

Progress on Hardy-type Inequalities

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Abstract

This paper surveys some of our recent progress on Hardy-type inequalities which consist of a well-known topic in Harmonic Analysis. In the first section, we recall the original probabilistic motivation dealing with the stability speed in terms of the L^2 -theory. A crucial application of a result by Fukushima and Uemura (2003) is included. In the second section, the non-linear case (a general Hardy-type inequality) is handled with a direct and analytic proof. In the last section, it is illustrated that the basic estimates presented in the first two sections can still be improved considerably.

This paper mainly concerns with the following Hardy-type inequality

$$\left(\int_{-M}^N |f - \pi(f)|^q d\mu \right)^{1/q} \leq A \left(\int_{-M}^N |f'|^p d\nu \right)^{1/p}, \quad (1)$$

where $p, q \in (1, \infty)$, μ and ν are Borel measures on an interval $[M, N]$ ($M, N \leq \infty$). Here, we assume that $\mu[-M, N] < \infty$ and define a probability measure $\pi = (\mu[-M, N])^{-1}\mu$. Then $\pi(f) := \int f d\pi$. The functions f are assumed to be absolutely continuous on $(-M, N)$ and belong to $L^q(\mu)$. For simplicity, we may also write the inequality as

$$\|f - \pi(f)\|_{L^q(\mu)} \leq A \|f'\|_{L^p(\nu)}.$$

To save our notation, assume the constant A to be optimal. The linear case that $p = q = 2$ is discussed in the next section, where a result by Fukushima and Uemura [14] plays an important role. The general case is studied in Section 2. In the last section, we show the possibility for improving further the basic estimates of the optimal constant.

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1 Linear case: $p = q = 2$.

Let us recall the original probabilistic problem. Throughout this section, we fix $p = q = 2$. Consider a second-order elliptic operator on $(-M, N)$:

$$L = a(x) \frac{d^2}{dx^2} + b(x) \frac{d}{dx}, \quad a(x) > 0 \text{ on } (-M, N).$$

Then the two measures used in inequality (1) are as follows

$$\mu(dx) = \frac{e^{C(x)}}{a(x)} dx, \quad \nu(dx) = e^{C(x)} dx, \quad C(x) := \int_{\theta}^x \frac{b}{a}, \quad (2)$$

here in the last integral and in what follows, the Lebesgue measure dx is omitted, $\theta \in (-M, N)$ is a reference point. Denote by $\{P_t\}_{t \geq 0}$ the (maximal) semigroup generated by L on $L^2(\mu)$. Here ‘‘maximal’’ means the Dirichlet form having the maximal domain which we learnt earlier from Fukushima [12] (cf. [1; §6.7]). We are interested in the stability speed, for instance, the L^2 -exponential convergence rate ε :

$$\|P_t f - \pi(f)\|_{L^2(\mu)} \leq \|f - \pi(f)\|_{L^2(\mu)} e^{-\varepsilon t}, \quad t \geq 0.$$

Then, it turns out that the largest rate ε_{\max} coincides with A^{-1} given in (1) (cf. [1; Theorem 9.1]).

We can state one of our recent results as follows.

Theorem 1.1 [4; Theorem 10.2] Let $a > 0$, a and b be continuous on $[-M, N]$ (or $(-M, N]$ if $M = \infty$, for instance). Assume that $\mu(-M, N) < \infty$. Then for the optimal constant A , we have the basic estimates: $\kappa \leq A \leq 2\kappa$, where

$$\kappa^{-2} = \inf_{-M < x < y < N} [\mu(-M, x)^{-1} + \mu(y, N)^{-1}] \hat{\nu}(x, y)^{-1},$$

$$\mu(x, y) = \int_x^y d\mu, \text{ and } \hat{\nu}(dx) = e^{-C(x)} dx.$$

The continuity assumption on a and b is not essential and will be removed in the next section. To understand the proof of this theorem, assume that $M, N < \infty$. Then the general case can be done by an approximating procedure. The optimal constant actually describes an eigenvalue $\lambda_1 (= A^{-2})$ defined by

$$\lambda_1 = \inf \left\{ \|f'\|_{L^2(\nu)}^2 : \pi(f) = 0, \|f\|_{L^2(\mu)} = 1 \right\}.$$

