Fourier spectrum characterization of Hardy Space

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Abstract

Fourier spectrum of the boundary values of functions in the complex Hardy Space $H^p(\mathbb{C}_+)$ is characterized for $p \in (0,2)$. First, functions f in the $L^p(\mathbb{R})$ can be decomposed into a sum $g + h$, where g and h are the non-tangential boundary limits of function in $H^p(\mathbb{C}_+)$ and $H^p(\mathbb{C}_-)$ in the sense of $L^p(\mathbb{R})$, where $H^p(\mathbb{C}_+)$ and $H^p(\mathbb{C}_-)$ are the Hardy spaces in the upper and lower complex plane $\mathbb{C}_+$ and $\mathbb{C}_-$, respectively. Secondly, as an application, extend the Paley-Wiener Theorem is extended, originally for square-integrable functions, to the $H^p(\mathbb{C}_+)$ cases with $0 < p < 2$. Finally, Fourier spectrum characterization is given for $L^p(\mathbb{R})$ functions.

Key Words The Paley-Wiener Theorem, Hardy Space

1 Introduction

The classical Hardy spaces $H^p(\mathbb{C}_+),\ 0 < p < +\infty$, are defined to consist of those functions f , holomorphic in the upper half plane $\mathbb{C}_+ = \{z = x + iy : y > 0\}$ with the property that $M_p(f, y)$ is uniformly bounded for $y > 0$, where

$$M_p(f,y) = \left(\int_{-\infty}^{\infty} |f(x+iy)|^p dx\right)^{\frac{1}{p}}.$$

Since $|f|^p$ is subharmonic for $f \in H^p(\mathbb{C}_+)$, the function $M_p(f, y)$ decreases in $(0, \infty)$,

$$\|f\|_{H^p_+} = \sup\{M_p(f,y) : 0 < y < \infty\} = \lim_{y\to 0} M_p(f,y).$$

If $f(x)$ is the non-tangential boundary limits of the function $f \in H^p(\mathbb{C}_+)$, then $f(x) \in L^p(\mathbb{R})$, and

$$\|f\|_p = \left(\int_{-\infty}^{\infty} |f(x)|^p dx\right)^{\frac{1}{p}} = \|f\|_{H^p_+}.$$

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Similarly, $H^p(\mathbb{C}_-)$, $0 < p < +\infty$, are defined to consist of those functions g , holomorphic in the lower half plane $\mathbb{C}_- = \{z = x+iy : y < 0\}$ with the property that $M_p(f,y)$ is uniformly bounded for $y < 0$, where

$$M_p(g,y) = \left(\int_{-\infty}^{\infty} |g(x+iy)|^p dx \right)^{\frac{1}{p}}.$$

Since $|g|^p$ is subharmonic for $g \in H^p(\mathbb{C}_-)$, the function $M_p(g,y)$ increases in $(-\infty, 0)$,

$$\|g\|_{H^p_-} = \sup\{M_p(g,y) : y < 0\} = \lim_{y<0, y\to 0} M_p(g,y).$$

If $g(x)$ is the non-tangential boundary limits of the function $g \in H^p(\mathbb{C}_-)$, then $g(x) \in L^p(\mathbb{R})$, and

$$\|g\|_p = \left(\int_{-\infty}^{\infty} |g(x)|^p dx \right)^{\frac{1}{p}} = \|f\|_{H^p_-}.$$

Since there is an isometric isomorphism between their non-tangential boundary limits and the functions in the Hardy spaces, we denote by $H^p_+(\mathbb{R})$ and $H^p_-(\mathbb{R})$ their the non-tangential boundary limits, respectively, that is

$$H^p_+(\mathbb{R}) = \Big\{ f : f \text{ is the non-tangential boundary limit of a function in } H^p(\mathbb{C}_+) \Big\},$$

$$H^p_-(\mathbb{R}) = \Big\{ f : f \text{ is the non-tangential boundary limit of a function in } H^p(\mathbb{C}_-) \Big\}.$$

The Fourier transform of a function $f \in L^1(\mathbb{R})$ is defined, for $x \in \mathbb{R}$, by

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(t) e^{-ixt} dt.$$

We recall that the Fourier transformation for tempered distribution, T, is defined through the relation

$$(\widehat{T}, \varphi) = (T, \hat{\varphi})$$

for φ in the Schwarz class $\mathbb{S}$, which coincides with the traditional definitions of Fourier transformations for functions in $L^p(\mathbb{R}), 1 \leq p \leq 2$. A measurable function f such that

$$\frac{f(x)}{(1+x^2)^k} \in L^p(\mathbb{R}), \quad 1 \leq p \leq \infty,$$

for some positive integer integer k, is called a tempered L^p function (when $p = \infty$ such a function is often also called a slowly increasing function). The Fourier transform is a one to one mapping of $\mathbb{S}$ onto $\mathbb{S}$.

Our results are as follow:

Theorem 1 Suppose that $0 < p < 1$ and $f \in L^p(\mathbb{R})$. Then, there exist a positive constant A_p and two sequences of rational functions $\{P_k(z)\}$ and $\{Q_k(z)\}$ such that $P_k \in H^p(\mathbb{C}_+)$, $Q_k \in H^p(\mathbb{C}_-)$ and

$$\sum_{k=1}^{\infty}\left(\|P_k\|_{H^p_+}^p + \|Q_k\|_{H^p_-}^p\right) \le A_p\|f\|_p^p, \tag{1}$$

$$\lim_{n\to\infty} ||f - \sum_{k=1}^{n}(P_k + Q_k)||_p = 0. \tag{2}$$

so

$$g(z) = \sum_{k=1}^{\infty} P_k(z) \in H^p(\mathbb{C}_+), \quad h(z) = \sum_{k=1}^{\infty} Q_k(z) \in H^p(\mathbb{C}_-), \tag{3}$$

and $g(x)$ and $h(x)$ are the non-tangential boundary limits of functions for $g \in H^p(\mathbb{C}_+)$ and $h \in H^p(\mathbb{C}_-)$, respectively, $f(x) = g(x) + h(x)$ almost everywhere, and

