applying Lemma 3.6, (2.1) and the Littlewood-Paley theory, we have that, for any fixed $0 < \tau < 1/2$,

$$\begin{aligned} M_1 &\leq C \frac{2^{\tau l}}{\tau} \|b\|_{BMO} \sum_{|k| \leq 2} \left\| \left(\sum_{j \in \mathbb{Z}} |T_{|\Omega|,j}(|\Delta_{l-j+k} S_{l-j}^2 f|)|^2 \right)^{1/2} \right\|_{L^p} \\ &\leq C \frac{2^{\tau l}}{\tau} \|b\|_{BMO} \|\Omega\|_{L^1} \sum_{|k| \leq 2} \left\| \left(\sum_{j \in \mathbb{Z}} |\Delta_{j+k} S_j^2 f|^2 \right)^{1/2} \right\|_{L^p} \\ &\leq C \frac{2^{\tau l}}{\tau} \|b\|_{BMO} \|\Omega\|_{L^1} \left\| \left(\sum_{j \in \mathbb{Z}} |S_j f|^2 \right)^{1/2} \right\|_{L^p} \\ &\leq C \frac{2^{\tau l}}{\tau} \|b\|_{BMO} \|\Omega\|_{L^1} \|f\|_{L^p}, \end{aligned} \tag{3.19}$$

where C is independent of l and τ .

(b) *The estimate of M_2 .* Since $|k| \leq 2$, $\Delta_{i+k}S_{l-j}g = 0$ for $g \in \mathcal{S}'(\mathbb{R}^n)$ when $|i - (l - j)| \geq 8$. Thus

$$\begin{aligned}
& R(b, T_j S_{l-j} f) - T_j(R(b, S_{l-j} f))(x) \\
&= \sum_{i \in \mathbb{Z}} \sum_{|k| \leq 2} (\Delta_i b)(x) (T_j \Delta_{i+k} S_{l-j} f)(x) - T_j \left(\sum_{i \in \mathbb{Z}} \sum_{|k| \leq 2} (\Delta_i b)(\Delta_{i+k} S_{l-j} f) \right)(x) \\
&= \sum_{k=-2}^2 \sum_{|i-(l-j)| \leq 7} \left((\Delta_i b)(x) (T_j \Delta_{i+k} S_{l-j} f)(x) - T_j((\Delta_i b)(\Delta_{i+k} S_{l-j} f))(x) \right) \\
&= \sum_{k=-2}^2 \sum_{|i-(l-j)| \leq 7} [\Delta_i b, T_j](\Delta_{i+k} S_{l-j} f)(x).
\end{aligned}$$

By the equality above and using Lemma 3.6, (2.1), (3.10) and the Littlewood-Paley theory, we have

$$\begin{aligned}
M_2 &\leq C \|\Omega\|_{L^1} \sup_{i \in \mathbb{Z}} \|\Delta_i(b)\|_{L^\infty} \sum_{|k| \leq 7} \left\| \left(\sum_{j \in \mathbb{Z}} |T_{j,|\Omega|}(|\Delta_{l-j+k} S_{l-j} f|)|^2 \right)^{1/2} \right\|_{L^p} \\
&\leq C \|b\|_{BMO} \|\Omega\|_{L^1} \left\| \left(\sum_{j \in \mathbb{Z}} |S_j^2 f|^2 \right)^{1/2} \right\|_{L^p} \\
&\leq C \|b\|_{BMO} \|\Omega\|_{L^1} \|f\|_{L^p}.
\end{aligned} \tag{3.20}$$

(c) *The estimate of M_3 .* Finally, we give the estimate of M_3 . Note that $S_{l-j}((\Delta_i g)(G_{i-3} h)) = 0$ for $g, h \in \mathcal{S}'(\mathbb{R}^n)$ if $|i - (l - j)| \geq 5$. Thus we get

$$\begin{aligned}
& S_{l-j}(\pi_b(T_j S_{l-j} f) - T_j(\pi_b(S_{l-j} f))) \\
&= S_{l-j} \left(\sum_{i \in \mathbb{Z}} (\Delta_i b)(G_{i-3} T_j S_{l-j} f) - T_j \left(\sum_{i \in \mathbb{Z}} (\Delta_i b)(G_{i-3} S_{l-j} f) \right) \right)(x) \\
&= \sum_{|i-(l-j)| \leq 4} \left\{ S_{l-j}((\Delta_i b)(G_{i-3} T_j S_{l-j} f))(x) - S_{l-j} T_j((\Delta_i b)(G_{i-3} S_{l-j} f))(x) \right\} \\
&= \sum_{|i-(l-j)| \leq 4} S_{l-j}([\Delta_i b, T_j](G_{i-3} S_{l-j} f)).
\end{aligned}$$

Applying Proposition 5.1.4 in [22, p.343], it is easy to see that

$$\left\| \left(\sum_{j \in \mathbb{Z}} |G_{j+k} S_j f_j|^2 \right)^{1/2} \right\|_{L^p} \leq \left\| \left(\sum_{j \in \mathbb{Z}} |f_j|^2 \right)^{1/2} \right\|_{L^p} \quad \text{for } k \in [-10, 10].$$

Thus, by the Littlewood-Paley theory, Lemma 3.6 and (3.10) we get

$$\begin{aligned}
M_3 &\leq C \sup_{i \in \mathbb{Z}} \|\Delta_i(b)\|_{L^\infty} \sum_{|k| \leq 4} \left\| \left(\sum_{j \in \mathbb{Z}} |T_{|\Omega|,j}(|G_{l-j+k-3} S_{l-j} f|)|^2 \right)^{1/2} \right\|_{L^p} \\
&\leq C \|b\|_{BMO} \|\Omega\|_{L^1} \left\| \left(\sum_{j \in \mathbb{Z}} |S_j f|^2 \right)^{1/2} \right\|_{L^p} \\
&\leq C \|b\|_{BMO} \|\Omega\|_{L^1} \|f\|_{L^p}.
\end{aligned} \tag{3.21}$$

By (3.15), (3.19)-(3.21), we get

$$L_3 \leq C \max\left\{2, \frac{2^{\tau l}}{\tau}\right\} \|b\|_{BMO} \|\Omega\|_{L^1} \|f\|_{L^p} \quad \text{for } l \in \mathbb{Z}.$$

Combining this with (3.14), we complete the proof of (3.13).