Let g be the eigenfunction corresponding to λ_1 :

$$Lg = -\lambda_1 g, \quad g \neq 0.$$

To ignore a constant (say $\pi(g)$) from g , making derivative on both sides of the equation and replacing g' by f , we get

$$L_S f = -\lambda_1 f,$$

where

$$L_S = a(x) \frac{d^2}{dx^2} + (a'(x) + b(x)) \frac{d}{dx} + b'(x)$$

which is a Schrödinger operator. Because $g'(-M) = 0 = g'(N)$, the boundary condition for L_S becomes $f(-M) = 0 = f(N)$. This leads to the principal eigenvalue of L_S :

$$\lambda_S = \sup_{f \in \mathcal{F}} \inf_{x \in (-M, N)} \frac{-L_S f}{f}(x),$$

where

$$\mathcal{F} = \{f \in \mathcal{C}^1[-M, N] \cap \mathcal{C}^2(-M, N) : f(-M) = f(N) = 0, f|_{(-M, N)} > 0\}.$$

Note that the zero-point of g in the original eigenequation for L is located inside of the interval $(-M, N)$, not explicitly known; the zero-points for the eigenvalue λ_S are located at the boundaries $-M$ and N only. This is the advantage of L_S . However, there is an extra term b' in operator L_S which costs some trouble as usual. To avoid this, we rewrite L as

$$L = \frac{d}{d\mu} \frac{d}{d\hat{\nu}}.$$

Then we define a dual operator of L as follows.

$$L^* = \frac{d}{d\mu^*} \frac{d}{d\hat{\nu}^*} := \frac{d}{d\hat{\nu}} \frac{d}{d\mu},$$

i.e. an exchange of μ and $\hat{\nu}$. More explicitly,

$$L^* = a(x) \frac{d^2}{dx^2} + (a'(x) - b(x)) \frac{d}{dx}.$$

Next, define

$$\lambda_0^* = \sup_{f^* \in \mathcal{F}} \inf_{x \in (-M, N)} \frac{-L^* f^*}{f^*}(x).$$

In view of the next result which is crucial in proving Theorem 1.1, it is clear that we have thus removed the extra term b' in the operator L_S .

Proposition 1.2 We have $A^{-2} = \lambda_1 = \lambda_S = \lambda_0^*$.

Here, in proving $\lambda_1 = \lambda_S$, we have used a mathematical tool — the coupling technique (cf. [8] or [2]). We have also used another tool — dual technique in proving $\lambda_S = \lambda_0^*$. It is interesting that they are the main tools used in the study on interacting particle systems (cf. [1] and references therein). To obtain the basic estimates listed in Theorem 1.1, we need one more mathematical tool — the capacity method. The next result is taken from Fukushima & Uemura [14] and [2, 3], see also [13].

Theorem 1.3 For a regular transient Dirichlet form $(D, \mathcal{D}(D))$ with locally compact state space (E, \mathcal{E}) , the optimal constant $A_{\mathbb{B}}$ in the Poincaré-type inequality

$$\|f^2\|_{\mathbb{B}} \leq A_{\mathbb{B}}^2 D(f), \quad f \in \mathcal{C}_K^\infty(E)$$

satisfies $B_{\mathbb{B}} \leq A_{\mathbb{B}} \leq 2B_{\mathbb{B}}$, where $\|\cdot\|_{\mathbb{B}}$ is the norm in a normed linear space \mathbb{B} and

$$B_{\mathbb{B}}^2 = \sup_{\text{compact } K} \text{Cap}(K)^{-1} \|\mathbb{1}_K\|_{\mathbb{B}}.$$