$$\|f\|_p^p \le \|g\|_p^p + \|h\|_p^p \le A_p\|f\|_p^p,$$

that is , in the sense of $L^p(\mathbb{R})$

$$L^p(\mathbb{R}) = H^p_+(\mathbb{R}) + H^p_-(\mathbb{R}).$$

Theorem 2 If $0 < p \le 1,\ f \in H^p(\mathbb{C}_+)$, then there exist a positive constant A_p, depending only on p, and a slowly increasing continuous function F whose support is in $[0, \infty)$, satisfying that, for φ in the Schwarz class $\mathbb{S}$,

$$(F, \varphi) = \lim_{y>0, y\to 0} \int_{\mathbb{R}} f(x + iy)\hat{\varphi}(x)dx,$$

and that

$$|F(t)| \leqslant A_p\|f\|_{H^p_+}|t|^{\frac{1}{p}-1}, \quad (t \in \mathbb{R}) \tag{4}$$

and

$$f(z) = \frac{1}{\sqrt{2\pi}}\int_0^{\infty} F(t)e^{itz}dt \qquad (z \in \mathbb{C}_+).$$

Theorem 3 If $0 < p \le 1,\ g \in H^p(\mathbb{C}_-)$, then there exist a positive constant A_p, depending only on p, and a slowly increasing continuous function G whose support is in $(-\infty, 0]$, satisfying that, for φ in the Schwarz class $\mathbb{S}$,

$$(G, \varphi) = \lim_{y<0, y\to 0} \int_{\mathbb{R}} g(x + iy)\hat{\varphi}(x)dx,$$

and that (4) holds and that

$$g(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{0} G(t)e^{itz}dt \qquad (z \in \mathbb{C}_-).$$

Theorem 4 Let $0 < p < 1$, $f \in L^p(\mathbb{R})$. Then $f \in H^p_+(\mathbb{R})$ if and only if there exists a sequence of functions $\{f_n\}$ satisfying $f_n \in L^2(\mathbb{R}) \bigcap L^p(\mathbb{R})$, $\text{supp}\hat{f}_n \subset [0, +\infty)$, and

$$\lim_{n\to\infty} ||f - f_n||_p = 0, \tag{5}$$

i.e., $H^p_+(\mathbb{R})$ is the $L^p(\mathbb{R})-$closure of $L^p(\mathbb{R}) \bigcap H^2_+(\mathbb{R})$.

Theorem 5 (the Paley-Wiener Theorem for $1 \le p \le 2$) Suppose $1 \le p \le 2$, $f \in L^p(\mathbb{R})$. Then $f \in H^p_+(\mathbb{R})$ if and only if $\text{supp}\hat{f} \subset [0, +\infty)$. If the condition is satisfied, then

$$f(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \hat{f}(t) e^{itz} dt \in H^p(\mathbb{C}_+), \tag{6}$$

$f(x)$ are the non-tangential boundary limits of function $f(z)$.

2 Proof of theorem 1

We need the following Lemmas.

Lemma 1 Suppose that $0 < p < 1$ and R is a rational function with $R \in L^p(\mathbb{R})$, we have the following conclusion.

1) if $R(z)$ is analytic in the upper plane $\mathbb{C}_+$, then $R \in H^p(\mathbb{C}_+)$.

2) if $R(z)$ is analytic in the lower plane $\mathbb{C}_-$, then $R \in H^p(\mathbb{C}_-)$.

Proof If $0 < p < 1$, $R(z) = \frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are polynomials with the degrees $m = \text{deg}P$ and $n = \text{deg}Q$, respectively, and without the common zeros, and there exists a constant $c \neq 0$ such that

$$\lim_{z\to\infty} R(z) z^{n-m} = c.$$

Therefore, there exists a constant $M_0 > 1$ such that

$$\frac{|c|}{2}|z|^{m-n} \le |R(z)| \le 2|c||z|^{m-n}, \quad |z| > M_0,$$

$R \in L^p([M_0, \infty))$ implies that $p(m-n) < -1$, and so for $y \ge 0$,

$$\int_{|x|>M_0} |R(x+iy)|^p dx \leqslant (2|c|)^p \int_{|x|>R} |x+iy|^{p(m-n)} dx$$

$$\leqslant (2|c|)^p \int_{|x|>M_0} |x|^{p(m-n)} dx = \frac{2^{p+1}|c|^p}{p(n-m)-1} = M_1 < \infty,$$

where M_1 is a positive constant.

Similarly for $y > R$,

$$\int_{|x|\le M_0} |R(x+iy)|^p dx \leqslant (2|c|)^p \int_{|x|\le M_0} |x+iy|^{p(m-n)} dx$$

$$\leqslant 2(2|c|)^p M_0^{p(m-n)+1} = M_2 < \infty.$$

Therefore if $R(z)$ is analytic in the upper half plane $\mathbb{C}_+$, then $Q(z) \neq 0$ for $z \in \mathbb{C}_+$. If $Q(x) \neq 0$ for $x \in \mathbb{R}$, then $R(z)$ is continuous in the rectangle $E_0 = [-M_0, M_0] \times [0, M_0]$, and so $R \in H^p(\mathbb{C}_+)$. Otherwise, the null set $N_Q = \{a \in \mathbb{R} : Q(a) = 0\}$ of Q in $\mathbb{R}$ is a finite set, let $N_Q = \{a_1, a_2, \cdots, a_q\}$ with $a_1 < a_2 < \cdots < a_q$, and $P(a_k) \neq 0,\ k = 1, 2, \cdots, q$. Therefore there exists a polynomial $Q_1(z)$ with $Q_1(a_k) \neq 0,\ k = 1, 2, \cdots, q$, and positive integers $l_k(\ k = 1, 2, \cdots, q)$ such that

$$Q(z) = (z - a_1)^{l_1}(z - a_2)^{l_2} \cdots (z - a_q)^{l_q} Q_1(z),$$

so there exist positive constants δ, ε_0 and $M_3 > \varepsilon_0$ such that

$$\varepsilon_0 \leq |R(z)(z - a_k)^{l_k}| \leq M_3,$$

for $z = x + iy \in I_k = \{z + x + iy : 0 < |x - a_k| \leq \delta,\ 0 \leq y \leq \delta,\ k = 1, 2, \cdots, q\}$. Therefore,

$$\int_{|x-a_k|\leq\delta} |R(x)|^p dx \geq \varepsilon_0^p \int_{|x-a_k|\leq\delta} |x - a_k|^{-pl_k} dx,$$

