

On the exterior Dirichlet problem for Hessian equations

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Abstract

In this paper, we establish a theorem on the existence of the solutions of the exterior Dirichlet problem for Hessian equations with prescribed asymptotic behavior at infinity. This extends a result of Caffarelli and Li in [3] for the Monge-Ampère equation to Hessian equations.

1 Introduction

In this paper, we consider the solvability of the Dirichlet problem for Hessian equations

$$\sigma_k(\lambda(D^2u)) = 1 \tag{1.1}$$

on exterior domains $\mathbb{R}^n \setminus D$, where D is a bounded open set in \mathbb{R}^n , $n \geq 3$, $\lambda(D^2u)$ denotes the eigenvalues $\lambda_1, \dots, \lambda_n$ of the Hessian matrix of u . Here

$$\sigma_k(\lambda) = \sum_{1 \leq i_1 < \dots < i_k \leq n} \lambda_{i_1} \cdots \lambda_{i_k}$$

is the k -th elementary symmetric function of n variations, $k = 1, \dots, n$. Note that the case $k = 1$ corresponds to the Poisson's equation, which is a linear equation. There have been extensive literatures on the exterior Dirichlet problem for linear elliptic equations of second order, see [19] and the references therein. For $2 \leq k \leq n$, the Hessian equation (1.1) is an important class of fully nonlinear elliptic equations. Especially, for $k = n$, we have the Monge-Ampère equation $\det(D^2u) = 1$.

For the Monge-Ampère equation, a classical theorem of Jörgens ([17]), Calabi ([5]), and Pogorelov ([20]) states that any classical convex solution of $\det(D^2u) = 1$ in \mathbb{R}^n must be a quadratic polynomial. A simpler and more analytic proof, along the lines of affine geometry, was later given by Cheng and Yau [6]. Caffarelli [1] extended the result for classical solutions to viscosity solutions. Another proof of this theorem was given by Jost and Yin in [18]. Trudinger and Wang [24] proved that if Ω is an

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open convex subset of \mathbb{R}^n and u is a convex C^2 solution of $\det(D^2u) = 1$ in Ω with $\lim_{x \rightarrow \partial\Omega} u(x) = \infty$, then $\Omega = \mathbb{R}^n$ and u is quadratic.

Caffarelli and the third author [3] extended the Jörgens-Calabi-Pogorelov theorem to exterior domains. They proved that if u is a convex viscosity solution of $\det(D^2u) = 1$ outside a bounded subset of \mathbb{R}^n , $n \geq 3$, then there exist a $n \times n$ real symmetric positive definite matrix A , a vector $b \in \mathbb{R}^n$, and a constant $c \in \mathbb{R}$ such that

$$\limsup_{|x| \rightarrow \infty} \left(|x|^{n-2} \left| u(x) - \left(\frac{1}{2} x^T A x + b \cdot x + c \right) \right| \right) < \infty. \quad (1.2)$$

With this prescribed asymptotic behavior at infinity, an existence result for the exterior Dirichlet problem for the Monge-Ampère equation in \mathbb{R}^n , $n \geq 3$, was also established in [3]. In this paper, we will extend the existence theorem to the Dirichlet problem for Hessian equations (1.1) with $2 \leq k \leq n - 1$ on exterior domains, with an appropriate asymptotic behavior at infinity. In dimension two, similar problems were studied by Ferrer, Martínez and Milán in [12, 13] using complex variable method. See also Delanoë [11].

We remark that for the case that $A = c^* I$, where

$$c^* = (C_n^k)^{-1/k}, \quad C_n^k = \frac{n!}{(n-k)!k!},$$

I is the $n \times n$ identity matrix and $1 \leq k \leq n$, the exterior Dirichlet problem of Hessian equation (1.1) has been investigated in [9, 10]. For interior domains, there have been many well known results on the solvability of Hessian equations. For instance, Caffarelli, Nirenberg and Spruck [4] established the classical solvability of the Dirichlet problem, Trudinger [23] proved the existence and uniqueness of weak solutions, and Urbas [25] demonstrated the existence of viscosity solutions. Jian [16] studied the Hessian equations with infinite Dirichlet boundary value conditions.

For readers' convenience, we recall the definition of viscosity solutions to Hessian equations (see [2, 25] and the references therein). We say that a function $u \in C^2(\mathbb{R}^n \setminus \overline{D})$ is admissible (or k -convex) if $\lambda(D^2u) \in \overline{\Gamma}_k$ in $\mathbb{R}^n \setminus \overline{D}$, where Γ_k is the connected component of $\{\lambda \in \mathbb{R}^n \mid \sigma_k(\lambda) > 0\}$ containing

$$\Gamma^+ = \{\lambda \in \mathbb{R}^n \mid \lambda_i > 0, i = 1, \dots, n\}.$$

It is well known that Γ_k is a convex symmetric cone with vertex at the origin. Moreover,

$$\Gamma_k = \{\lambda \in \mathbb{R}^n \mid \sigma_j(\lambda) > 0, \text{ for all } j = 1, \dots, k\}.$$

See [4, 22]. Clearly, $\Gamma_k \subseteq \Gamma_j$ for $k \geq j$, and Γ_1 is the half space $\{\lambda \in \mathbb{R}^n \mid \lambda_1 + \dots + \lambda_n > 0\}$, while $\Gamma_n = \Gamma^+$. We use the following definitions, which can be found in [21].

Let $\Omega \subset \mathbb{R}^n$, we use $\text{USC}(\Omega)$ and $\text{LSC}(\Omega)$ to denote respectively the set of upper and lower semicontinuous real valued functions on Ω .

Definition 1.1. A function $u \in \text{USC}(\mathbb{R}^n \setminus \overline{D})$ is said to be a viscosity subsolution of equation (1.1) in $\mathbb{R}^n \setminus \overline{D}$ (or say that u satisfies $\sigma_k(\lambda(D^2u)) \geq 1$ in $\mathbb{R}^n \setminus \overline{D}$ in the viscosity sense), if for any function $\psi \in C^2(\mathbb{R}^n \setminus \overline{D})$ and point $\bar{x} \in \mathbb{R}^n \setminus \overline{D}$ satisfying

$$\psi(\bar{x}) = u(\bar{x}) \quad \text{and} \quad \psi \geq u \text{ on } \mathbb{R}^n \setminus \overline{D},$$

we have

$$\sigma_k(\lambda(D^2\psi(\bar{x}))) \geq 1.$$

A function $u \in \text{LSC}(\mathbb{R}^n \setminus \bar{D})$ is said to be a viscosity supersolution of (1.1) in $\mathbb{R}^n \setminus \bar{D}$ (or say that u satisfies $\sigma_k(\lambda(D^2u)) \leq 1$ in $\mathbb{R}^n \setminus \bar{D}$ in the viscosity sense), if for any k -convex function $\psi \in C^2(\mathbb{R}^n \setminus \bar{D})$ and point $\bar{x} \in \mathbb{R}^n \setminus \bar{D}$ satisfying

$$\psi(\bar{x}) = u(\bar{x}) \quad \text{and} \quad \psi \leq u \quad \text{on} \quad \mathbb{R}^n \setminus \bar{D},$$

we have

$$\sigma_k(\lambda(D^2\psi(\bar{x}))) \leq 1.$$

A function $u \in C^0(\mathbb{R}^n \setminus \bar{D})$ is said to be a viscosity solution of (1.1), if it is both a viscosity subsolution and supersolution of (1.1).