The space \mathbb{B} can be very general, for instance $L^p(\mu)$ ($p \geq 1$) or the Orlicz spaces. In the present context, $D(f) = \int_{-M}^N f'^2 e^C = \|f'\|_{L^2(\nu)}^2$, $\mathcal{D}(D)$ is the closure of $\mathcal{C}_K^\infty(-M, N)$ with respect to the norm $\|\cdot\|_D$: $\|f\|_D^2 = \|f\|^2 + D(f)$, and

$$\text{Cap}(K) = \inf \{D(f) : f \in \mathcal{C}_K^\infty(-M, N), f|_K \geq 1\}.$$

Note that we have the universal factor 2 here and the isoperimetric constant $B_{\mathbb{B}}$ has a very compact form. We now need to compute the capacity only. The problem is that the capacity is usually not explicitly computable. For instance, at the moment, we do not know how to compute it for Schrödinger operators even for the elliptic operators having killings. Very lucky, we are able to compute the capacity for the one-dimensional elliptic operators. The result has a simple expression:

$$B_{\mathbb{B}}^2 = \sup_{-M < x < y < N} [\hat{\nu}(-M, x)^{-1} + \hat{\nu}(y, N)^{-1}]^{-1} \|\mathbb{1}_{(x, y)}\|_{\mathbb{B}}.$$

It looks strange to have double inverse here. So, making inverse in both sides, we get

$$B_{\mathbb{B}}^{-2} = \inf_{-M < x < y < N} [\hat{\nu}(-M, x)^{-1} + \hat{\nu}(y, N)^{-1}] \|\mathbb{1}_{(x, y)}\|_{\mathbb{B}}^{-1}.$$

Applying this result to $\mathbb{B} = L^1(\mu)$, we obtain the solution to the case having double Dirichlet boundaries: $\lambda_0 = A_{L^1(\mu)}^{-2}$ and

$$\kappa_0^{-2} = B_{L^1(\mu)}^{-2} = \inf_{-M < x < y < N} [\hat{\nu}(-M, x)^{-1} + \hat{\nu}(y, N)^{-1}] \mu(x, y)^{-1}.$$

Applying the last result to the dual process, we have not only

$$(\kappa_0^*)^2 \leq \lambda_1 = \lambda_S = \lambda_0^* \leq 4(\kappa_0^*)^2$$

but also

$$\begin{aligned} (\kappa_0^*)^{-2} &= \inf_{x < y} [\hat{\nu}^*(-M, x)^{-1} + \hat{\nu}^*(y, N)^{-1}] \mu^*(x, y)^{-1} \\ &= \inf_{x < y} [\mu(-M, x)^{-1} + \mu(y, N)^{-1}] \hat{\nu}(x, y)^{-1} \\ &= \kappa^{-2}. \end{aligned}$$

We have thus arrived at the assertion of Theorem 1.1. Refer to [4; §10] and [5] for more details.

To conclude this section, we remark that the use of the capacity is natural in the higher dimensions, since in which the boundary may be very complicated. However, it seems unnecessary to use it in the present one-dimensional situation. This leads to a direct proof of Theorem 1.1, given in the next section, without using the three mathematical tools just mentioned above.

2 Non-linear case

We now return to our general inequality (1). First, we need a measure $\hat{\nu}$, as in the last section, deduced from ν . Let $\nu^\#$ be the absolutely continuous part of ν with respect to the Lebesgue measure. Then, define

$$\hat{\nu}(dx) = \hat{\nu}_p(dx) = \left(\frac{d\nu^\#}{dx} \right)^{-1/(p-1)} dx, \quad p > 1.$$

Next, we need a universal factor

$$k_{q,p} = \left[\frac{q-p}{pB\left(\frac{p}{q-p}, \frac{p(q-1)}{q-p}\right)} \right]^{1/p-1/q} \leq 2 \quad \text{if } q \geq p,$$

where $B(\alpha, \beta)$ is the Beta function. In particular (as the limit of $q \downarrow p$),

$$k_{p,p} = p^{1/p}(p^*)^{1/p^*},$$

where p^* is the conjugate of $p \in (1, \infty)$: $1/p + 1/p^* = 1$.