$R \in L^p([a_k - \delta, a_k + \delta])$ implies that $pl_k < 1$, and so for $y \in [0, \delta]$,

$$\int_{|x-a_k|\leq\delta} |R(x + iy)|^p dx \leqslant M_3^p \int_{|x-a_k|\leq\delta} |x + iy - a_k|^{-pl_k} dx$$

$$\leqslant M_3^p \int_{|x-a_k|\leq\delta} |x - a_k|^{-pl_k} dx = \frac{2M_3^p \delta^{1-pl_k}}{1 - pl_k} < \infty.$$

Since the poles $R(z)$ in the closed upper half plane is n N_q, so $R(z)$ is continuous in the bounded closed set

$$\{z \in E_0 : z \notin I_k,\ k = 1, 2, \cdots, q\}.$$

Therefore

$$\int_{|x|\leq M_0} |R(x + iy)|^p dx$$

is bounded on $[0, M_0]$. This proves that $R \in H^p(\mathbb{C}_+)$. Similarly, we can prove 2) in Lemma 1

Lemma 2 If $0 < p \leq 1$, $f \in L^p(\mathbb{R})$, then, for $\varepsilon > 0$, there exists a sequence of rational functions $\{R_k(z)\}$, Their poles are contained in $\{i, -i\}$, such that

$$\sum_{k=1}^{\infty} ||R_k||_p^p \leq (1 + \varepsilon)\|f\|_p^p \tag{7}$$

and

$$\lim_{n\to\infty} ||f - \sum_{k=1}^{n} R_k||_p = 0. \tag{8}$$

Proof For case $0 < p < 1$, we can assume that $\|f\|_p^p > 0$. The fractional linear mapping

$$z = \alpha(w) = i\frac{1-w}{1+w}$$

is a comformal mapping from the unit disc $U = \{w : |w| < 1\}$ to the upper half plane $\mathbb{C}_+$, its inverse mapping is

$$\beta(z) = \frac{i-z}{z+i}.$$

Let $x = \alpha(e^{i\theta}),\ \theta \in [-\pi, \pi]$, then $x = \tan\frac{\theta}{2}$ and $dx = \frac{d\theta}{1+\cos\theta}$, so

$$\int_{-\infty}^{\infty} |f(x)|^p dx = \int_{-\pi}^{\pi} |f(\tan\frac{\theta}{2})|^p \frac{d\theta}{1+\cos\theta} < \infty.$$

Therefore the function

$$g(\theta) = \frac{f(\tan\frac{\theta}{2})}{(1+\cos\theta)^{\frac{1}{p}}} \in L^p([-\pi, \pi]).$$

The trigonometric polynomials is dense in $L^p([-\pi, \pi])$, so there exists a sequence of rational functions $\{r_k(w)\}$, whose poles are contained in $\{0\}$, i.e., each $r_k(e^{i\theta})$ can be write as $\sum_{j=-m_k}^{m_k} c_{k,j}e^{ij\theta}$, which are trigonometric polynomials, such that

$$\lim_{k\to\infty} ||g(\theta) - r_k(e^{i\theta})||_{L^p([-\pi,\pi])} = 0.$$

Furthermore, for any $\varepsilon > 0$, the sequence of rational functions $\{r_k(w)\}$ can be choose such that

$$||g(\theta) - r_k(e^{i\theta})||^p_{L^p([-\pi,\pi])} \le \frac{A_\varepsilon}{4^{k+3}}.$$

where $A_\varepsilon = \|f\|_p^p\varepsilon$, Let

$$M_k = \sum_{j=-m_k}^{m_k} |c_{k,j}| + 1.$$

Since $0 < p < 1$, there exists a positive integer l_p such that $1 < p2^{l_p} \le 2$. taking $m = 2^{l_p-1}$, then m is a positive integer satisfying $1 < 2pm \le 2$. Thus we have $0 \ge 2(pm-1) > -1$. The function

$$g_1(\theta) = (2\sin^2\theta)^{pm-1} \in L^1\left[0, \frac{\pi}{2}\right].$$

Let

$$C_1 = \int_0^{\frac{\pi}{2}} g_1(\theta)d\theta.$$

The function $g_2(x) = x^{\frac{1}{p}-m}$ is continuous in the interval $[0, 2]$, then there exist a sequence of polynomials $\{q_k(x)\}$ such that

$$|g_2(x) - q_k(x)| < \frac{A_\varepsilon}{M_k^p C_1 4^{k+3}}$$

so

$$I_k = \int_0^{\frac{\pi}{2}} |(2\sin^2\theta)^{\frac{1}{p}-m} - q_k(2\sin^2\theta)|^p g_1(\theta)d\theta \le \frac{A_\varepsilon}{M_k 4^{k+3}}.$$

Let

$$s_k(e^{i\theta}) = r_k(e^{i\theta})q_k(1+\cos\theta)(1+\cos\theta)^m,$$

then $s_k(e^{i\theta})$ is a trigonometric polynomial, and satisfying

$$J_k = \int_{-\pi}^{\pi} \left| r_k(e^{i\theta}) - \frac{s_k(e^{i\theta})}{(1+\cos\theta)^{\frac{1}{p}}} \right|^p d\theta$$

$$\le M_k^p \int_{-\pi}^{\pi} |1 - q_k(1+\cos\theta)(1+\cos\theta)^{m-\frac{1}{p}}|^p d\theta$$

$$= M_k^p \int_{-\pi}^{\pi} |(1+\cos\theta)^{\frac{1}{p}-m} - q_k(1+\cos\theta)|^p (1+\cos\theta)^{pm-1} d\theta$$

$$= M_k^p \int_{-\pi}^{\pi} |g_2(1+\cos\theta) - q_k(1+\cos\theta)|^p (1+\cos\theta)^{pm-1} d\theta$$

so

$$J_k \le \frac{1}{C_1 4^{k+3}} \int_{-\pi}^{\pi} (1+\cos\theta)^{pm-1} d\theta$$