It is well known that a function $u \in C^2(\mathbb{R}^n \setminus \bar{D})$ is a viscosity solution (respectively, subsolution, supersolution) of (1.1) if and only if it is a k -convex classical solution (respectively, subsolution, supersolution).

Definition 1.2. Let $\varphi \in C^0(\partial D)$. A function $u \in \text{USC}(\mathbb{R}^n \setminus D)$ ($u \in \text{LSC}(\mathbb{R}^n \setminus D)$) is said to be a viscosity subsolution (supersolution) of the Dirichlet problem

$$\begin{cases} \sigma_k(\lambda(D^2u)) = 1, & \text{in } \mathbb{R}^n \setminus \bar{D}, \\ u = \varphi, & \text{on } \partial D, \end{cases} \quad (1.3)$$

if u is a viscosity subsolution (supersolution) of (1.1) in $\mathbb{R}^n \setminus \bar{D}$ and $u \leq (\geq) \varphi$ on ∂D . A function $u \in C^0(\mathbb{R}^n \setminus D)$ is said to be a viscosity solution of (1.3) if it is both a subsolution and a supersolution.

Let

$$\mathcal{A}_k = \left\{ A \mid A \text{ is a real } n \times n \text{ symmetric positive definite matrix, with } \sigma_k(\lambda(A)) = 1 \right\}.$$

Our main result is

Theorem 1.1. *Let D be a smooth, bounded, strictly convex open subset of \mathbb{R}^n , $n \geq 3$, and let $\varphi \in C^2(\partial D)$. Then for any given $b \in \mathbb{R}^n$ and any given $A \in \mathcal{A}_k$ with $2 \leq k \leq n$, there exists some constant c_* , depending only on n, b, A, D and $\|\varphi\|_{C^2(\partial D)}$, such that for every $c > c_*$ there exists a unique viscosity solution $u \in C^0(\mathbb{R}^n \setminus D)$ of (1.3) and*

$$\limsup_{|x| \rightarrow \infty} \left(|x|^{\theta(n-2)} \left| u(x) - \left(\frac{1}{2} x^T A x + b \cdot x + c \right) \right| \right) < \infty, \quad (1.4)$$

where $\theta \in \left[\frac{k-2}{n-2}, 1 \right]$ is a constant depending only on n, k , and A .

Remark 1.1. For the two cases (i) $k = n$, the Monge-Ampère equations with any $A \in \mathcal{A}_n$; and (ii) $2 \leq k \leq n - 1$, (1.4) with $A = c^* I \in \mathcal{A}_k$, Theorem 1.1 has been proved by Caffarelli-Li [3] and Dai-Bao [10], respectively, where $\theta = 1$. Moreover, for the symmetric case $A = c^* I$, Wang-Bao [26] have proved that for $2 \leq k \leq n$, there exists a $\bar{c}(k, n)$ such that there is no classical radial solution of (1.3) and (1.4) if $c < \bar{c}(k, n)$.

Recall that any real symmetric matrix A has an eigen-decomposition $A = O^T \Lambda O$ where O is an orthogonal matrix, and Λ is a diagonal matrix. That is, A may be regarded as a real diagonal matrix Λ that has been re-expressed in some new coordinate system, and the eigenvalues $\lambda(A) = \lambda(\Lambda)$. Let

$$y = Ox, \quad \text{and} \quad v(y) = u(O^{-1}y),$$

then (1.3) and (1.4) become

$$\begin{cases} \sigma_k(\lambda(D_y^2 v)) = 1, & \text{in } \mathbb{R}^n \setminus \overline{D}, \\ v = \varphi(O^{-1}y), & \text{on } \partial \overline{D} \end{cases}$$

and

$$\limsup_{|y| \rightarrow \infty} \left(|O^{-1}y|^{\theta(n-2)} \left| v(y) - \left(\frac{1}{2}y^T \Lambda y + bO^{-1} \cdot y + c \right) \right| \right) < \infty,$$

where \overline{D} is transformed from D under $y = Ox$. So, without loss of generality, we always assume that A is diagonal in this paper.

If A is diagonal and $A \in \mathcal{A}_n$, then $\sigma_n(\lambda(A)) = 1$, and we can find a diagonal matrix Q with $\det Q = 1$ such that $QAQ = I \in \mathcal{A}_n$. Clearly, $\lambda(I)$ is not necessarily the same as $\lambda(A)$, but under transformation $y = Qx$, we still have

$$\det(D_x^2 u) = \det(QD_y^2 uQ) = \det(D_y^2 u).$$

Therefore, when the Monge-Ampère equation is considered, Caffarelli and Li [3] can assume without loss of generality that $A = I$. However, when $2 \leq k \leq n-1$, if A is diagonal and $A \in \mathcal{A}_k$, $\sigma_k(\lambda(A)) = 1$, although we can also find a diagonal matrix Q such that $QAQ = c^*I \in \mathcal{A}_k$, it is clear that $\lambda(A) \neq \lambda(c^*I)$ unless $A = c^*I$, and for Hessian operator

$$\sigma_k(\lambda(QD_y^2 uQ)) \neq \sigma_k(\lambda(Q))\sigma_k(\lambda(D_y^2 u))\sigma_k(\lambda(Q)).$$

So, in order to prove Theorem 1.1, we are only allowed to assume that A is diagonal, but we can not further assume that $A = c^*I$.

Definition 1.3. For a diagonal matrix $A = \text{diag}(a_1, a_2, \dots, a_n)$, we call u a generalized symmetric function with respect to A , if u is a function of

$$s = \frac{1}{2}x^T Ax = \frac{1}{2} \sum_{i=1}^n a_i x_i^2.$$

If u is a generalized symmetric function with respect to A and u is a solution (respectively, subsolution, supersolution) of the Hessian equation (1.1), then we call u a generalized symmetric solution (respectively, subsolution, supersolution) of (1.1).