Theorem 2.1 [6; Theorem 2.6] Let $\mu(-M, N) < \infty$. Then the optimal constant A in the Hardy-type inequality (1) satisfies

- (1) the upper estimate $A \leq k_{2,p}B^*$ for $1 < p \leq 2 \leq q < \infty$ once the pure point part of μ (denoted by μ_{pp}) vanishes, and
- (2) the lower estimate $A \geq B_*$ for $1 < p, q < \infty$, where

$$B^* = \sup_{x \leq y} \frac{\hat{\nu}(x, y)^{(p-1)/p}}{\left\{ \mu(-M, x)^{\frac{p}{q(1-p)}} + \mu(y, N)^{\frac{p}{q(1-p)}} \right\}^{(p-1)/p}},$$

$$B_* = \sup_{x \leq y} \frac{\hat{\nu}(x, y)^{(p-1)/p}}{\left\{ \mu(-M, x)^{\frac{1}{1-q}} + \mu(y, N)^{\frac{1}{1-q}} \right\}^{(q-1)/q}}.$$

Moreover, $B_* \leq B^* \leq 2^{1/p-1/q}B_*$ once $q \geq p$.

Here are some remarks on Theorem 2.1.

- (a) The isoperimetric constants B^* and B_* are expressed explicitly in measures μ and $\hat{\nu}$.
- (b) The boundaries $-M$ and N symmetric in the formulas of B^* and B_* .
- (c) Even through $B^* \geq B_*$ in general, but the rough ratio $k_{q,p}2^{1/p-1/q}$ of the upper and lower bounds is still ≤ 2 .
- (d) When $q = p$, we have

$$B^* = B_* = \sup_{x \leq y} \frac{\hat{\nu}(x, y)^{(p-1)/p}}{\left\{ \mu(-M, x)^{\frac{1}{1-p}} + \mu(y, N)^{\frac{1}{1-p}} \right\}^{(p-1)/p}}.$$

- (e) Ignoring the $\mu(-M, x)$ -term in the expression of B^* or B_* , we obtain

$$B^+ = \sup_y \hat{\nu}(-M, y]^{1/p^*} \mu(y, N)^{1/q}.$$

Similarly, Ignoring the $\mu(y, N)$ -term in the expression of B^* or B_* , we obtain

$$B^- = \sup_x \hat{\nu}(x, N)^{1/p^*} \mu(-M, x)^{1/q}.$$

We have thus returned to one of the main results in the study of Hardy-type inequalities.

Theorem 2.2 (1920—1992) For the Hardy-type inequalities

$$\|f\|_{L^q(\mu)} \leq A^+ \|f'\|_{L^p(\nu)}, \quad f(-M) = 0$$

and

$$\|f\|_{L^q(\mu)} \leq A^- \|f'\|_{L^p(\nu)}, \quad f(N) = 0,$$

we have the basic estimates $B^\pm \leq A^\pm \leq k_{q,p} B^\pm$. Moreover, the factor $k_{q,p}$ is sharp.

There is a long history about the development of Theorem 2.2. The reader is urged to refer to [15] and [6] for a long list of references including five books.

Having the experience in proving Theorem 1.1 and known the history of Theorem 2.2, it is hardly imaginable how to find a direct proof of Theorem 1.1, or even much more general Theorem 2.2, without using capacity. To have a test, let us introduce the proof of a hard part — the upper estimate of Theorem 2.2.