Since $(1+\cos\theta) = 2\cos^2\frac{\theta}{2}$, so

$$J_k \le \frac{A_\varepsilon}{C_1 4^{k+3}} \int_{-\pi}^{\pi} (1+\cos\theta)^{pm-1} d\theta = \frac{A_\varepsilon}{C_1 4^{k+2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\cos\theta)^{pm-1} d\theta$$

so

$$J_k \le \frac{A_\varepsilon}{C_1 4^{k+2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2\sin\theta)^{pm-1} d\theta < \frac{A_\varepsilon}{4^{k+2}}.$$

Finally, let

$$g_k(\theta) = \frac{s_k(e^{i\theta})}{(1+\cos\theta)^{\frac{1}{p}}}$$

we obtain that

$$T_k = ||g - g_k||^p_{L^p([-\pi,\pi])}$$

$$\le ||g - r_k(e^{i\cdot})||^p_{L^p([-\pi,\pi])} + ||r_k(e^{i\cdot}) - g_k||^p_{L^p([-\pi,\pi])} \le \frac{2A_\varepsilon}{4^{k+3}}.$$

and

$$T_k = ||g - g_k||^p_{L^p([-\pi,\pi])} = \int_{-\pi}^{\pi} \left| f(\tan\frac{\theta}{2}) - s_k(e^{i\theta}) \right|^p \frac{d\theta}{1+\cos\theta}$$

$$= \int_{-\infty}^{\infty} \left| f(x) - s_k\left(\frac{i-x}{x+i}\right) \right|^p dx \le \frac{2A_\varepsilon}{4^{k+3}}.$$

the function

$$Q_k(z) = s_k\left(\frac{i-z}{z+i}\right)$$

is a rational function, the pole of $Q_k(z)$ is contained in $\{i,-i\}$, such that

$$||Q_k||_p^p = \int_{-\infty}^{\infty} |Q_k(x)|^p dx = \int_{-\infty}^{\infty}\left|s_k\left(\frac{i-x}{x+i}\right)\right|^p dx \leq \|f\|_p^p + \frac{3A_\varepsilon}{4^{k+3}}$$

and

$$||f-Q_k||_p^p = \int_{-\infty}^{\infty}\left|f(x) - s_k\left(\frac{i-x}{x+i}\right)\right|^p dx \leq \frac{2A_\varepsilon}{4^{k+3}}.$$

Therefore, the sequence of rational functions $\{Q_k(z)\}$ can be choose such that

$$||Q_k - Q_{k-1}||_p^p \leq \frac{A_\varepsilon}{4^{k+2}}. \quad (k = 2, 3, \cdots)$$

Let

$$R_1(z) = Q_1(z),\ R_k(z) = Q_k(z) - Q_{k-1}(z), \quad (k = 2, 3, \cdots)$$

the sequence $\{R_k(z)\}$ is a sequence of rational functions , the poles of each $R_k(z)$ is contained in $\{i,-i\}$, satisfying (7) and (8).

For case $p = 1$,

if $f \in L(\mathbb{R}) \cap \{f : \int_{-\infty}^{\infty} f(x)dx = 0\}$, and $R(x) \in A \cap \{R : \int_{-\infty}^{\infty} R(x)dx = 0\}$, where $A = \{R(x) = \frac{\sum\limits_{k=0}^{s}\alpha_k x^k}{(1+x^2)^l} : s < 2l\}$. Let $f = f^+ - f^-,\quad f^+ = \max\{f, 0\},\quad f^- = \max\{-f, 0\}$. Because A is dense in $L(\mathbb{R})$,(in fact),for any $\varepsilon_1 > 0$, there exist $R_1 \in A, R_2 \in A$ such that

$$\|R_1 - f^+\|_1 < \frac{\varepsilon_1}{2}, \|R_2 - f^-\|_1 < \frac{\varepsilon_1}{2}.$$

So that

$$\|R_1 - R_2 - f\|_1 = \|R_1 - R_2 - (f^+ - f^-)\|_1 \leq \|R_1 - f^+\|_1 + \|R_2 - f^-\|_1 < \varepsilon_1.$$

Since $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{\infty} f^+(x) - f^-(x)dx = 0$, that $\int_{-\infty}^{\infty} f^+(x)dx \int_{-\infty}^{\infty} f^-(x)dx$. Therefore, for any $\varepsilon_2 > 0$,

$$\begin{array}{rl} \int_{-\infty}^{\infty} R_1(x) - R_2(x)dx = & \int_{-\infty}^{\infty} R_1(x) - R_2(x)dx + \int_{-\infty}^{\infty} f(x)dx \\ = & \int_{-\infty}^{\infty} R_1(x) - f^-(x)dx + \int_{-\infty}^{\infty} f^+(x) - R_2(x)dx \\ = & \int_{-\infty}^{\infty} R_1(x) - f^+(x)dx + \int_{-\infty}^{\infty} f^-(x) - R_2(x)dx := \varepsilon_3 \end{array}.$$

We can let $R(x) = R_1(x) - R_2(x) - \frac{\varepsilon_3}{2\pi(1+x^2)}$, then $R(x) \in A$, and

$$\begin{array}{rl} \int_{-\infty}^{\infty} R(x)dx = & \int_{-\infty}^{\infty} R_1(x) - R_2(x) - \frac{\varepsilon_3}{2\pi(1+x^2)}dx \\ = & \int_{-\infty}^{\infty} R_1(x) - R_2(x)dx - \int_{-\infty}^{\infty} \frac{\varepsilon_3}{2\pi(1+x^2)}dx = \varepsilon_3 - \varepsilon_3 = 0 \end{array}$$

Thus, $R(x) \in A \cap \{R : \int_{-\infty}^{\infty} R(x)dx = 0\}$.
Moreover, for $\varepsilon_4 > 0$

$$\begin{aligned}\int_{-\infty}^{\infty} |R(x) - f(x)|dx = & \int_{-\infty}^{\infty} |R_1(x) - R_2(x) - f(x) - \tfrac{\varepsilon_3}{2\pi(1+x^2)}|dx \\ \le & \int_{-\infty}^{\infty} |R_1(x) - R_2(x) - f(x)|dx + \int_{-\infty}^{\infty} \tfrac{\varepsilon_3}{2\pi(1+x^2)}dx \le \varepsilon_4.\end{aligned}$$

The following proof of case of $p = 1$ is similarly to the case of $0 < p < 1$.
Thus, the proof of Lemma 2 is completed.