4 Proof of Theorem 1

Let $\phi \in C_0^\infty(\mathbb{R}^n)$ be a radial function such that $0 \leq \phi \leq 1$, $\text{supp}\phi \subset \{1/2 \leq |\xi| \leq 2\}$ and

$$\sum_{l \in \mathbb{Z}} \phi^3(2^{-l}\xi) = 1, \quad |\xi| \neq 0.$$

Define the multiplier operator S_l by

$$\widehat{S_l f}(\xi) = \phi(2^{-l}\xi)\widehat{f}(\xi).$$

Let $K_j(x) = \frac{\Omega(x)}{|x|^n} \chi_{\{2^j < |x| \leq 2^{j+1}\}}$. Define the operator

$$T_j f(x) = K_j * f(x) = \int_{2^j < |y| \leq 2^{j+1}} \frac{\Omega(y)}{|y|^n} f(x-y) dy,$$

and the multiplier T_j^l by $\widehat{T_j^l f}(\xi) = T_j \widehat{S_{l-j} f}(\xi) = \phi(2^{j-l}\xi)\widehat{K_j}(\xi)\widehat{f}(\xi)$. With the notations above, it is easy to see that

$$[b, T_\Omega]f(x) = \sum_{l \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} [b, S_{l-j} T_j S_{l-j}^2]f(x) = \sum_{l \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} [b, S_{l-j} T_j^l S_{l-j}]f(x) := \sum_{l \in \mathbb{Z}} V_l f(x),$$

where $V_l f(x) = \sum_{j \in \mathbb{Z}} [b, S_{l-j} T_j^l S_{l-j}]f(x)$. Then by the Minkowski inequality, we get

$$\|[b, T_\Omega]f\|_{L^p} \leq \left\| \sum_{l=-\infty}^{\lfloor \log \sqrt{2} \rfloor} V_l f \right\|_{L^p} + \left\| \sum_{l=\lfloor \log \sqrt{2} \rfloor + 1}^{\infty} V_l f \right\|_{L^p}. \quad (4.1)$$

Now, we will estimate the two cases respectively.

Case 1. The estimate of $\left\| \sum_{l=-\infty}^{\lfloor \log \sqrt{2} \rfloor} V_l f \right\|_{L^p}$.

Since $\Omega \in L^1(S^{n-1})$ satisfies (1.1) and (1.2), by a well-known Fourier transform estimate of Duoandikoetxea and Rubio de Francia (See [15, p.551-552]), it is easy to show that

$$|\widehat{K_j}(\xi)| \leq C \|\Omega\|_{L^1} 2^j |\xi|.$$

A trivial computation gives that

$$\|\nabla \widehat{K_j}\|_{L^\infty} \leq C 2^j \|\Omega\|_{L^1}.$$

Set $m_j(\xi) = \widehat{K_j}(\xi)$, $m_j^l(\xi) = m_j(\xi)\phi(2^{j-l}\xi)$, and recall that T_j^l by

$$\widehat{T_j^l f}(\xi) = m_j^l(\xi)\widehat{f}(\xi).$$

Straightforward computations lead to

$$\|m_j^l(2^{-j}\cdot)\|_{L^\infty} \leq C \|\Omega\|_{L^1} 2^l, \quad \|\nabla m_j^l(2^{-j}\cdot)\|_{L^\infty} \leq C \|\Omega\|_{L^1},$$

$$\text{supp}\{m_j^l(2^{-j}\xi)\} \subset \{|\xi| \leq 2^{l+2}\}.$$

Let $\widehat{T_j^l}$ be the operator defined by

$$\widehat{\widehat{T_j^l} f}(\xi) = m_j^l(2^{-j}\xi)\widehat{f}(\xi).$$

Denote by $T_{j;b,1}^l f = [b, T_j^l]f$ and $T_{j;b,0}^l f = T_j^l f$. Similarly, denote by $\tilde{T}_{j;b,1}^l f = [b, \tilde{T}_j^l]f$ and $\tilde{T}_{j;b,0}^l f = \tilde{T}_j^l f$. Thus via the Plancherel theorem and Lemma 3.4 states that for any fixed $0 < v < 1$, $k \in \{0, 1\}$,

$$\|\tilde{T}_{j;b,k}^l f\|_{L^2} \leq C \|b\|_{BMO}^k \|\Omega\|_{L^1} 2^{vl} \|f\|_{L^2}, \quad l \leq [\log \sqrt{2}].$$

Dilation-invariance says that

$$\|T_{j;b,k}^l f\|_{L^2} \leq C \|b\|_{BMO}^k \|\Omega\|_{L^1} 2^{vl} \|f\|_{L^2}, \quad l \leq [\log \sqrt{2}]. \quad (4.2)$$

First, we will give the L^2 -norm estimate of $V_l f$ by using the inequality (4.2). Recalling that $V_l f(x) = \sum_{j \in \mathbb{Z}} [b, S_{l-j} T_j^l S_{l-j}] f(x)$, for any $j, l \in \mathbb{Z}$, we may write

$$[b, S_{l-j} T_j^l S_{l-j}] f = [b, S_{l-j}](T_j^l S_{l-j} f) + S_{l-j}[b, T_j^l](S_{l-j} f) + S_{l-j} T_j^l([b, S_{l-j}] f).$$