The idea is starting from Theorem 2.2. For this, we split the interval $(-M, N)$ into two parts: $(-M, \theta)$ and (θ, N) ,



Denote by A_θ^- the optimal constant on the left subinterval $(-M, \theta)$ and by A_θ^+ the one on the right subinterval (θ, N) with the same boundary condition $f(\theta) = 0$. Then, we can rewrite Theorem 2.2 as follows.

Known Theorem Let $q \geq p$. Then we have $B_\theta^\pm \leq A_\theta^\pm \leq k_{q,p} B_\theta^\pm$, where

$$B_\theta^- = \sup_{x < \theta} \hat{\nu}(x, \theta)^{1/p^*} \mu(-M, x)^{1/q}, \quad B_\theta^+ = \sup_{y > \theta} \hat{\nu}(\theta, y)^{1/p^*} \mu(y, N)^{1/q}.$$

Proof of the upper estimate: $k_{2,p} B^* \geq A$.

Rewrite B^* as

$$\begin{aligned} B^* &= \sup_{x \leq y} \frac{\hat{\nu}(x, y)^{(p-1)/p}}{\left\{ \mu(-M, x)^{\frac{p}{q(1-p)}} + \mu(y, N)^{\frac{p}{q(1-p)}} \right\}^{(p-1)/p}} \\ &= \left\{ \sup_{x \leq y} \frac{\hat{\nu}(x, y)}{\mu(-M, x)^{\frac{p}{q(1-p)}} + \mu(y, N)^{\frac{p}{q(1-p)}}} \right\}^{(p-1)/p} \\ &=: \left\{ \sup_{x \leq y} \frac{\hat{\nu}(x, y)}{\varphi(x) + \psi(y)} \right\}^{1/p^*}. \end{aligned}$$

By proportional property, we have

$$\frac{\hat{\nu}(x, y)}{\varphi(x) + \psi(y)} = \frac{\hat{\nu}(x, \theta) + \hat{\nu}(\theta, y)}{\varphi(x) + \psi(y)} \geq \left\{ \frac{\hat{\nu}(x, \theta)}{\varphi(x)} \wedge \frac{\hat{\nu}(\theta, y)}{\psi(y)} \right\}, \quad \theta \in (x, y).$$

Here $a \wedge b = \min\{a, b\}$ and similarly $a \vee b = \max\{a, b\}$. Hence (omit what in $\{\dots\}$)

$$\frac{\hat{\nu}(x, y)}{\varphi(x) + \psi(y)} \geq \sup_{\theta \in (x, y)} \{\dots\}.$$

Then

$$\begin{aligned} \sup_{x \leq y} \frac{\hat{\nu}(x, y)}{\varphi(x) + \psi(y)} &\geq \sup_{x \leq y} \sup_{\theta \in (x, y)} \{\dots\} \\ &= \sup_{\theta} \sup_{(x, y) \ni \theta} \{\dots\} \\ &= \left\{ \left[\sup_{x \leq \theta} \frac{\hat{\nu}(x, \theta)}{\varphi(x)} \right] \wedge \left[\sup_{y > \theta} \frac{\hat{\nu}(\theta, y)}{\psi(y)} \right] \right\}. \end{aligned}$$

Making power $1/p^*$ on both sides, by definition of B_θ^\pm , we obtain

$$B^* \geq \sup_{\theta} (B_\theta^- \wedge B_\theta^+).$$

Here a problem appears: we need \vee rather than \wedge on the right-hand side. To overcome this, we assume that $\mu_{pp} = 0$. Then, there exists $\bar{\theta} \in (-M, N)$ such that $B_{\bar{\theta}}^- = B_{\bar{\theta}}^+$. Therefore

$$B^* \geq \sup_{\theta} (B_\theta^- \wedge B_\theta^+) \geq B_{\bar{\theta}}^- = B_{\bar{\theta}}^- \vee B_{\bar{\theta}}^+$$

and then

$$\begin{aligned}
k_{q,p}B^* &\geq (k_{q,p}B_{\bar{\theta}}^-) \vee (k_{q,p}B_{\bar{\theta}}^+). \\
&\geq A_{\bar{\theta}}^- \vee A_{\bar{\theta}}^+ \quad (\text{Known Theorem}) \\
&\geq \inf_{\bar{\theta}} (A_{\bar{\theta}}^- \vee A_{\bar{\theta}}^+) \\
&\geq A \quad (\text{splitting technique}).
\end{aligned}$$