Lemma 3 Suppose that $0 < p \le 1$ and that $R \in L^p(\mathbb{R})$ is a rational function whose poles of $Q(z)$ is contained in $\{i, -i\}$, then there exist a rational functions P and Q such that $P \in H^p(\mathbb{C}_+)$, $Q \in H^p(\mathbb{C}_-)$ such that $R(z) = P(z) + Q(z)$ and

$$\|P\|_{H^p_+}^p + \|Q\|_{H^p_-}^p \le \left(1 + \frac{4\pi}{1-p}\right) \|R\|_p^p.$$

Proof If $0 < p < 1$, $R \in L^p(\mathbb{R})$ and R is a rational function, the poles of $R(z)$ is contained in $\{i, -i\}$, then $R(z)$ can be written as

$$R(z) = \sum_{k=-n}^{k=n} c_k(\beta(z))^k, \quad \text{where} \quad \beta(z) = \frac{i-z}{z+i}.$$

Therefore $\beta(x) = e^{i\theta(x)}$, where $\theta(x) = \arg(i - x) - \arg(x + i) \in (-\pi, \pi)$ for $x \in \mathbb{R}$. Define, for each $\varphi \in \mathbb{R}$,

$$P(z, \varphi) = \frac{(\beta(z))^m R(z)}{(\beta(z))^m - e^{i\varphi}}, \quad Q(z, \varphi) = \frac{(\beta(z))^{-m} R(z)}{(\beta(z))^{-m} - e^{-i\varphi}},$$

where m is any positive integer greater than the positive integer n. By Fubini's theorem,

$$\begin{aligned} I = \int_{-\pi}^{\pi}\int_{-\infty}^{+\infty} |P(x,\varphi)|^p dx d\varphi &= \int_{-\pi}^{\pi}\int_{-\infty}^{+\infty} \frac{|\beta(x)|^{mp}|R(x)|^p}{|(\beta(x))^m - e^{i\varphi}|^p} dx d\varphi \\ &= \int_{-\infty}^{+\infty}\int_{-\pi}^{\pi} \frac{|R(x)|^p}{|1 - e^{i(\varphi - m\theta(x))}|^p} d\varphi dx. \end{aligned}$$

Observe that

$$\int_{-\pi}^{\pi} \frac{2^p d\varphi}{|1 - e^{i\varphi - im\theta(x)}|^p} = \int_{-\pi}^{\pi} \frac{2^p d\varphi}{|1 - e^{i\varphi}|^p} = \int_{-\pi}^{\pi} \frac{d\varphi}{\sin^p \frac{\varphi}{2}} \le 4\int_0^{\frac{\pi}{2}} \frac{d\varphi}{(\frac{2}{\pi}\varphi)^p} \le \frac{2\pi}{1-p},$$

we obtain that

$$I \le \frac{2^{1-p}\pi}{1-p}\int_{-\infty}^{+\infty} |R(x)|^p dx.$$

Thus, there is a real number φ such that

$$\int_{-\infty}^{+\infty} |P(x,\varphi)|^p dx \le \frac{2\pi}{1-p}\int_{-\infty}^{+\infty} |R(x)|^p dx.$$

For this specially chosen real number φ, define $P(z) = P(z, \varphi)$, $Q(z) = Q(z, \varphi)$, then $R(z) = P(z) + Q(z)$. Since $m > n$, the functions P and Q are rational functions, the poles of $P(z)$ is contained in $\{-i\} \bigcup \{x_k : k = 0, 1, 2, \cdots, n-1\}$, the poles of $Q(z)$ is contained in $\{i\} \bigcup \{x_k : k = 0, 1, 2, \cdots, n-1\}$, where

$$x_k = \alpha(e^{\frac{i}{n}(\varphi+2k\pi)}) = \tan^2(\frac{1}{2n}(\varphi + 2k\pi))$$

are real numbers, so $P(z)$ is analytic in the upper plane $\mathbb{C}_+$, $Q(z)$ is analytic in the lower plane $\mathbb{C}_-$ and

$$\int_{-\infty}^{+\infty} |P(x)|^p dx \leq \frac{2\pi}{1-p} \int_{-\infty}^{+\infty} |R(x)|^p dx$$

$$\int_{-\infty}^{+\infty} |Q(x)|^p dx \leq \left(1 + \frac{2\pi}{1-p}\right) \int_{-\infty}^{+\infty} |R(x)|^p dx.$$

By Lemma 1, $P \in H^p(\mathbb{C}_+)$, $Q \in H^p(\mathbb{C}_-)$.

If $p = 1$, suppose that R is a rational function such that $R(x) \in A \cap \{R : \int_{-\infty}^{\infty} R(x)dx = 0\}$, where $A = \{R(x) = \frac{\sum\limits_{k=0}^{s} \alpha_k x^k}{(1+x^2)^l} : s < 2l\}$. Then $R(z)$ can be written as

$$R(z) = \frac{\sum\limits_{k=0}^{s} \alpha_k z^k}{(1+z^2)^l}$$

where $2l - 1 > s$.
Moreover, there is

$$R(z) = \frac{a_l}{(z-i)^l} + \cdots + \frac{a_2}{(z-i)^2} + \frac{a_1}{(z-i)} + \frac{b_l}{(z+i)^l} + \cdots + \frac{b_2}{(z+i)^2} + \frac{b_1}{(z+i)},$$

where $a_j, b_j, (j = 1, 2, ..., l)$ are constant complex number.
Let

$$p(z) = a_l(z+i)^l + \cdots + a_1(z-i)^{l-1}(z+i)^l + b_l(z-i)^l + \cdots + b_1(z+i)^{l-1}(z-i)^l,$$