Thus,

$$\begin{aligned} \|V_l f\|_{L^2} &\leq \left\| \sum_{j \in \mathbb{Z}} [b, S_{l-j}](T_j^l S_{l-j} f) \right\|_{L^2} + \left\| \sum_{j \in \mathbb{Z}} S_{l-j} T_j^l([b, S_{l-j}] f) \right\|_{L^2} \\ &\quad + \left\| \sum_{j \in \mathbb{Z}} S_{l-j}[b, T_j^l](S_{l-j} f) \right\|_{L^2} \\ &:= Q_1 + Q_2 + Q_3. \end{aligned} \quad (4.3)$$

For Q_1 , by Lemma 3.3(iii), (4.2) for $k = 0$ and the Littlewood-Paley theory, we get

$$\begin{aligned} Q_1 &\leq C \|b\|_{BMO} \left\| \left(\sum_{j \in \mathbb{Z}} |T_j^l S_{l-j} f|^2 \right)^{1/2} \right\|_{L^2} \\ &\leq C \|b\|_{BMO} 2^{vl} \|\Omega\|_{L^1} \left\| \left(\sum_{j \in \mathbb{Z}} |S_{l-j} f|^2 \right)^{1/2} \right\|_{L^2} \\ &\leq C \|b\|_{BMO} 2^{vl} \|\Omega\|_{L^1} \|f\|_{L^2}. \end{aligned} \quad (4.4)$$

For Q_2 , by the Littlewood-Paley theory, (4.2) for $k = 0$ and Lemma 3.3(i), we get

$$\begin{aligned} Q_2 &\leq C \left\| \left(\sum_{j \in \mathbb{Z}} |T_j^l([b, S_{l-j}] f)|^2 \right)^{1/2} \right\|_{L^2} \\ &\leq C 2^{vl} \|\Omega\|_{L^1} \left\| \left(\sum_{j \in \mathbb{Z}} |[b, S_{l-j}] f|^2 \right)^{1/2} \right\|_{L^2} \\ &\leq C \|b\|_{BMO} 2^{vl} \|\Omega\|_{L^1} \|f\|_{L^2}. \end{aligned} \quad (4.5)$$

About Q_3 , by(4.2) for $k = 1$ and the Littlewood-Paley theory, we have

$$\begin{aligned} Q_3 &\leq C \left\| \left(\sum_{j \in \mathbb{Z}} |[b, T_j^l](S_{l-j} f)|^2 \right)^{1/2} \right\|_{L^2} \\ &\leq C 2^{vl} \|\Omega\|_{L^1} \left\| \left(\sum_{j \in \mathbb{Z}} |S_{l-j} f|^2 \right)^{1/2} \right\|_{L^2} \\ &\leq C \|b\|_{BMO} 2^{vl} \|\Omega\|_{L^1} \|f\|_{L^2}. \end{aligned} \quad (4.6)$$

Combining (4.4) with (4.5) and (4.6), we have

$$\|V_l f\|_{L^2} \leq C \|b\|_{BMO} 2^{vl} \|\Omega\|_{L^1} \|f\|_{L^2}, \quad l \leq [\log \sqrt{2}]. \quad (4.7)$$

On the other hand, since $T_j^l f(x) = T_j S_{l-j} f(x)$, then $V_l f(x) = \sum_{j \in \mathbb{Z}} [b, S_{l-j} T_j S_{l-j}^2] f(x)$. Applying Lemma 3.7, we get for $1 < p < \infty$

$$\|V_l f\|_{L^p} \leq C \|b\|_{BMO} \|\Omega\|_{L^1} \|f\|_{L^p}, \quad l \leq [\log \sqrt{2}]. \quad (4.8)$$

Interpolating between (4.7) and (4.8), there exists a constant $0 < \beta < 1$, such that

$$\|V_l f\|_{L^p} \leq C 2^{\beta v l} \|\Omega\|_{L^1} \|b\|_{BMO} \|f\|_{L^p}, \quad l \leq [\log \sqrt{2}]. \quad (4.9)$$

Then by the Minkowski inequality, we get for $1 < p < \infty$

$$\begin{aligned} \left\| \sum_{l=-\infty}^{[\log \sqrt{2}]} V_l f \right\|_{L^p} &\leq \sum_{l=-\infty}^{[\log \sqrt{2}]} \|V_l f\|_{L^p} \\ &\leq C \sum_{l=-\infty}^{[\log \sqrt{2}]} 2^{\beta v l} \|b\|_{BMO} \|\Omega\|_{L^1} \|f\|_{L^p} \\ &\leq C \|b\|_{BMO} \|\Omega\|_{L^1} \|f\|_{L^p}. \end{aligned} \quad (4.10)$$

Case 2. The estimate of $\left\| \sum_{l=1+[\log \sqrt{2}]}^{\infty} V_l f \right\|_{L^p}$.

Recalling that $V_l f(x) = \sum_{j \in \mathbb{Z}} [b, S_{l-j} T_j S_{l-j}^2] f(x)$. We will give the delicate L^2 norm of $V_l f$ and the L^p ($1 < p < \infty$) norm of $V_l f$ respectively. It is easy to see that if $\Omega \in F_\alpha(S^{n-1})$ for $\alpha > 1$ satisfies (1.1) and (1.2),

$$|\widehat{K}_j(\xi)| \leq C \log^{-\alpha-1}(|2^j \xi| + 2), \quad \|\nabla \widehat{K}_j\|_{L^\infty} \leq C 2^j.$$

Set $m_j(\xi) = \widehat{K}_j(\xi)$, $m_j^l(\xi) = \phi(2^{j-l}\xi) m_j(\xi)$. Let T_j^l be the operator defined by $\widehat{T_j^l f}(\xi) = m_j^l(\xi) \widehat{f}(\xi)$. Straightforward computations lead to

$$\begin{aligned} \|m_j^l(2^{-j}\cdot)\|_{L^\infty} &\leq C \log^{-\alpha-1}(2 + 2^l), \quad \|\nabla m_j^l(2^{-j}\cdot)\|_{L^\infty} \leq C, \\ \text{supp}\{m_j^l(2^{-j}\xi)\} &\subset \{|\xi| \leq 2^{l+2}\}. \end{aligned}$$