Here each step holds for all $q \geq p$ except the last one. In which, some additional work is required, due to the appearance of $f - \pi(f)$ rather than f only. We prove the conclusion first for $q = 2 \geq p$ and then extend it to $q \geq 2$, even to a large class of normed linear space \mathbb{B} , as used in Theorem 1.3, using a known lifting procedure (cf. [2; §6.3]). Note that in the proof above, we use a bridge θ to combine the known results on two subintervals together. But then remove it, otherwise, $\bar{\theta}$ for instance, may not be computable. Nevertheless, it should be understandable that the present analytic proof does not use the coupling, duality, or capacitary techniques.

Actually, much more topics are studied in [6]: the bilateral Dirichlet boundaries, logarithmic Sobolev inequalities, Nash inequalities, and so on.

3 Improvements of the basic estimates

Note that for two given numbers having smaller ratio, their difference can be still quite big. Hence, it is meaningful to improve the basic estimates introduced in the last two sections. In this section, we show the possibility in doing so by a simplest example: $\mu = dx$ and $\nu = dx$ on $(0, 1)$. We need to consider the following Hardy-type inequality

$$\|f\|_{L^q(\mu)} \leq A \|f'\|_{L^p(\nu)}, \quad f(0) = 0 \quad (3)$$

only since the other cases (the ergodic case in particular) can be reduced to this one by symmetry. The basic estimates for the optimal constant A in (3) are given in Theorem 2.2.

Example 3.1 Let $\mu = dx$ and $\nu = dx$ on $(0, 1)$. Then the optimal constant A in (3) is given as follows.

(1) When $p = q = 2$, we have $A = 2/\pi$.

(2) When $p = q \in (1, \infty)$, we have

$$A = \frac{p}{\pi(p-1)^{1/p}} \sin \frac{\pi}{p}.$$

(3) For general $p, q \in (1, \infty)$, we have

$$A = \frac{p^{\frac{1}{q}} q^{1-\frac{1}{p}} (pq + p - q)^{\frac{1}{p}-\frac{1}{q}}}{(p-1)^{\frac{1}{p}} \text{B}\left(\frac{1}{q}, 1 - \frac{1}{p}\right)}.$$

This simplest example already shows that it is nontrivial from the special case $p = q = 2$ to the general one.

Proposition 3.2 [7] For the optimal constant A in the last example, we have the following improved estimates:

$$B \leq \bar{\delta}_1 \leq A \leq A^* \leq \delta_1 \leq k_{q,p}B,$$

where

$$B = \frac{p^{1/q}((p-1)q)^{1-1/p}}{(pq+p-q)^{1-1/p+1/q}},$$

$$\bar{\delta}_1 = \frac{p^{1/q}((p-1)(q+1))^{1-1/p}}{(pq+p-q)^{1-1/p+1/q}},$$

$$A^* = \left[\frac{p^*}{q} \right]^{\frac{1}{q}} \left[\frac{p^*+q}{\pi p^*} \sin \frac{\pi p^*}{p^*+q} \right]^{\frac{1}{p^*} + \frac{1}{q}} \quad \boxed{= A \text{ if } q = p}$$

$$\delta_1 = \frac{1}{(q\gamma^*/p^* + 1)^{\frac{1}{q}}} \left[\sup_{x \in (0,1)} \frac{1}{x^{\gamma^*}} \int_0^x (1 - y^{q/\gamma^* p^* + 1})^{\frac{p^*}{q}} dy \right]^{\frac{1}{p^*}}, \quad \gamma^* := \frac{q}{p^* + q}$$

The results in Proposition 3.2 are shown by Figures 1–4.