In this paper we often abuse notations slightly by writing $u(x) = u(\frac{1}{2}x^T Ax)$ for a generalized symmetric function with respect to A . Clearly, for diagonal matrix $A = \text{diag}(a_1, a_2, \dots, a_n) \in \mathcal{A}_k$, and real constants μ_1, μ_2 , with $\mu_1^k = 1$,

$$\omega(s) = \mu_1 s + \mu_2, \quad s = \frac{1}{2} \sum_{i=1}^n a_i x_i^2 \tag{1.5}$$

satisfies the Hessian equation (1.1) and $\omega''(s) \equiv 0$.

First, we will derive a formula of $\sigma_k(\lambda(M))$ for matrices M of the form

$$M = (p_i \delta_{ij} - \beta q_i q_j)_{n \times n}, \quad (1.6)$$

where $p = (p_1, p_2, \dots, p_n)$, $q = (q_1, q_2, \dots, q_n)$ and $\beta \in \mathbb{R}$.

Proposition 1.2. *If M is a $n \times n$ matrix of the form (1.6) for $p = (p_1, p_2, \dots, p_n)$, $q = (q_1, q_2, \dots, q_n)$ and $\beta \in \mathbb{R}$, then we have*

$$\sigma_k(\lambda(M)) = \sigma_k(p) - \beta \sum_{i=1}^n q_i^2 \sigma_{k-1;i}(p), \quad (1.7)$$

where $\sigma_{k-1;i}(p) = \sigma_{k-1}(p)|_{p_i=0}$.

For any $A = \text{diag}(a_1, a_2, \dots, a_n)$, suppose $\omega \in C^2(\mathbb{R}^n)$ is a generalized symmetric function with respect to A , that is,

$$\omega(x) = \omega\left(\frac{1}{2} \sum_{i=1}^n a_i x_i^2\right),$$

then

$$D_i \omega(x) = \omega'(s) a_i x_i,$$

$$D_{ij} \omega(x) = \omega'(s) a_i \delta_{ij} + \omega''(s) (a_i x_i) (a_j x_j).$$

We have the following lemma.

Lemma 1.3. *For any $A = \text{diag}(a_1, a_2, \dots, a_n)$, if $\omega \in C^2(\mathbb{R}^n)$ is a generalized symmetric function with respect to A , then, with $a = (a_1, a_2, \dots, a_n)$,*

$$\sigma_k(\lambda(D^2 \omega)) = \sigma_k(a) (\omega')^k + \omega'' (\omega')^{k-1} \sum_{i=1}^n \sigma_{k-1;i}(a) (a_i x_i)^2. \quad (1.8)$$

If $A = c^* I$, $2 \leq k \leq n$, then there exist a family of radially symmetric functions

$$\bar{\omega}_k(s) = \int_1^s \left(1 + \alpha t^{-\frac{n}{2}}\right)^{\frac{1}{k}} dt, \quad \alpha > 0, s > 0,$$

satisfying

$$\sigma_k(\lambda(D^2 \omega)) = 1, \quad \text{in } \mathbb{R}^n \setminus \{0\}.$$

Such radially symmetric solutions play an important role to the solvability of the exterior Dirichlet problems studied by Caffarelli-Li [3] and by Dai-Bao [10]. However, for any given $A \in \mathcal{A}_k$ with $2 \leq k \leq n-1$, it is not enough to prove Theorem 1.1 only using these radially symmetric functions. Due to the invariance of (1.1) for $k = n$, the Monge-Ampère equation, under affine transformations, $\bar{\omega}_n(\frac{1}{2} x^T A x)$ is a solution of (1.1) in $\mathbb{R}^n \setminus \{0\}$ for $A \in \mathcal{A}_n$. So the Monge-Ampère equation has generalized symmetric solutions with respect to A for every $A \in \mathcal{A}_n$. A natural question is that whether (1.1) with $2 \leq k \leq n-1$ has generalized symmetric solutions with respect to A for every $A \in \mathcal{A}_k$ besides those of the form (1.5).

For this, we have

Proposition 1.4. For $A = \text{diag}(a_1, a_2, \dots, a_n) \in \mathcal{A}_k$, $1 \leq k \leq n$, and $0 < \alpha < \beta < \infty$, if there exists an $\omega \in C^2(\alpha, \beta)$ with $\omega'' \not\equiv 0$ in (α, β) , such that $\omega(x) = \omega(\frac{1}{2} \sum_{i=1}^n a_i x_i^2)$ is a generalized symmetric solution of the Hessian equation (1.1) in $\{x \in \mathbb{R}^n \mid \alpha < \frac{1}{2} \sum_{i=1}^n a_i x_i^2 < \beta\}$, then

$$k = n \quad \text{or} \quad a_1 = a_2 = \dots = a_n = c^*,$$

where $c^* = (C_n^k)^{-1/k}$, $C_n^k = \frac{n!}{(n-k)!k!}$, and vice versa.

This means that for $A = \text{diag}(a_1, a_2, \dots, a_n) \in \mathcal{A}_k$, $2 \leq k \leq n-1$, $\omega(\frac{1}{2} x^T A x)$ is in general not a solution of (1.1).

To prove Theorem 1.1 for $2 \leq k \leq n-1$, it suffices to obtain enough subsolutions with appropriate properties. We construct such subsolutions which are generalized symmetric functions with respect to A . This is the main new ingredient in our proof of the theorem.

This paper is set out as follows. In the next section we construct a family of generalized symmetric smooth k -convex subsolutions of (1.1) in $\mathbb{R}^n \setminus \{0\}$. In Section 3, we prove Theorem 1.1 using Perron's method.

2 Generalized symmetric solutions and subsolutions

In this section, we first derive formula (1.7) and (1.8), then prove Proposition 1.4, and finally construct a family of generalized symmetric smooth k -convex subsolutions of (1.1).

For $A = \text{diag}(a_1, a_2, \dots, a_n)$, we denote $\lambda(A) = (a_1, a_2, \dots, a_n) := a$. If $A \in \mathcal{A}_k$, then we have $a_i > 0$ ($i = 1, 2, \dots, n$) and $\sigma_k(a) = 1$. Here we introduce some notations. For any fixed t -tuple $\{i_1, \dots, i_t\}$, $1 \leq t \leq n-k$, we define

$$\sigma_{k;i_1 \dots i_t}(a) = \sigma_k(a)|_{a_{i_1} = \dots = a_{i_t} = 0},$$

that is, $\sigma_{k;i_1 \dots i_t}$ is the k -th order elementary symmetric function of the $n-t$ variables $\{a_i \mid i \in \{1, 2, \dots, n\} \setminus \{i_1, i_2, \dots, i_t\}\}$. The following properties of the functions σ_k will be used in this paper:

$$\sigma_k(a) = \sigma_{k;i}(a) + a_i \sigma_{k-1;i}(a), \quad i = 1, 2, \dots, n, \quad (2.1)$$

and

$$\sum_{i=1}^n a_i \sigma_{k-1;i}(a) = k \sigma_k(a). \quad (2.2)$$

Now we prove Proposition 1.2 to derive a formula of $\sigma_k(\lambda(M))$ for matrices M of the form (1.6).