then

$$R(z) = \frac{p(z)}{(z+i)^l(z-i)^l}.$$

Since $\int_{-\infty}^{\infty} R(x)dx = 0$, that $\int_{-\infty}^{\infty} R(x)dx = \int_{-\infty}^{\infty} \frac{p(x)}{(1+x^2)^l}dx = 0$, so $\int_{-\infty}^{\infty} p(x)dx = 0$.
Then we can get that $a_1 + b_1 = 0$.
Moreover, $\int_{-\infty}^{\infty} R(x)dx = 0$ and $\int_{-\infty}^{\infty} \frac{1}{(x-i)^k}dx = 0$, $\int_{-\infty}^{\infty} \frac{1}{(x+i)^k}dx = 0$ for $k \geq 2$ assures that

$$\int_{-\infty}^{\infty} \frac{a_1}{x-i} + \frac{b_1}{x-i}dx = \int_{-\infty}^{\infty} \frac{a_1}{x-i} - \frac{a_1}{x-i}dx = 2a_1 \int_{-\infty}^{\infty} \frac{i}{1+x^2}dx = 2\pi i a_1 = 0,$$

so that $a_1 = 0$, and $b_1 = -a_1 = 0$. Therefore, $R(z)$ can be written as

$$R(z) = \frac{a_l}{(z-i)^l} + \cdots + \frac{a_2}{(z-i)^2} + \frac{b_l}{(z+i)^l} + \cdots + \frac{b_2}{(z+i)^2}.$$

If let $P(z) = \frac{b_l}{(z+i)^l} + \cdots + \frac{b_2}{(z+i)^2}$ and $Q(z) = \frac{a_l}{(z-i)^l} + \cdots + \frac{a_2}{(z-i)^2}$,
then

$$R(z) = Q(z) + P(z).$$

It is easy to know that $Q(z)$ is analytic in $\mathbb{C}_-$, and $P(z)$ is analytic in $\mathbb{C}_+$. And moreover,

$$\begin{aligned} \int_{-\infty}^{\infty} |P(x+iy)|dx = & \int_{-\infty}^{\infty} |\tfrac{b_l}{(z+i)^l} + \cdots + \tfrac{b_2}{(z+i)^2}|dx \\ \leq & \int_{-\infty}^{\infty} \sum_{k=2}^{l} |\tfrac{|b_k|}{|z+i|^k}|dx \\ \leq & \int_{-\infty}^{\infty} \sum_{k=2}^{l} |\tfrac{|b_k|}{|x|^k}|dx < \infty. \end{aligned}.$$

Thus,

$$\sup_{y>0} \int_{-\infty}^{\infty} |P(x+iy)|dx < \infty.$$

Therefore, $P(z) \in H^1(\mathbb{C}_+)$.
It is similarly to prove that $Q(z) \in H^1(\mathbb{C}_-)$.
So the proof of Lemma 3 is completed.

Proof of Theorem 1 According to Lemma 1 and 2, there exist two sequences of rational functions $\{P_k(z)\}$ and $\{Q_k(z)\}$ such that $P_k \in H^p(\mathbb{C}_+)$, $Q_k \in H^p(\mathbb{C}_-)$,

$$\sum_{k=1}^{\infty} \left(\|P_k\|_p^p + \|Q_k\|_p^p\right) \leq 2\left(1 + \frac{2\pi}{1-p}\right)\|f\|_p^p$$

and

$$\lim_{n\to\infty} ||f - \sum_{k=1}^{n}(P_k + Q_k)||_{L^p} = 0.$$

Since

$$\|P_k\|_{H_+^p}^p = \|P_k\|_p^p, \quad \|Q_k\|_{H_-^p}^p = \|Q_k\|_p^p,$$

we see that (1) and (2) hold. For any $\delta > 0$,

$$|P_k(x+iy+i\delta)|^p \leq C_p\|P_k\|_p^p\delta^{-1}, |Q_k(x-iy-i\delta)|^p \leq C_p\|Q_k\|_p^p\delta^{-1},$$

where $C_p^p = \frac{2}{\pi} \leq 1$. (1) implies that (3) holds. Therefore the non-tangential boundary limits $g(x)$ and $h(x)$ of functions for $g \in H^p(\mathbb{C}_+)$ and $h \in H^p(\mathbb{C}_-)$ exist almost everywhere, respectively, (2) implies that $f(x) = g(x)+h(x)$ almost everywhere.

3 Proof of Theorems 2,3, 4 and 5

Proof of Theorem 2 Recall the Paley-Wiener Theorem asserting that $g \in H^2(\mathbb{C}_+)$ if and only if $\hat{g} \in L^2(\mathbb{R})$ with the support supp$\hat{g} \subset [0, \infty)$, such that

$$g(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \hat{g}(t) e^{itz} dt \qquad (z \in \mathbb{C}_+).$$

The following equality

$$\int_0^\infty |\hat{g}(t)|^2 dt = \|g\|^2_{H^2_+}$$

holds.

If $0 < p \le 1$, $f \in H^p(\mathbb{C}_+)$, for $\delta > 0$, let $f_\delta(z) = f(z + i\delta)$. Then $|f|^p$ is subharmonic and for $y > 0$,

$$|f_\delta(x + iy)| \leqslant C_p \|f\|_{H^p_+} \delta^{-\frac{2}{p}},$$

where $C_p^p = \frac{2}{\pi} \le 1$. Therefore

$$\int_{-\infty}^\infty |f_\delta(x+iy)|^2 dx \leqslant \int_{-\infty}^\infty |f_\delta(x+iy)|^p |f_\delta(x+iy)|^{2-p} dx \leqslant C_p^{2-p} \|f\|^2_{H^p_+} \delta^{1-\frac{2}{p}},$$

$$\int_{-\infty}^\infty |f_\delta(x+iy)| dx = \int_{-\infty}^\infty |f_\delta(x+iy)|^p |f_\delta(x+iy)|^{1-p} dx \leqslant C_p^{1-p} \|f\|_{H^p_+} \delta^{1-\frac{1}{p}}.$$