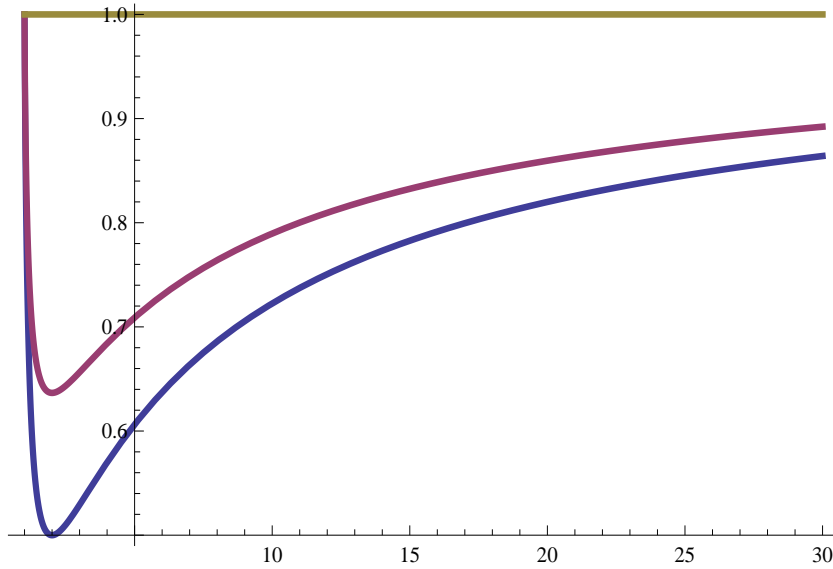


Figure 1 The basic estimates of A : $p = q \in (1, 30)$.

Note that in the case that $p = q$, we have $A^* = A$. So there are five curves only in Figure 2.

The improvements are surprisingly effective. Note that a suitable convex mean of the new upper and lower bounds should provides a quite precise app-

ximation of A . However, the convex means of the basic estimates do not have this property. When $p = q$, much more refined results can be found from [9, 10, 11].

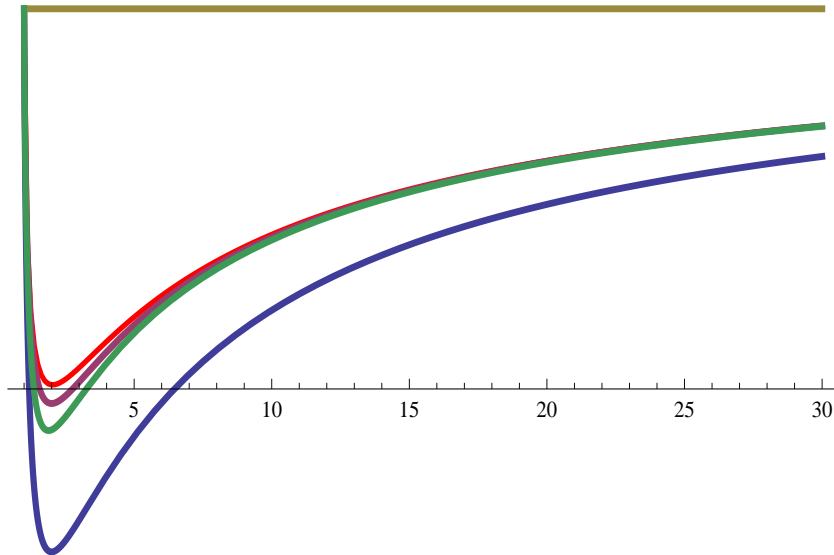


Figure 2 The basic estimates of A and their improvements: $p = q \in (1, 30)$.