Let \widetilde{T}_j^l be the operator defined by

$$\widehat{\widetilde{T}_j^l f}(\xi) = m_j^l(2^{-j}\xi) \widehat{f}(\xi).$$

Denote by $T_{j;b,1}^l f = [b, T_j^l] f$ and $T_{j;b,0}^l f = T_j^l f$. Similarly, denote by $\widetilde{T}_{j;b,1}^l f = [b, \widetilde{T}_j^l] f$ and $\widetilde{T}_{j;b,0}^l f = \widetilde{T}_j^l f$. Thus via the Plancherel theorem and Lemma 3.4 with $\sigma = 2^l$ states that for any fixed $0 < v < 1$, $k \in \{0, 1\}$,

$$\|\widetilde{T}_{j;b,k}^l f\|_{L^2} \leq C \|b\|_{BMO}^k C \log^{(-\alpha-1)v+1}(2 + 2^l) \|f\|_{L^2}, \quad l \geq 1 + [\log \sqrt{2}]. \quad (4.11)$$

Dilation-invariance says that

$$\|T_{j;b,k}^l f\|_{L^2} \leq C \|b\|_{BMO}^k \log^{(-\alpha-1)v+1}(2 + 2^l) \|f\|_{L^2}, \quad l \geq 1 + [\log \sqrt{2}]. \quad (4.12)$$

Applying (4.12), Lemma 3.3 and the Littlewood-Paley theory, similar to the proof of (4.7), we get

$$\|V_l f\|_{L^2} \leq C \|b\|_{BMO} \log^{(-\alpha-1)v+1}(2 + 2^l) \|f\|_{L^2}, \quad l \geq 1 + [\log \sqrt{2}], \quad (4.13)$$

On the other hand, by Lemma 3.7, for any fixed $0 < \tau < 1/2$, $1 < p < \infty$,

$$\|V_l f\|_{L^p} \leq C \|b\|_{BMO} \frac{2^{\tau l}}{\tau} \|\Omega\|_{L^1} \|f\|_{L^p}, \quad l \geq 1 + [\log \sqrt{2}],$$

where C is independent of τ and l . Take $\tau = 1/l$, we get

$$\|V_l f\|_{L^p} \leq Cl \|b\|_{BMO} \|\Omega\|_{L^1} \|f\|_{L^p}, \quad l \geq 1 + [\log \sqrt{2}],$$

where C is independent of l . Which says that for any r satisfying $1 < r < \infty$, we have

$$\|V_l f\|_{L^r} \leq Cl \|b\|_{BMO} \|f\|_{L^r}, \quad l \geq 1 + [\log \sqrt{2}]. \quad (4.14)$$

Now for any $p \geq 2$, we take r sufficient large such that $r > p$. Using the Riesz-Thorin interpolation theorem between (4.13) and (4.14), we have that for any $l \geq 1 + [\log \sqrt{2}]$,

$$\|V_l f\|_{L^p} \leq C \|b\|_{BMO} l^{1-\theta} \log^{((-\alpha-1)v+1)\theta} (2+2^l) \|f\|_{L^p},$$

where $\theta = \frac{2(r-p)}{p(r-2)}$. We can see that if $r \mapsto \infty$, then θ goes to $2/p$ and $\log^{((-\alpha-1)v+1)\theta} (2+2^l)$ goes to $\log^{((-\alpha-1)v+1)2/p} (2+2^l)$. Therefore, we get

$$\|V_l f\|_{L^p} \leq C \|b\|_{BMO} l^{1-2/p} \log^{((-\alpha-1)v+1)\frac{2}{p}} (2+2^l) \|f\|_{L^p}, \quad l \geq 1 + [\log \sqrt{2}], \quad p \geq 2. \quad (4.15)$$

Then by the Minkowski inequality, for $2 \leq p < \alpha + 1$, we get

$$\begin{aligned} & \left\| \sum_{l=1+[\log \sqrt{2}]}^{\infty} V_l f \right\|_{L^p} \\ & \leq C \|b\|_{BMO} \sum_{l=1+[\log \sqrt{2}]}^{\infty} l^{1-2/p} l^{((-\alpha-1)v+1)\frac{2}{p}} \|f\|_{L^p} \\ & \leq C \|b\|_{BMO} \|f\|_{L^p}. \end{aligned} \quad (4.16)$$

If $1 < p < 2$, by duality, we get for $p > \frac{\alpha+1}{\alpha}$

$$\left\| \sum_{l=1+[\log \sqrt{2}]}^{\infty} V_l f \right\|_{L^p} \leq C \|b\|_{BMO} \|f\|_{L^p}. \quad (4.17)$$

Combining (4.16) with (4.17), we get for $\frac{\alpha+1}{\alpha} < p < \alpha + 1$,

$$\left\| \sum_{l=1+[\log \sqrt{2}]}^{\infty} V_l f \right\|_{L^p} \leq C \|b\|_{BMO} \|f\|_{L^p}.$$

This completes the proof of Theorem 1.

5 Proof of Theorem 2

Let $\alpha > 2$, K_j and the operator T_j be the same as in the proof of Theorem 1. Define

$$\begin{aligned} [b, T_{\Omega}^s] f(x) &= \int_{|x-y| > 2^s} (b(x) - b(y)) \frac{\Omega(x-y)}{|x-y|^n} f(y) dy \\ &= \sum_{j=s}^{\infty} \int_{2^j < |x-y| \leq 2^{j+1}} (b(x) - b(y)) \frac{\Omega(x-y)}{|x-y|^n} f(y) dy \\ &= \sum_{j=s}^{\infty} [b, T_j] f(x), \end{aligned}$$

where

$$[b, T_j]f(x) = \int_{2^j \leq |x-y| \leq 2^{j+1}} (b(x) - b(y)) \frac{\Omega(x-y)}{|x-y|^n} f(y) dy. \quad (5.1)$$