Proof of Proposition 1.2. If $\beta = 0$, (1.7) is obvious. If $\beta \neq 0$, we work with

$$\widehat{M} = \frac{1}{\beta} M = (\hat{p}_i \delta_{ij} - q_i q_j), \quad \hat{p} = \frac{p}{\beta}.$$

Therefore we only need to prove Proposition 1.2 for $\beta = 1$, which we assume in the rest of the proof.

Denote

$$D_n(\{p_1, p_2, \dots, p_n\}; \{q_1, q_2, \dots, q_n\}; \lambda) := \det(\lambda I - M). \quad (2.3)$$

By direct computations, we have

$$\begin{aligned} & D_n(\{p_1, p_2, \dots, p_n\}; \{q_1, q_2, \dots, q_n\}; \lambda) \\ &= \begin{vmatrix} \lambda - p_1 + q_1^2 & q_1 q_2 & \cdots & q_1 q_{n-1} & q_1 q_n \\ q_2 q_1 & \lambda - p_2 + q_2^2 & \cdots & q_2 q_{n-1} & q_2 q_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ q_{n-1} q_1 & q_{n-1} q_2 & \cdots & \lambda - p_{n-1} + q_{n-1}^2 & q_{n-1} q_n \\ q_n q_1 & q_n q_2 & \cdots & q_n q_{n-1} & \lambda - p_n + q_n^2 \end{vmatrix} \\ &= \begin{vmatrix} \lambda - p_1 + q_1^2 & q_1 q_2 & \cdots & q_1 q_{n-1} & 0 \\ q_2 q_1 & \lambda - p_2 + q_2^2 & \cdots & q_2 q_{n-1} & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ q_{n-1} q_1 & q_{n-1} q_2 & \cdots & \lambda - p_{n-1} + q_{n-1}^2 & 0 \\ q_n q_1 & q_n q_2 & \cdots & q_n q_{n-1} & \lambda - p_n \end{vmatrix} \\ &+ \begin{vmatrix} \lambda - p_1 + q_1^2 & q_1 q_2 & \cdots & q_1 q_{n-1} & q_1 q_n \\ q_2 q_1 & \lambda - p_2 + q_2^2 & \cdots & q_2 q_{n-1} & q_2 q_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ q_{n-1} q_1 & q_{n-1} q_2 & \cdots & \lambda - p_{n-1} + q_{n-1}^2 & q_{n-1} q_n \\ q_n q_1 & q_n q_2 & \cdots & q_n q_{n-1} & q_n^2 \end{vmatrix} \\ &= (\lambda - p_n) D_{n-1}(\{p_1, p_2, \dots, p_{n-1}\}; \{q_1, q_2, \dots, q_{n-1}\}; \lambda) \\ &+ q_n \begin{vmatrix} \lambda - p_1 + q_1^2 & q_1 q_2 & \cdots & q_1 q_{n-1} & q_1 q_n \\ q_2 q_1 & \lambda - p_2 + q_2^2 & \cdots & q_2 q_{n-1} & q_2 q_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ q_{n-1} q_1 & q_{n-1} q_2 & \cdots & \lambda - p_{n-1} + q_{n-1}^2 & q_{n-1} q_n \\ q_1 & q_2 & \cdots & q_{n-1} & q_n \end{vmatrix}. \end{aligned}$$

For the second term, multiplying its last row by $-q_i$ ($i \neq n$) and adding to the i th row, respectively, we obtain

$$\begin{aligned} & \begin{vmatrix} \lambda - p_1 + q_1^2 & q_1 q_2 & \cdots & q_1 q_{n-1} & q_1 q_n \\ q_2 q_1 & \lambda - p_2 + q_2^2 & \cdots & q_2 q_{n-1} & q_2 q_n \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ q_{n-1} q_1 & q_{n-1} q_2 & \cdots & \lambda - p_{n-1} + q_{n-1}^2 & q_{n-1} q_n \\ q_1 & q_2 & \cdots & q_{n-1} & q_n \end{vmatrix} \\ &= \begin{vmatrix} \lambda - p_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda - p_2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \lambda - p_{n-1} & 0 \\ q_1 & q_2 & \cdots & q_{n-1} & q_n \end{vmatrix} \\ &= q_n (\lambda - p_1) (\lambda - p_2) \cdots (\lambda - p_{n-1}). \end{aligned}$$

Hence

$$\begin{aligned}
& D_n(\{p_1, p_2, \dots, p_n\}; \{q_1, q_2, \dots, q_n\}; \lambda) \\
&= (\lambda - p_n)D_{n-1}(\{p_1, p_2, \dots, p_{n-1}\}; \{q_1, q_2, \dots, q_{n-1}\}; \lambda) \\
&\quad + q_n^2(\lambda - p_1)(\lambda - p_2) \cdots (\lambda - p_{n-1}).
\end{aligned} \tag{2.4}$$

We will deduce from (2.4), by induction, that for $n \geq 2$,

$$D_n(\{p_1, p_2, \dots, p_n\}; \{q_1, q_2, \dots, q_n\}; \lambda) = \prod_{i=1}^n (\lambda - p_i) + \sum_{j=1}^n \left(q_j^2 \prod_{i \neq j} (\lambda - p_i) \right). \tag{2.5}$$

For $n = 2$,

$$\begin{aligned}
D_2(\{p_1, p_2\}; \{q_1, q_2\}; \lambda) &= \begin{vmatrix} \lambda - p_1 + q_1^2 & q_1 q_2 \\ q_1 q_2 & \lambda - p_2 + q_2^2 \end{vmatrix} \\
&= (\lambda - p_1)(\lambda - p_2) + q_1^2(\lambda - p_2) + q_2^2(\lambda - p_1).
\end{aligned}$$

That is, (2.5) holds for $n = 2$. We now assume (2.5) holds for $n - 1 \geq 2$. Then by (2.4) and the induction hypothesis,

$$\begin{aligned}
& D_n(\{p_1, p_2, \dots, p_n\}; \{q_1, q_2, \dots, q_n\}; \lambda) \\
&= (\lambda - p_n)D_{n-1}(\{p_1, p_2, \dots, p_{n-1}\}; \{q_1, q_2, \dots, q_{n-1}\}; \lambda) \\
&\quad + q_n^2(\lambda - p_1)(\lambda - p_2) \cdots (\lambda - p_{n-1}) \\
&= (\lambda - p_n) \left(\prod_{i=1}^{n-1} (\lambda - p_i) + \sum_{j=1}^{n-1} \left(q_j^2 \prod_{i \neq j, i \leq n-1} (\lambda - p_i) \right) \right) \\
&\quad + q_n^2(\lambda - p_1)(\lambda - p_2) \cdots (\lambda - p_{n-1}) \\
&= \prod_{i=1}^n (\lambda - p_i) + \sum_{j=1}^n \left(q_j^2 \prod_{i \neq j} (\lambda - p_i) \right).
\end{aligned}$$