Next, since $q \geq p$, we may write $q = p + r$ for some $r \geq 0$. In Figures 3 and 4, there are six curves, three of them are upper estimates and two of them are lower ones. The third curve from the bottom is the exact one; the top and the bottom curves consist of the basic estimates of the exact one. The other three curves are the improvements of the basic estimates.

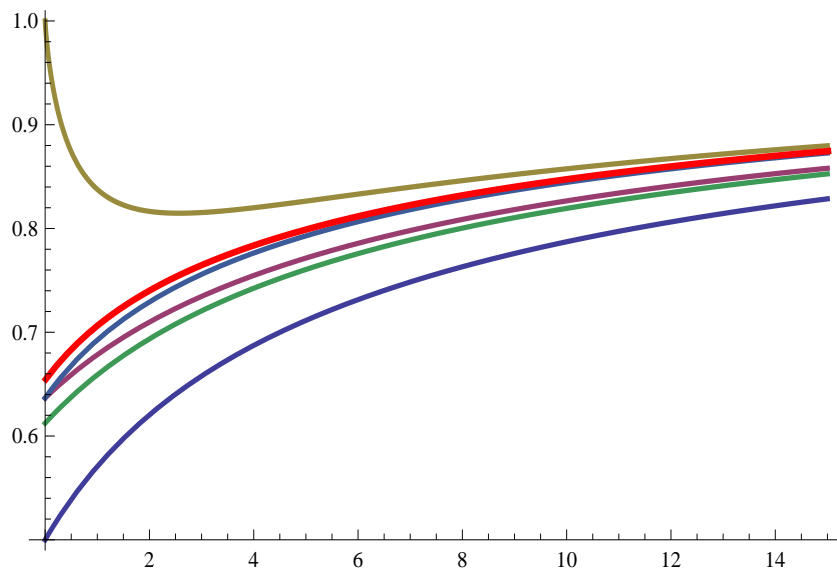


Figure 3 The basic estimates and their improvements: $p = 2$, $r \in (0, 15)$.

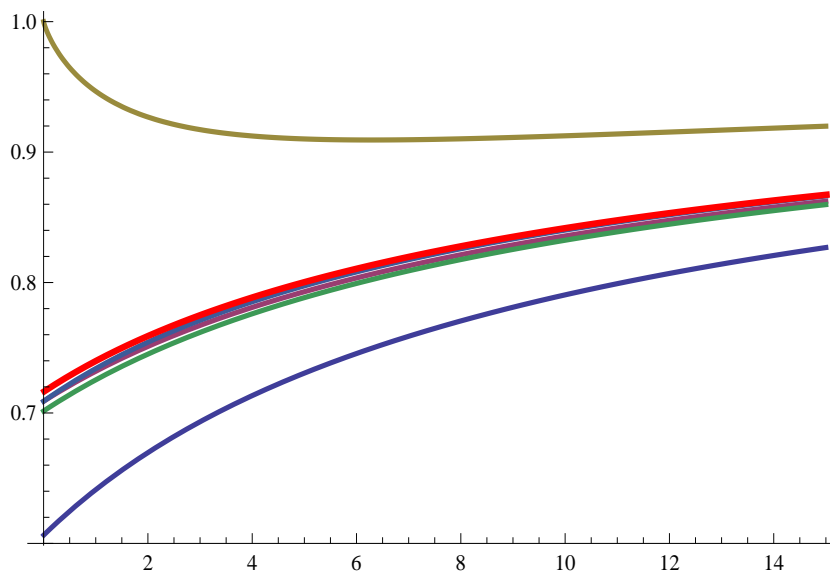


Figure 4 The basic estimates and their improvements: $p = 5$, $r \in (0, 15)$.

Note that in Figure 4, the new upper bounds and lower bounds are almost overlapped with A . In general, they are closer when p or $q \geq p$ is larger.

Finally, we mention that the main results in this note: Theorems 1.1 and 2.1, and Proposition 3.2 are new addition to the context of Hardy-type inequalities.

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