So, we get

$$\sup_{s>0} |[b, T_\Omega^s]f(x)| \leq \sup_{s \in \mathbb{Z}} \left| \sum_{j=s}^{\infty} [b, T_j]f(x) \right|,$$

Thus, to prove Theorem 2, it suffices to estimate the L^p norm of $\sup_{s \in \mathbb{Z}} \left| \sum_{j=s}^{\infty} [b, T_j]f(x) \right|$. Taking a radial Schwartz function $\widehat{\Phi}(\xi) = 1$ for $|\xi| \leq 1$ and $\widehat{\Phi}(\xi) = 0$ for $|\xi| > 2$, and define Φ_s by $\widehat{\Phi}_s(\xi) = \widehat{\Phi}(2^s \xi)$. Write

$$\begin{aligned} \sum_{j=s}^{\infty} [b, T_j]f(x) &= \left[\Phi_s * \left([b, T_\Omega]f - \sum_{j=-\infty}^{s-1} [b, T_j]f \right) \right](x) + \left[\sum_{j=s}^{\infty} [b, T_j]f(x) - \Phi_s * \left(\sum_{j=s}^{\infty} [b, T_j]f \right) \right](x) \\ &:= L_s f(x) + J_s f(x). \end{aligned}$$

Observed that

$$\Phi_s * \left(\sum_{j=-\infty}^{s-1} [b, T_j]f \right)(x) = [b, \Phi_s * \sum_{j=-\infty}^{s-1} K_j]f(x) - [b, W_s] \left(\sum_{j=-\infty}^{s-1} T_j f \right)(x),$$

where W_s is a convolution operator with its convolution kernel Φ_s . Observe that

$$\left| \Phi_s * \sum_{j=-\infty}^{s-1} K_j(x) \right| \leq C \|\Omega\|_{L^1} 2^{-ns} / (1 + |2^{-s}x|^{n+1})$$

(see [15]) and $\sum_{j=-\infty}^{s-1} T_j f(x) = T_\Omega f(x) - \sum_{j=s}^{\infty} T_j f(x)$. It follows that

$$\sup_{s \in \mathbb{Z}} |L_s f(x)| \leq CM([b, T_\Omega]f)(x) + C[b, M]f(x) + [b, M](T_\Omega f)(x) + [b, M](T_\Omega^* f)(x).$$

Then by Theorem 1, the L^p ($\frac{\alpha}{\alpha-1} < p < \alpha$) boundedness of T_Ω, T_Ω^* with kernel function $\Omega \in F_\alpha$ for $\alpha > 2$ (see [23]) and $[b, M]$ (see [21]), we get for $\frac{\alpha}{\alpha-1} < p < \alpha$,

$$\|\sup_{s \in \mathbb{Z}} |L_s f|\|_{L^p} \leq C \|b\|_{BMO} \|f\|_{L^p}. \quad (5.2)$$

To estimate $\sup_{s \in \mathbb{Z}} |J_s f(x)|$, write

$$\sum_{j=s}^{\infty} [b, T_j]f(x) - \Phi_s * \left(\sum_{j=s}^{\infty} [b, T_j]f \right)(x) = \sum_{j=s}^{\infty} [b, T_j]f(x) - [b, \Phi_s * \sum_{j=s}^{\infty} K_j]f(x) + [b, W_s] \left(\sum_{j=s}^{\infty} T_j f \right)(x).$$

Thus we get

$$\sup_{s \in \mathbb{Z}} |J_s f(x)| \leq \sup_{s \in \mathbb{Z}} \left| \sum_{j=s}^{\infty} [b, (\delta - \Phi_s) * K_j]f(x) \right| + [b, M](T_\Omega^* f)(x),$$

where δ is Dirac mass at the origin. Since for $\frac{\alpha}{\alpha-1} < p < \alpha$, (see [23])

$$\|[b, M](T_\Omega^* f)\|_{L^p} \leq C \|b\|_{BMO} \|f\|_{L^p}. \quad (5.3)$$

Thus, to give the estimate of L^p norm for the term $\sup_{s \in \mathbb{Z}} |J_s f(x)|$, it suffices to give the estimate of L^p norm for the term $\sup_{s \in \mathbb{Z}} \left| \sum_{j=s}^{\infty} [b, (\delta - \Phi_s) * K_j] f(x) \right|$. Note that

$$\sup_{s \in \mathbb{Z}} \left| \sum_{j=s}^{\infty} [b, (\delta - \Phi_s) * K_j] f(x) \right| \leq \sum_{j=0}^{\infty} \sup_{s \in \mathbb{Z}} |[b, (\delta - \Phi_s) * K_{j+s}] f(x)|.$$

Let $U_{s,j} f(x) = (\delta - \Phi_s) * K_{s+j} * f$ and $[b, U_{s,j}] f(x) = [b, (\delta - \Phi_s) * K_{s+j}] f$. Then

$$\sup_{s \in \mathbb{Z}} \left| \sum_{j=s}^{\infty} [b, (\delta - \Phi_s) * K_j] f(x) \right| \leq \sum_{j=0}^{\infty} \sup_{s \in \mathbb{Z}} |[b, U_{s,j}] f(x)|. \quad (5.4)$$

It is easy to see that

$$\begin{aligned} & \sup_{s \in \mathbb{Z}} |[b, U_{s,j}] f(x)| \\ & \leq C \sup_{s \in \mathbb{Z}} |[b, T_{s+j}] f(x)| + C \sup_{s \in \mathbb{Z}} \left(W_s |[b, T_{s+j}] f + C[b, W_s](T_{s+j} f) \right)(x) \\ & \leq C \sup_{s \in \mathbb{Z}} |[b, T_{s+j}] f(x)| + CM(\sup_{s \in \mathbb{Z}} |[b, T_{s+j}] f)(x) + C[b, M](M_{\Omega} f)(x) \\ & \leq C[b, M_{\Omega}] f(x) + CM([b, M_{\Omega}] f)(x) + C[b, M](M_{\Omega} f)(x). \end{aligned}$$