We have proved that (2.5) holds for $n \geq 2$. Recall the Veite theorem that for any $n \times n$ matrix U ,

$$\det(\lambda I - U) = \sum_{i=0}^n (-1)^i \sigma_i(\lambda(U)) \lambda^{n-i}. \tag{2.6}$$

In particular, if $U = \text{diag}(p_1, p_2, \dots, p_n)$,

$$\prod_{i=1}^n (\lambda - p_i) = \sum_{i=0}^n (-1)^i \sigma_i(p) \lambda^{n-i}, \tag{2.7}$$

here $p = (p_1, p_2, \dots, p_n)$. Using (2.3) and (2.7), (2.5) is written as

$$\begin{aligned}
\det(\lambda I - M) &= \sum_{i=0}^n (-1)^i \sigma_i(p) \lambda^{n-i} + \sum_{j=1}^n \left(q_j^2 \sum_{i=1}^n (-1)^{i-1} \sigma_{i-1;j}(p) \lambda^{n-i} \right) \\
&= \sum_{i=0}^n (-1)^i \left(\sigma_i(p) - \sum_{j=1}^n q_j^2 \sigma_{i-1;j}(p) \right) \lambda^{n-i},
\end{aligned}$$

here we used standard conventions that $\sigma_0(p) = 1$ and $\sigma_{-1}(p) = 0$. Thus, (1.7) follows from (2.6). The proof of Proposition 1.2 is completed. \square

Proof of Lemma 1.3. For any $A = \text{diag}(a_1, a_2, \dots, a_n)$, if $\omega \in C^2(\mathbb{R}^n)$ is a generalized symmetric function with respect to A , that is

$$\omega(x) = \omega\left(\frac{1}{2} \sum_{i=1}^n a_i x_i^2\right),$$

then

$$\begin{aligned} D_i \omega(x) &= \omega'(s) a_i x_i, \\ D_{ij} \omega(x) &= \omega'(s) a_i \delta_{ij} + \omega''(s) (a_i x_i) (a_j x_j). \end{aligned} \quad (2.8)$$

Comparing (1.6) and (2.8), letting $\beta = -\omega''(s)$, $p_i = \omega'(s) a_i$ and $q_i = a_i x_i$, and substituting them into (1.7), we have (1.8). \square

Symmetric solutions. For $A = c^* I$ and $2 \leq k \leq n$,

$$\bar{\omega}_k(s) = \int_1^s \left(1 + \alpha t^{-\frac{n}{2}}\right)^{\frac{1}{k}} dt, \quad \alpha > 0, \quad s > 0, \quad (2.9)$$

satisfies the ordinary differential equation

$$\sigma_k(\lambda(D^2 \omega)) = (\omega'(s))^k + 2s \frac{k}{n} \omega''(s) (\omega'(s))^{k-1} = 1, \quad s > 0. \quad (2.10)$$

Therefore, $\bar{\omega}_k\left(\frac{c^*}{2}|x|^2\right)$ is a solution of (1.1) in $\mathbb{R}^n \setminus \{0\}$. In order to prove Proposition 1.4, for every $a = (a_1, a_2, \dots, a_n) \in \Gamma^+$, we denote

$$A_k^i(a) = a_i \sigma_{k-1;i}(a), \quad i = 1, 2, \dots, n. \quad (2.11)$$

From the property of σ_k , (2.2), we have

$$\sum_{i=1}^n A_k^i(a) = k \sigma_k(a). \quad (2.12)$$

Proof of Proposition 1.4. To better illustrate the idea of the proof, we start with $k = 1$. For $s \in (\alpha, \beta)$, $1 \leq i \leq n$, let $x = (0, \dots, 0, \sqrt{\frac{2s}{a_i}}, 0, \dots, 0)$. We have, using $A \in \mathcal{A}_1$,

$$1 = \Delta \omega(x) = \omega'(s) \sum_{j=1}^n a_j + \omega''(s) \sum_{j=1}^n a_j^2 x_j^2 = \omega'(s) + 2s \omega''(s) a_i.$$

Since $\omega'' \neq 0$ in (α, β) , there exists some $\bar{s} \in (\alpha, \beta)$ such that $\omega''(\bar{s}) \neq 0$. It follows that

$$a_i = \frac{1 - \omega'(\bar{s})}{2\bar{s}\omega''(\bar{s})}$$

is independent of i . Since $A \in \mathcal{A}_1$, $1 = \sum_{i=1}^n a_i$. So $a_1 = a_2 = \dots = a_n = \frac{1}{n}$. Proposition 1.4 for $k = 1$ is established.

Now we consider the case $2 \leq k \leq n$. For $s \in (\alpha, \beta)$, $1 \leq i \leq n$, let $x = (0, \dots, 0, \sqrt{\frac{2s}{a_i}}, 0, \dots, 0)$, we have, using Lemma 1.3,

$$\begin{aligned} 1 &= \sigma_k(\lambda(D^2\omega(x))) \\ &= \sigma_k(a)(\omega'(s))^k + \omega''(s)(\omega'(s))^{k-1}\sigma_{k-1;j}(a)(a_j x_j)^2 \\ &= (\omega'(s))^k + 2s\omega''(s)(\omega'(s))^{k-1}\sigma_{k-1;i}(a)a_i. \end{aligned}$$

It is clear from the above that $\omega'(s) \neq 0$, $\forall s \in (\alpha, \beta)$. Since $\omega'' \not\equiv 0$ in (α, β) , there exists some $\bar{s} \in (\alpha, \beta)$ such that $\omega''(\bar{s}) \neq 0$. It follows that

$$A_k^i(a) = \sigma_{k-1;i}(a)a_i = \frac{1 - (\omega'(\bar{s}))^k}{2\bar{s}\omega''(\bar{s})(\omega'(\bar{s}))^{k-1}}$$

is independent of i . For $2 \leq k \leq n-1$, for any $i_1, i_2 \in \{1, 2, \dots, n\}$, by (2.11) and (2.1), we have

$$\begin{aligned} 0 &= A_k^{i_1}(a) - A_k^{i_2}(a) \\ &= a_{i_1}\sigma_{k-1;i_1}(a) - a_{i_2}\sigma_{k-1;i_2}(a) \\ &= a_{i_1}(a_{i_2}\sigma_{k-2;i_1i_2}(a) + \sigma_{k-1;i_1i_2}(a)) - a_{i_2}(a_{i_1}\sigma_{k-2;i_1i_2}(a) + \sigma_{k-1;i_1i_2}(a)) \\ &= (a_{i_1} - a_{i_2})\sigma_{k-1;i_1i_2}(a). \end{aligned} \tag{2.13}$$