Therefore, the support supp$\hat{f}_\delta$ of $\hat{f}_\delta$ is in $[0, \infty)$,

$$f_\delta(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \hat{f}_\delta(s) e^{itz} dt,$$

Since for $y > 0$, $f_\delta(x + iy) = (P_y * f_\delta)(x)$. where

$$P_y(x) = \mathrm{Re}\left(\frac{i}{\pi z}\right) = \frac{y}{\pi(x^2 + y^2)}$$

is the Poisson kernel on the upper plane $\mathbb{C}_+$. Because that $f_\delta \in L^2(\mathbb{R})$, $P_y \in L^1(\mathbb{R})$ and $\hat{P}_y(s) = e^{-|s|y}$, for almost $s \in \mathbb{R}$, $\hat{f}_{\delta+y}(s) = \hat{f}_\delta(s) e^{-|s|y}$. So for almost $s \in \mathbb{R}$, $\hat{f}_{\delta+y}(s) e^{|s|(\delta+y)} = \hat{f}_\delta(s) e^{|s|\delta}$. i.e., the function $F(s) = \hat{f}_\delta(s) e^{|s|\delta}$ is independent of $\delta > 0$, with the support supp$F \subset [0, \infty)$ and

$$\int_{-\infty}^\infty |F(s)|^2 e^{-2|s|\delta} dt = \int_{-\infty}^\infty |\hat{f}_\delta(x)|^2 dx$$

$$= \int_{-\infty}^\infty |f_\delta(x)|^2 dx \leqslant C_p^{2-p} \|f\|^2_{H^p_+} \delta^{1-\frac{2}{p}}$$

and for any $\delta > 0$,

$$|F(s)| = |\hat{f}_\delta(s)| e^{|s|\delta} \leqslant \|f_\delta\|_1 e^{|s|\delta} \leqslant C_p^{1-p} \|f\|_{H^p_+} e^{|s|\delta} \delta^{-B_p},$$

where $B_p = \frac{1}{p} - 1 \geqslant 0$. Since

$$\inf\{|s|\delta - B_p \log \delta : \delta > 0\} = B_p - B_p(\log B_p - \log |s|),$$

therefore

$$|F(s)| \leqslant C_p^{1-p} \|f\|_{H_+^p} B_p^{-B_p} e^{B_p} |s|^{B_p}.$$

Thus F is a slowly increasing continuous function F whose support supp$F$ is in $[0, \infty)$. F can be regard as a tempered distribution, which can be defined by

$$(F, \hat{\varphi}) = \int_{\mathbb{R}} F(x)\hat{\varphi}(x)dx$$

for φ in the Schwarz class $\mathbb{S}$. So

$$\lim_{\delta\to 0} \int_{\mathbb{R}} f_\delta(x)\varphi(-x)dx = \lim_{\delta\to 0} \int_0^{+\infty} \hat{f}_\delta(x)\hat{\varphi}(x)dx,$$

$$= \lim_{\delta\to 0} \int_0^{+\infty} F(x)e^{-\delta x}\hat{\varphi}(x)dx = (F, \hat{\varphi}).$$

This completes the proof of Theorem 2.

Proof of Theorem 3. Since $g(z) \in H^p(\mathbb{C}_-)$ if and if $f(z) = \overline{g(\bar{z})} \in H^p(\mathbb{C}_+)$, we can see Theorem 3 holds.

Proof of the necessity of Theorem 4 and 5. If $0 < p \leq 2, f \in H_+^p(\mathbb{R})$, then there exists $f \in H^p(\mathbb{C}_+)$ such that $f(x)$ is a non-tangential boundary limit of the function $f(z)$. $|f|^p$ is subharmonic in $\mathbb{C}_+$, so

$$|f(x+iy)| \leq \frac{2}{\pi y}\|f\|_{H_+^p} \quad z = x+iy \in \mathbb{C}_+.$$

Let $f_n(z) = f(z + \frac{i}{n})$, then $f_n \in H^p(\mathbb{C}_+) \bigcap H^2(\mathbb{C}_+)$, so $f_n \in L^2(\mathbb{R}) \bigcap L^p(\mathbb{R})$, supp$\hat{f}_n \subset [0, +\infty)$, and

$$\lim_{n\to\infty} ||f - f_n||_p = \lim_{n\to\infty} ||f - f_n||_{H_+^p} = 0.$$

The necessity of Proof of Theorem 4 is proved. If $1 \leq p \leq 2$, the the Paley-Wiener Theorem asserts that

$$f_n(z) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \hat{f}_n(t)e^{itz}dt \quad (z \in \mathbb{C}_+) \tag{9}$$

is in $H^2(\mathbb{C}_+)$, and the following equality

$$\int_0^\infty |\hat{f}_n(t)|^2 dt = \|f_n\|_{H_+^2}^2$$

hold. By Hölder's inequality, for any $\varphi \in S(\mathbb{R})$,

$$\begin{aligned} |(\hat{f}_n, \varphi) - (\hat{f}, \varphi)| = & \ |\textstyle\int_{\mathbb{R}}(f(x+\frac{i}{n}) - f(x))\hat{\varphi}(x)dx| \\ \leq & \ ||f_n - f||_p ||\hat{\varphi}||_q \to 0, \quad y \to 0, \end{aligned}$$

where $\frac{1}{p}+\frac{1}{q}=1$. Therefore, for any $\varphi\in S(\mathbb{R})$, whose support of $\hat{\varphi}$, $supp\varphi\subset(-\infty,0]$, we have $(\hat{f},\varphi)=0$, so

$$supp\hat{f}\subset[0,\infty).$$

(6) can be derived from (9) by letting $n\to\infty$ and Hölder's inequality. The necessity of theorem 5 is proved.