Applying Theorem 3, the L^p ($1 < p < \infty$) boundedness of M , M_{Ω} with kernel function $\Omega \in L^1(S^{n-1})$ (see [22]) and $[b, M]$ (see [21]), we have for $\frac{\alpha}{\alpha-1} < p < \alpha$,

$$\| \sup_{s \in \mathbb{Z}} |[b, U_{s,j}] f \|_{L^p} \leq C(\| [b, M_{\Omega}] f \|_{L^p} + \| b \|_{BMO} \| M_{\Omega} f \|_{L^p}) \leq C \| b \|_{BMO} \| f \|_{L^p}. \quad (5.5)$$

On the other hand, set

$$B_{s,j}(\xi) = (1 - \widehat{\Phi}_s(\xi)) \widehat{K}_{s+j}(\xi), \quad B_{s,j}^l(\xi) = (1 - \widehat{\Phi}_s(\xi)) \widehat{K}_{s+j}(\xi) \phi(2^{s-l} \xi).$$

Define the operator $U_{s,j}^l$ by $\widehat{U_{s,j}^l f}(\xi) = \widehat{U_{s,j} f}(\xi) \phi(2^{s-l} \xi)$, and denote by $[b, U_{s,j}^l]$ the commutator of $U_{s,j}^l$. Then it is clear that

$$[b, U_{s,j}] f(x) = \sum_{l \in \mathbb{Z}} [b, U_{s,j}^l S_{l-s}^2] f(x).$$

By the Minkowski inequality, we get

$$\begin{aligned} \| \sup_{s \in \mathbb{Z}} |[b, U_{s,j}] f \|_{L^2} & \leq \left\| \left(\sum_{s \in \mathbb{Z}} |[b, U_{s,j}] f|^2 \right)^{1/2} \right\|_{L^2} \\ & \leq \left\| \left(\sum_{s \in \mathbb{Z}} \left| \sum_{l \in \mathbb{Z}} [b, U_{s,j}^l S_{l-s}^2] f \right|^2 \right)^{1/2} \right\|_{L^2} \\ & \leq \sum_{l \in \mathbb{Z}} \left\| \left(\sum_{s \in \mathbb{Z}} |[b, U_{s,j}^l] S_{l-s}^2 f|^2 \right)^{1/2} \right\|_{L^2} \\ & \quad + \sum_{l \in \mathbb{Z}} \left\| \left(\sum_{s \in \mathbb{Z}} |U_{s,j}^l [b, S_{l-s}^2] f|^2 \right)^{1/2} \right\|_{L^2} \\ & := I_1 + I_2. \end{aligned} \quad (5.6)$$

To complete the proof we will estimate each term separately. Denote by $U_{s,j;b,1}^l f = [b, U_{s,j}^l] f$ and $U_{s,j;b,0}^l f = U_{s,j}^l f$. Obviously, if we can prove that for any $0 < v < 1$, $k \in \{0, 1\}$, there exists a constant $0 < \beta < 1$, such that

$$\| U_{s,j;b,k}^l f \|_{L^2} \leq C 2^{-\beta j} \| b \|_{BMO}^k 2^l \| f \|_{L^2}, \quad \text{for } l \leq [\log \sqrt{2}] \quad (5.7)$$

and

$$\|U_{s,j;b,k}^l f\|_{L^2} \leq C \|b\|_{BMO}^k \log^{(-\alpha-1)v+1}(2^{l+j} + 2) \|f\|_{L^2}, \quad \text{for } l \geq [\log \sqrt{2}] + 1 \quad (5.7')$$

then we may finish the estimate of I_1 and I_2 . We consider first I_1 . In fact, by (5.7) and (5.7') for $k = 1$ and the Littlewood-Paley theory, we get

$$\begin{aligned} I_1 &\leq \sum_{l=-\infty}^{[\log \sqrt{2}]} \left\| \left(\sum_{s \in \mathbb{Z}} |[b, U_{s,j}^l] S_{l-s}^2 f|^2 \right)^{1/2} \right\|_{L^2} + \sum_{l=[\log \sqrt{2}]+1}^{\infty} \left\| \left(\sum_{s \in \mathbb{Z}} |[b, U_{s,j}^l] S_{l-s}^2 f|^2 \right)^{1/2} \right\|_{L^2} \\ &\leq C 2^{-\beta j} \|b\|_{BMO} \left(\sum_{l=-\infty}^{[\log \sqrt{2}]} 2^{l-1} \left\| \left(\sum_{s \in \mathbb{Z}} |S_{l-s}^2 f|^2 \right)^{1/2} \right\|_{L^2} \right) \\ &\quad + C \|b\|_{BMO} \left(\sum_{l=[\log \sqrt{2}]+1}^{\infty} \log^{(-\alpha-1)v+1}(2^{l+j} + 2) \left\| \left(\sum_{s \in \mathbb{Z}} |S_{l-s}^2 f|^2 \right)^{1/2} \right\|_{L^2} \right). \end{aligned}$$

Since $(l+j)^2 \geq l(j+1)$, we get

$$I_1 \leq C(j+1)^{\frac{(-\alpha-1)v+1}{2}} \|b\|_{BMO} \|f\|_{L^2}. \quad (5.8)$$