Since $a_i > 0$, $i = 1, 2, \dots, n$, it follows that $\sigma_{k-1;i_1i_2}(a) \neq 0$. By the arbitrariness of i_1, i_2 , we have $a_1 = a_2 = \dots = a_n$. Using $\sigma_k(a) = 1$, we have

$$a_1 = a_2 = \dots = a_n = (C_n^k)^{-1/k}.$$

Proposition 1.4 is proved. \square

Generalized symmetric subsolutions. From Proposition 1.4, we see that there is no generalized symmetric solutions of (1.1) with $\omega''(s) \not\equiv 0$ in remaining cases. We will construct a family of generalized symmetric smooth functions satisfying

$$\omega'(s) > 0, \quad \omega''(s) \leq 0,$$

and

$$\sigma_k(\lambda(D^2\omega)) \geq 1, \quad \text{and} \quad \sigma_m(\lambda(D^2\omega)) \geq 0, \quad 1 \leq m \leq k-1.$$

For $A = \text{diag}(a_1, a_2, \dots, a_n) \in \mathcal{A}_k$, denote $a = (a_1, a_2, \dots, a_n)$, and consider

$$h_k(a) := \max_{1 \leq i \leq n} A_k^i(a). \tag{2.14}$$

Since $A_n^i(a) = a_i\sigma_{n-1;i}(a) = \sigma_n(a)$ for every i , we have $h_n(a) = 1$. By (2.11), (2.1) and (2.12), we have, for $1 \leq k \leq n-1$,

$$A_k^i(a) = a_i\sigma_{k-1;i}(a) < \sigma_k(a) = 1, \quad \forall i,$$

and

$$nh_k(a) \geq \sum_{i=1}^n A_k^i(a) = k\sigma_k(a) = k.$$

We see from the above that

$$\frac{k}{n} \leq h_k(a) < 1, \quad (2.15)$$

with “ = ” holds if and only if $A_k^i(a)$ is independent of i , i.e., in view of (2.13), $a_1 = a_2 = \cdots = a_n = c^*$. For $n \geq 3$ and $2 \leq k \leq n$, in view of (2.15) and $h_n(a) = 1$, we have

$$\frac{k}{2h_k(a)} > 1. \quad (2.16)$$

By a simple computation, the following ordinary differential equation

$$\begin{cases} (\omega'(s))^k + 2h_k(a)s\omega''(s)(\omega'(s))^{k-1} = 1, & s > 0, \\ \omega'(s) > 0, & \omega''(s) \leq 0 \end{cases} \quad (2.17)$$

has a family of solutions

$$\omega_\alpha(s) = \beta + \int_{\bar{s}}^s \left(1 + \alpha t^{-\frac{k}{2h_k(a)}}\right)^{\frac{1}{k}} dt, \quad \alpha > 0, \quad s > 0, \quad (2.18)$$

where $\beta \in \mathbb{R}$ and $\bar{s} > 0$. It follows from (2.16) that

$$\begin{aligned} \omega_\alpha(s) &= \beta + s - \bar{s} + \int_{\bar{s}}^s \left(\left(1 + \alpha t^{-\frac{k}{2h_k(a)}}\right)^{\frac{1}{k}} - 1 \right) dt \\ &= s + \mu(\alpha) + O\left(s^{\frac{(2-n)\theta}{2}}\right), \quad \text{as } s \rightarrow \infty, \end{aligned} \quad (2.19)$$

where

$$\mu(\alpha) = \beta - \bar{s} + \int_{\bar{s}}^{\infty} \left(\left(1 + \alpha t^{-\frac{k}{2h_k(a)}}\right)^{\frac{1}{k}} - 1 \right) dt < \infty,$$

and

$$\theta = \frac{1}{n-2} \left(\frac{k}{h_k(a)} - 2 \right).$$

We see from (2.15) that $\theta \in \left(\frac{k-2}{n-2}, 1\right]$ if $2 \leq k \leq n-1$, and $\theta = 1$ if $k = n$.

Proposition 2.1. *For $n \geq 3$ and $2 \leq k \leq n$, $A \in \mathcal{A}_k$, let $\omega_\alpha(x) = \omega_\alpha\left(\frac{1}{2}x^T Ax\right)$ be given in (2.18). Then ω_α is a smooth k -convex subsolution of (1.1) in $\mathbb{R}^n \setminus \{0\}$ satisfying*

$$\omega_\alpha(x) = \frac{1}{2}x^T Ax + \mu(\alpha) + O(|x|^{\theta(2-n)}), \quad \text{as } |x| \rightarrow \infty. \quad (2.20)$$

Proof. Obviously, (2.20) follows from (2.19). By computation,

$$\omega'_\alpha(s) = \left(1 + \alpha s^{-\frac{k}{2h_k(a)}}\right)^{\frac{1}{k}} > 1,$$

$$\omega''_\alpha(s) = -\frac{1}{2h_k(a)s} \cdot \frac{\alpha}{s^{\frac{k}{2h_k(a)} + \alpha}} \cdot \omega'_\alpha(s) < 0. \quad (2.21)$$

It is clear from Lemma 1.3, (2.14) and (2.17) that

$$\sigma_k(\lambda(D^2u)) \geq \sigma_k(a)(\omega'_\alpha)^k + h_k(a)\omega''_\alpha(\omega'_\alpha)^{k-1}2s = 1, \quad \text{in } \mathbb{R}^n \setminus \{0\}.$$

By Lemma 1.3, (2.21) and (2.14), we have, for any $1 \leq m \leq k - 1$,

$$\begin{aligned}\sigma_m(\lambda(D^2u)) &= \sigma_m(a)(\omega'_\alpha)^m + \omega''_\alpha(\omega'_\alpha)^{m-1} \sum_{i=1}^n \sigma_{m-1;i}(a)(a_i x_i)^2 \\ &= (\omega'_\alpha)^m \left(\sigma_m(a) - \frac{1}{2sh_k(a)} \cdot \frac{\alpha}{s^{\frac{k}{2h_k(a)} + \alpha}} \sum_{i=1}^n \sigma_{m-1;i}(a)(a_i x_i)^2 \right) \\ &\geq (\omega'_\alpha)^m \left(\sigma_m(a) - \frac{1}{2s} \cdot \frac{\alpha}{s^{\frac{k}{2h_k(a)} + \alpha}} \sum_{i=1}^n \frac{\sigma_{m-1;i}(a)(a_i x_i)^2}{a_i \sigma_{k-1;i}(a)} \right).\end{aligned}$$