Proof of the sufficiency of Theorem 4. Let $0<p<1$, if there exists a sequence $\{f_n\}$ satisfying $f_n\in L^2(\mathbb{R})\bigcap L^p(\mathbb{R})$ and $\mathrm{supp}\hat{f}_n\subset[0,+\infty)$, the Paley-Wiener Theorem asserts that

$$f_n(z)=\frac{1}{\sqrt{2\pi}}\int_0^{\infty}\hat{f}_n(t)e^{itz}dt=\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{f_n(t)}{t-z}dt\qquad(z\in\mathbb{C}_+)$$

is in $H^2(\mathbb{C}_+)$. Therefore, $f_n\in H^p(\mathbb{C}_+)$ by Corollary 4.3 in [6], and $||f_m-f_n||_p=||f_m-f_n||_{H^p_+}$. Thus there exists a $f\in H^p(\mathbb{C}_+)$ such that

$$\lim_{n\to\infty}||f-f_n||_p=\lim_{n\to\infty}||f-f_n||_{H^p_+},$$

so $f(x)\in H^p_+(\mathbb{R})$.

Proof of the sufficiency of Theorem 5 . Let $1\le p\le 2$, If $f(x)\in L^p$ and $supp\hat{f}\subset[0,\infty)$,

$$|\chi_{[0,\infty)}(t)e^{2\pi iz\cdot t}\hat{f}(t)|=\chi_{[0,\infty)}(t)|\hat{f}(t)|e^{-2\pi y\cdot t}\in L^1(\mathbb{R}^n).$$

$\chi_{[0,\infty)}(t)$ is the characteristic function of $[0,\infty)$, that is, $\chi_{[0,\infty)}(t)=1$, for $t\in[0,\infty)$, and otherwise zero. So the function

$$G(z)=\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}^n}e^{izt}\hat{f}(t)dt=\int_{\mathbb{R}}\chi_{[0,\infty)}(t)e^{izt}\hat{f}(t)dt$$

is holomorphic in $\mathbb{C}_+$. Therefore, to complete the proof of Theorem 4, it is sufficient to prove that $G(z)\in H^p(\mathbb{C}_+)$ and the boundary value of $G(z)$ is $f(x)$ as $y\to 0$. Fixed $z\in\mathbb{C}_+$, let

$$g_z(t)=\chi_{[0,\infty)}(t)\frac{e^{izt}}{\sqrt{2\pi}},\quad \widetilde{g}_z(t)=g_z(-t)$$

then $g_z\in L^1(\mathbb{R})\bigcap L^2(\mathbb{R})$, $\hat{g}_z(s)=\frac{1}{2\pi i(s-z)}$ and

$$\begin{array}{rl}G(z)= & \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}}\chi_{[0,\infty)}(t)e^{izt}\frac{1}{\sqrt{2\pi}}(\int_{\mathbb{R}}e^{-ist}F(s)ds)dt\\ = & \frac{1}{2\pi i}\int_{\mathbb{R}}\frac{f(s)ds}{s-z}.\end{array}.$$

For $z,w\in\mathbb{C}_+$, let

$$I(z,w)=\frac{1}{4\pi^2}\int_{\mathbb{R}}\frac{f(t)dt}{(t-z)(t-w)},$$

then

$$I(z,w)=\int_{\mathbb{R}}\hat{g}_z(t)f(t)\hat{g}_w(-t)dt.$$

Since, for $z, w \in \mathbb{C}_+$, $\sqrt{2\pi}\hat{g}_z(t)\hat{g}_w(-t) = \widehat{g_z * \widetilde{g}_w}(t)$, where

$$(g_z * \widetilde{g}_w)(t) = \int_{\mathbb{R}} g_z(\xi)\widetilde{g}_w(t-\xi)d\xi = \int_{\mathbb{R}} g_z(\xi)g_w(\xi - t)d\xi$$

$$= \frac{1}{2\pi}\int_{\mathbb{R}} \chi_{[0,\infty)}(\xi)e^{2\pi i z\xi}\chi_{[0,\infty)}(\xi - t)e^{2\pi i w(\xi - t)}d\xi,$$

so

$$I(z,w) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \hat{f}(s)\chi_{[0,\infty)}(s)(g_z * \widetilde{g}_w)(s)ds$$

$$= \frac{1}{(\sqrt{2\pi})^3}\int_{\mathbb{R}} \hat{f}(s)\chi_{[0,\infty)}(s)\int_{\mathbb{R}} \chi_{[0,\infty)}(\xi)e^{2\pi i z\cdot\xi}\chi_{[0,\infty)}(\xi - s)e^{2\pi i w\cdot(\xi - s)}d\xi ds.$$

Therefore, by Fubini's theorem and

$$\chi_{[0,\infty)}(t)\chi_{[0,\infty)}(t+s)\chi_{[0,\infty)}(s) = \chi_{[0,\infty)}(t)\chi_{[0,\infty)}(s),$$

$$\begin{aligned} I(z,w) = & \tfrac{1}{(\sqrt{2\pi})^3}\int_{\mathbb{R}}\int_{\mathbb{R}} \chi_{[0,\infty)}(s)\chi_{[0,\infty)}(t)\chi_{[0,\infty)}(t+s)e^{iz(s+t)}e^{iwt}\hat{f}(s)dsdt \\ = & \tfrac{1}{(\sqrt{2\pi})^3}\int_{\mathbb{R}}\int_{\mathbb{R}} \chi_{[0,\infty)}(t)\chi_{[0,\infty)}(s)e^{izs}e^{i(z+w)t}\hat{f}(s)dtds \\ = & \tfrac{i}{2\pi}\tfrac{G(z)}{z+w}. \end{aligned}$$

Therefore, for $z \in \mathbb{C}_+$, we have $-\bar{z} \in \mathbb{C}_+$,

$$I(z, -\bar{z}) = \frac{i}{2\pi}\frac{G(z)}{z - \bar{z}} = \frac{G(z)}{4\pi y}, \ z = x + iy, y > 0.$$

So

$$G(z) = \int_{\mathbb{R}} \frac{4\pi y f(t)dt}{(2\pi i)^2(t-z)(-t-\bar{z})} = \int_{\mathbb{R}} f(t)P(x-t,y)dt,$$

where $P(x,y) = \frac{y}{\pi(x^2+y^2)}$ is the Poisson Kernel associated with the upper half plane $\mathbb{C}_+$. Therefore the boundary value of $G(z)$ is $f(x)$ as $y \to 0$ and $G(z) \in H^p(\mathbb{C}_+)$. Thus, the Theorem 5 is proved.

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