We will now estimate I_2 . by (5.7) for $k = 0$, the Littlewood-Paley theory and Lemma 3.3 (ii), we get

$$\begin{aligned} I_2 &\leq \sum_{l=-\infty}^{[\log \sqrt{2}]} \left\| \left(\sum_{s \in \mathbb{Z}} |U_{s,j}^l [b, S_{l-s}^2] f|^2 \right)^{1/2} \right\|_{L^2} + \sum_{l=[\log \sqrt{2}]+1}^{\infty} \left\| \left(\sum_{s \in \mathbb{Z}} |U_{s,j}^l [b, S_{l-s}^2] f|^2 \right)^{1/2} \right\|_{L^2} \\ &\leq C 2^{-\beta j} \left(\sum_{l=-\infty}^{[\log \sqrt{2}]} 2^{l-1} \left\| \left(\sum_{s \in \mathbb{Z}} |[b, S_{l-s}^2] f|^2 \right)^{1/2} \right\|_{L^2} \right) \\ &\quad + C \left(\sum_{l=[\log \sqrt{2}]+1}^{\infty} \log^{(-\alpha-1)v+1}(2^{l+j} + 2) \left\| \left(\sum_{s \in \mathbb{Z}} |[b, S_{l-s}^2] f|^2 \right)^{1/2} \right\|_{L^2} \right) \\ &\leq C(j+1)^{\frac{(-\alpha-1)v+1}{2}} \|b\|_{BMO} \|f\|_{L^2}. \end{aligned} \quad (5.9)$$

Combining I_1 with I_2 , we get

$$\| \sup_{s \in \mathbb{Z}} [b, U_{s,j}] f \|_{L^2} \leq C(j+1)^{\frac{(-\alpha-1)v+1}{2}} \|b\|_{BMO} \|f\|_{L^2}. \quad (5.10)$$

Interpolating between (5.5) and (5.10), similar to the proof of (4.15), for $p \geq 2$, we get

$$\| \sup_{s \in \mathbb{Z}} [b, U_{s,j}] f \|_{L^p} \leq C(j+1)^{\frac{2}{p} \frac{(-\alpha-1)v+1}{2}} \|b\|_{BMO} \|f\|_{L^p}. \quad (5.11)$$

Then by (5.4), we get for $2 \leq p < \alpha$,

$$\begin{aligned} \left\| \sup_{s \in \mathbb{Z}} \left| \sum_{j=s}^{\infty} [b, (\delta - \Phi_s) * K_j] f(x) \right| \right\|_{L^p} &\leq \sum_{j=0}^{\infty} (j+1)^{\frac{2}{p} \frac{(-\alpha-1)v+1}{2}} \|b\|_{BMO} \|f\|_{L^p} \\ &\leq C \|b\|_{BMO} \|f\|_{L^p}. \end{aligned} \quad (5.12)$$

Similarly, for $p < 2$, we get

$$\| \sup_{s \in \mathbb{Z}} [b, U_{s,j}] f \|_{L^p} \leq C(j+1)^{\frac{2}{p'} \frac{(-\alpha-1)v+1}{2}} \|b\|_{BMO} \|f\|_{L^p}. \quad (5.13)$$

Then by (5.4), we get for $\frac{\alpha}{\alpha-1} < p < 2$,

$$\left\| \sup_{s \in \mathbb{Z}} \left| \sum_{j=s}^{\infty} [b, (\delta - \Phi_s) * K_j] f(x) \right| \right\|_{L^p} \leq \sum_{j=0}^{\infty} (j+1)^{\frac{2}{p'} \frac{(-\alpha-1)v+1}{2}} \|b\|_{BMO} \|f\|_{L^p} \quad (5.14)$$

$$\leq C \|b\|_{BMO} \|f\|_{L^p}.$$

This completes the proof of Theorem 2. Hence it remains to prove (5.7) and (5.7'). To this end, define multiplier $\tilde{U}_{s,j}^l$ by $\widehat{\tilde{U}_{s,j}^l f}(\xi) = B_{s,j}^l(2^{-s}\xi)\widehat{f}(\xi)$, and denote by $[b, \tilde{U}_{s,j}^l]$ the commutator of $\tilde{U}_{s,j}^l$. Define $\tilde{U}_{s,j;b,1}^l f = [b, \tilde{U}_{s,j}^l]f$ and $\tilde{U}_{s,j;b,0}^l f = \tilde{U}_{s,j}^l f$. Recall that

$$B_{s,j}(\xi) = (1 - \widehat{\Phi}_s(\xi))\widehat{K}_{s+j}(\xi), \quad B_{s,j}^l(\xi) = (1 - \widehat{\Phi}_s(\xi))\widehat{K}_{s+j}(\xi)\phi(2^{s-l}\xi).$$

It is easy to see that

$$|B_{s,j}(\xi)| \leq C2^{-j}|2^s\xi| \quad \text{for } |2^s\xi| \leq 1,$$

$$|B_{s,j}(\xi)| \leq C \log^{-\alpha-1}(|2^{s+j}\xi| + 2) \quad \text{for } |2^s\xi| > 1,$$

$$|\nabla B_{s,j}(\xi)| \leq C2^s2^j.$$

Since $\text{supp}(B_{s,j}^l(2^{-s}\xi)) \subset \{\xi : 2^{l-1} \leq |\xi| \leq 2^l\}$, we have the following estimates

$$|B_{s,j}(2^{-s}\xi)| \leq C2^{l-j} \quad \text{for } l \leq 0$$

$$|B_{s,j}(2^{-s}\xi)| \leq C \log^{-\alpha-1}(2^{l+j} + 2) \quad \text{for } l > 0,$$

$$|\nabla B_{s,j}^l(2^{-s}\xi)| \leq C2^j,$$

Applying Lemma 3.5 with $\sigma = 2^l$, $A = 1/2$ and the Plancherel theory, there exists a constant $0 < \beta < 1$, such that for any fixed $0 < v < 1$, $k \in \{0, 1\}$,

$$\|\tilde{U}_{s,j;b,k}^l f\|_{L^2} \leq C \|b\|_{BMO}^k 2^{-\beta j} 2^l \|f\|_{L^2}, \quad \text{for } l \leq [\log \sqrt{2}].$$

$$\|\tilde{U}_{s,j;b,k}^l f\|_{L^2} \leq C \|b\|_{BMO}^k \log^{(-\alpha-1)v+1}(2^{l+j} + 2) \|f\|_{L^2}, \quad \text{for } l \geq [\log \sqrt{2}] + 1.$$

Which implies that

$$\|U_{s,j;b,k}^l f\|_{L^2} \leq C \|b\|_{BMO}^k 2^{-\beta j} 2^l \|f\|_{L^2}, \quad \text{for } l \leq [\log \sqrt{2}].$$

$$\|U_{s,j;b,k}^l f\|_{L^2} \leq C \|b\|_{BMO}^k \log^{(-\alpha-1)v+1}(2^{l+j} + 2) \|f\|_{L^2}, \quad \text{for } l \geq [\log \sqrt{2}] + 1,$$

by dilation invariance. This establishes the proof of (5.7) and (5.7').