In order to show $\sigma_m(\lambda(D^2u)) \geq 0$, it suffices to prove, for each $1 \leq i \leq n$,

$$\sigma_m(a)\sigma_{k-1;i}(a) \geq \sigma_{m-1;i}(a). \quad (2.22)$$

Note that the Newtonian inequalities may be expressed as

$$\frac{\sigma_{k+1}(a)}{C_n^{k+1}} \cdot \frac{\sigma_{k-1}(a)}{C_n^{k-1}} \leq \left(\frac{\sigma_k(a)}{C_n^k} \right)^2,$$

for $1 \leq k \leq n - 1$. Since

$$\frac{C_n^{k-1} C_n^{k+1}}{C_n^k C_n^k} = \frac{(n-k)k}{(n-k+1)(k+1)} < 1,$$

it follows that

$$\frac{\sigma_{k+1}(a)}{\sigma_k(a)} \leq \frac{\sigma_k(a)}{\sigma_{k-1}(a)},$$

which shows that the Hessian quotient $\frac{\sigma_{k+1}(a)}{\sigma_k(a)}$ is decreasing with respect to k . So we have for any $m \leq k$, and each $1 \leq i \leq n$,

$$\sigma_{m;i}(a)\sigma_{k-1;i}(a) \geq \sigma_{m-1;i}(a)\sigma_{k;i}(a),$$

Then by the property (2.1), it follows that

$$\begin{aligned}\sigma_m(a)\sigma_{k-1;i}(a) &= (\sigma_{m;i}(a) + a_i \sigma_{m-1;i}(a)) \sigma_{k-1;i}(a) \\ &\geq \sigma_{m-1;i}(a) \cdot \sigma_{k;i}(a) + \sigma_{m-1;i}(a) \cdot a_i \sigma_{k-1;i}(a) \\ &= \sigma_{m-1;i}(a) \sigma_k(a) \\ &= \sigma_{m-1;i}(a).\end{aligned}$$

i.e. (2.22) is proved. Hence ω_α is a smooth k -convex subsolution of (1.1) in $\mathbb{R}^n \setminus \{0\}$. \square

3 Proof of Theorem 1.1

The following Lemma holds for any invertible and symmetric matrix A , and A is not necessarily diagonal or in \mathcal{A}_k , $2 \leq k \leq n$.

Lemma 3.1. *Let $\varphi \in C^2(\partial D)$. There exists some constant C , depending only on n , $\|\varphi\|_{C^2(\partial D)}$, the upper bound of A , the diameter and the convexity of D , and the C^2 norm of ∂D , such that, for every $\xi \in \partial D$, there exists $\bar{x}(\xi) \in \mathbb{R}^n$ satisfying*

$$|\bar{x}(\xi)| \leq C \quad \text{and} \quad w_\xi < \varphi \text{ on } \bar{D} \setminus \{\xi\},$$

where

$$w_\xi(x) := \varphi(\xi) + \frac{1}{2} \left((x - \bar{x}(\xi))^T A (x - \bar{x}(\xi)) - (\xi - \bar{x}(\xi))^T A (\xi - \bar{x}(\xi)) \right), \quad x \in \mathbb{R}^n.$$

Proof. Let $\xi \in \partial D$. By a translation and a rotation, we may assume without loss of generality that $\xi = 0$ and ∂D is locally represented by the graph of

$$x_n = \rho(x') = O(|x'|^2),$$

and φ locally has the expansion

$$\begin{aligned} \varphi(x', \rho(x')) &= \varphi(0) + \varphi_{x_1}(0)x_1 + \cdots + \varphi_{x_n}(0)x_n + O(|x'|^2) \\ &= \varphi(0) + \varphi_{x_1}(0)x_1 + \cdots + \varphi_{x_{n-1}}(0)x_{n-1} + O(|x'|^2), \end{aligned}$$

where $x' = (x_1, \dots, x_{n-1})$.

Since A is invertible, we can find $\bar{x} = \bar{x}(t) \in \mathbb{R}^n$ such that, for appropriate t to fit our need later,

$$A\bar{x}(t) = (-\varphi_{x_1}(0), \dots, -\varphi_{x_{n-1}}(0), t)^T.$$

Let

$$w(x) = \varphi(0) + \frac{1}{2} \left((x - \bar{x})^T A (x - \bar{x}) - \bar{x}^T A \bar{x} \right), \quad x \in \mathbb{R}^n.$$

Then

$$w(x) = \varphi(0) + \frac{1}{2} x^T A x - x^T A \bar{x} = \varphi(0) + \frac{1}{2} x^T A x + \sum_{\alpha=1}^{n-1} \varphi_{x_\alpha}(0) x_\alpha - t x_n. \quad (3.1)$$

It follows that

$$\begin{aligned} (w - \varphi)(x', \rho(x')) &= \frac{1}{2} x'^T A x' - t \rho(x') + O(|x'|^2) \\ &\leq C \left(|x'|^2 + \rho(x')^2 \right) - t \rho(x'), \end{aligned}$$

where C depends only on the upper bound of A , $\|\varphi\|_{C^2(\partial D)}$, and the C^2 norm of ∂D . By the strict convexity of ∂D , there exists some constant $\delta > 0$ depending only on D such that

$$\rho(x') \geq \delta |x'|^2, \quad \forall |x'| < \delta. \quad (3.2)$$

Clearly, for large t , we have

$$(w - \varphi)(x', \rho(x')) < 0, \quad \forall 0 < |x'| < \delta.$$

The largeness of t depends only on δ , A , $\|\varphi\|_{C^2(\partial D)}$, and the C^2 norm of ∂D .

On the other hand, by the strict convexity of ∂D and (3.2),

$$x_n \geq \delta^3, \quad \forall x \in \partial D \setminus \{(x', \rho(x')) \mid |x'| < \delta\}.$$

It follows from (3.1) that

$$w(x) \leq C - \delta^3 t, \quad \forall x \in \partial D \setminus \{(x', \rho(x')) \mid |x'| < \delta\},$$

where C depends only on $A, \text{diam}(D), \|\varphi\|_{C^2(\partial D)}$. By making t large (still under control), we have

$$w(x) - \varphi(x) < 0, \quad \forall x \in \partial D \setminus \{(x', \rho(x')) \mid |x'| < \delta\}.$$

Lemma 3.1 is established. \square

By an orthogonal transformation and by subtracting a linear function from u , we only need to prove Theorem 1.1 for the case that $A = \text{diag}(a_1, a_2, \dots, a_n)$ where $a_i > 0$ ($1 \leq i \leq n$), $b = 0$.