References

- [1] H. Al-Qassem and A. Al-Salman, *Rough Marcinkiewicz integral operators*, Int. J. Math. Math. Sci., **27** (2001), 495-503.

- [2] J. Alvarez, R. Bagby, D. Kurtz and C. Pérez, *Weighted estimates for commutators of linear operators*, Studia Math., **104** (1993), 195-209.
- [3] J. M. Bony, *Calcul symbolique et propagation des singularites pour les equations aux derivees partielles non lineaires*, Ann. Sci. Ecole Norm. Sup.(4), **14** (1981), 209-246.
- [4] M. Bramanti and M. Cerutti, *Commutators of singular integrals on homogeneous spaces*, Boll. Un. Mat. Ital. B(7), **10** (1996), 843-883.
- [5] A. P. Calderón, *Commutators of singular integrals*, Proc. Nat. Acad. Sci. U.S.A., **53** (1965), 1092-1099.
- [6] A. P. Calderón, *On commutators of singular integrals*, Studia Math., **53** (1975), 139-174.
- [7] D. X. Chen and S. Z. Lu, *L^p boundedness of parabolic Littlewood-Paley operator with rough kernel belonging to $F(S^{n-1})$* , Acta. Math. Sci., **31** (2011), 343-350.
- [8] J. Chen, D. Fan and Y. Pan, *A note on a Marcinkiewicz integral operator*, Math. Nachr., **227** (2001), 33C42.
- [9] L. C. Cheng and Y. Pan, *L^p bounds for singular integrals associated to surfaces of revolution*, J. Math. Anal. Appl. **265** (2002), 163-169.
- [10] J. Chen and C. Zhang, *Boundedness of rough singular integral on the Triebel-Lizorkin spaces*, J. Math. Anal. Appl. **337** (2008), 1048-1052.
- [11] F. Chiarenza, M. Frasca and P. Longo, *Interior $W^{2,p}$ estimates for nondivergence elliptic equations with discontinuous coefficients*, Ric. Math., **XL** (1991), 149-168.
- [12] F. Chiarenza, M. Frasca and P. Longo, *$W^{2,p}$ -solvability of the Dirichlet problem for nondivergence elliptic equations with VMO coefficients*, Trans. Amer. Math. Soc., **334** (1993) 841-853.
- [13] R. Coifman, P. Lions, Y. Meyer and S. Semmes, *Compensated compactness and Hardy spaces*, J. Math. Pures Appl., **72** (1993), 247-286.
- [14] R. Coifman, R. Rochberg and G. Weiss, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math., **103** (1976), 611-635.
- [15] J. Duoandikoetxea and J. L. Rubio de Francia, *Maximal and singular integral operators via Fourier transform estimates*, Invent. Math., **84** (1986), 541-561.
- [16] J. Duoandikoetxea, *Weighted norm inequalities for homogeneous singular integrals*, Trans. Amer. Math. Soc., **336** (1993), 869-880.
- [17] D. Fan, K. Guo and Y. Pan, *A note of a rough singular integral operator*, Math. Ineq. and Appl., **2** (1999), 73-81.

- [18] G. Di Fazio and M. A. Ragusa, *Interior estimates in morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients*, J. of Funct Analysis., **112** (1993), 241-256.
- [19] M. Frazier and B. Jawerth, *The φ -transform and applications to distribution spaces*, Lect. Notes Math., **1302**, Berlin, Springer, 1988, 223-246.
- [20] M. Frazier, B. Jawerth and G. Weiss, *Littlewood-Paley theory and the study of function Spaces*, CBMS Reg. Conf. Ser. **79**, Amer. Math. Soc., Providence, RI, 1991.
- [21] J. Garcia-Cuerva, E. Harboure, C. Segovia and J. L. Torre, *Weighted norm inequality for commutators of strongly singular integrals*, Indiana Univ. Math J., **40** (1991), 1397-1420.
- [22] L. Grafakos, *Classical and Modern Fourier Analysis*, Pearson Education, Inc. Upper Saddle River, New Jersey, 2004.
- [23] L. Grafakos and A. Stefanov, *Convolution Calderón-Zygmund singular integral operators with rough kernels*, Indiana Univ. Math. J., **47** (1998), 455-469.
- [24] L. Grafakos, P. Honzik, and D. Ryabogin, *On the p -independence boundedness property of Calderón-Zygmund theory*, J fr die reine und angewandte Mathematik (Crelles Journal), **2007**(2007), 227-234.
- [25] L. Greco and T. Iwaniec, *New inequalities for the Jacobian*, Ann. Inst. H. Poincaré **11** (1994), 17-35.
- [26] G. Hu, *$L^2(\mathbb{R}^n)$ boundedness for the commutators of convolution operators*, Nagoya Math. J., **163** (2001), 55-70.
- [27] G. Hu, *$L^p(\mathbb{R}^n)$ boundedness for the commutator of a homogeneous singular integral operator*, Studia Math., **154** (2003), 13-27.
- [28] D. S. Kurtz, *Littlewood-Paley and multiplier theorems on weighted L^p spaces*, Trans. Amer. Math. Soc., **259** (1980), 235-254.
- [29] R. Rochberg and G. Weiss, *Derivatives of analytic families of Banach spaces*, Ann. of Math., **118** (1983), 315-347.
- [30] E. M. Stein and G. Weiss, *Interpolation of operators with change of measures*, Trans. Amer. Math. Soc., **87** (1958), 159-172.