Proof of Theorem 1.1. Without loss of generality, we assume that $0 \in D$. For $s > 0$, let

$$E(s) := \left\{ x \in \mathbb{R}^n \mid \frac{1}{2} x^T A x < s \right\}.$$

Fix $\bar{s} > 0$ such that $\bar{D} \subset E(\bar{s})$. For $\alpha > 0, \beta \in \mathbb{R}$, set

$$\omega_\alpha(x) = \beta + \int_{\bar{s}}^{\frac{1}{2} x^T A x} \left(1 + \alpha t^{-\frac{k}{2h_k(a)}} \right)^{\frac{1}{k}} dt,$$

as in (2.18). We have by Proposition 2.1 that ω_α is a smooth k -convex subsolution of (1.1) in $\mathbb{R}^n \setminus \{0\}$, and

$$\omega_\alpha(x) = \frac{1}{2} x^T A x + \mu(\alpha) + O(|x|^{\theta(2-n)}), \quad \text{as } x \rightarrow \infty.$$

Here

$$\mu(\alpha) = \beta - \bar{s} + \int_{\bar{s}}^{\infty} \left(\left(1 + \alpha t^{-\frac{k}{2h_k(a)}} \right)^{\frac{1}{k}} - 1 \right) dt, \quad \theta \in \left[\frac{k-2}{n-2}, 1 \right].$$

Clearly, $\mu(\alpha)$ is strictly increasing in α , and

$$\lim_{\alpha \rightarrow \infty} \mu(\alpha) = \infty. \quad (3.3)$$

On the other hand,

$$\omega_\alpha \leq \beta, \quad \text{in } E(\bar{s}) \setminus \bar{D}, \quad \forall \alpha > 0. \quad (3.4)$$

Let

$$\begin{aligned} \beta &:= \min \{ w_\xi(x) \mid \xi \in \partial D, x \in \overline{E(\bar{s})} \setminus D \}, \\ \widehat{b} &:= \max \{ w_\xi(x) \mid \xi \in \partial D, x \in \overline{E(\bar{s})} \setminus D \}, \end{aligned}$$

where $w_\xi(x)$ is given by Lemma 3.1. We will fix the value of c_* in the proof. First we require that c_* satisfies $c_* > \widehat{b}$. It follows that

$$\mu(0) = \beta - \bar{s} < \beta \leq \widehat{b} < c_*.$$

Thus, in view of (3.3), for every $c > c_*$, there exists a unique $\alpha(c)$ such that

$$\mu(\alpha(c)) = c. \quad (3.5)$$

So $\omega_{\alpha(c)}$ satisfies

$$\omega_{\alpha(c)}(x) = \frac{1}{2}x^T Ax + c + O(|x|^{\theta(2-n)}), \quad \text{as } x \rightarrow \infty. \quad (3.6)$$

Set

$$\underline{w}(x) = \max \{w_\xi(x) \mid \xi \in \partial D\}.$$

It is clear by Lemma 3.1 that \underline{w} is a locally Lipschitz function in $\mathbb{R}^n \setminus D$, and $\underline{w} = \varphi$ on ∂D . Since w_ξ is a smooth convex solution of (1.1), \underline{w} is a viscosity subsolution of (1.1) in $\mathbb{R}^n \setminus \overline{D}$. We fix a number $\hat{s} > \bar{s}$, and then choose another number $\widehat{\alpha} > 0$ such that

$$\min_{\partial E(\hat{s})} \omega_{\widehat{\alpha}} > \max_{\partial E(\hat{s})} \underline{w}.$$

We require that c_* also satisfies $c_* \geq \mu(\widehat{\alpha})$. We now fix the value of c_* .

For $c \geq c_*$, we have $\alpha(c) = \mu^{-1}(c) \geq \mu^{-1}(c_*) \geq \widehat{\alpha}$, and therefore

$$\omega_{\alpha(c)} \geq \omega_{\widehat{\alpha}} > \underline{w}, \quad \text{on } \partial E(\hat{s}). \quad (3.7)$$

By (3.4), we have

$$\omega_{\alpha(c)} \leq \beta \leq \underline{w}, \quad \text{in } E(\bar{s}) \setminus \overline{D}. \quad (3.8)$$

Now we define, for $c > c_*$,

$$\underline{u}(x) = \begin{cases} \max \{ \omega_{\alpha(c)}(x), \underline{w}(x) \}, & x \in E(\hat{s}) \setminus D, \\ \omega_{\alpha(c)}(x), & x \in \mathbb{R}^n \setminus E(\hat{s}). \end{cases}$$

We know from (3.8) that

$$\underline{u} = \underline{w}, \quad \text{in } E(\bar{s}) \setminus \overline{D}, \quad (3.9)$$

and in particular

$$\underline{u} = \underline{w} = \varphi, \quad \text{on } \partial D. \quad (3.10)$$

We know from (3.7) that $\underline{u} = \omega_{\alpha(c)}$ in a neighborhood of $\partial E(\hat{s})$. Therefore \underline{u} is locally Lipschitz in $\mathbb{R}^n \setminus D$. Since both $\omega_{\alpha(c)}$ and \underline{w} are viscosity subsolutions of (1.1) in $\mathbb{R}^n \setminus \overline{D}$, so is \underline{u} .

For $c > c_*$,

$$\overline{u}(x) := \frac{1}{2}x^T Ax + c$$

is a smooth convex solution of (1.1). By (3.8),

$$\omega_{\alpha(c)} \leq \beta \leq \widehat{b} < c_* < \overline{u}, \quad \text{on } \partial D.$$

We also know by (3.6) that

$$\lim_{|x| \rightarrow \infty} (\omega_{\alpha(c)}(x) - \overline{u}(x)) = 0.$$

Thus, in view of the comparison principle for smooth k -convex solutions of (1.1), (see [4]), we have

$$\omega_{\alpha(c)} \leq \bar{u}, \quad \text{on } \mathbb{R}^n \setminus D. \quad (3.11)$$

By (3.7) and the above, we have, for $c > c_*$,

$$w_\xi \leq \bar{u}, \quad \text{on } \partial(E(\hat{s}) \setminus D), \quad \forall \xi \in \partial D.$$

By the comparison principle for smooth convex solutions of (1.1), we have

$$w_\xi \leq \bar{u}, \quad \text{in } E(\hat{s}) \setminus \bar{D}, \quad \forall \xi \in \partial D.$$

Thus

$$\underline{w} \leq \bar{u}, \quad \text{in } E(\hat{s}) \setminus \bar{D}.$$

This, combining with (3.11), implies that

$$\underline{u} \leq \bar{u}, \quad \text{in } \mathbb{R}^n \setminus D.$$

For any $c > c_*$, let \mathcal{S}_c denote the set of $v \in \text{USC}(\mathbb{R}^n \setminus D)$ which are viscosity subsolutions of (1.1) in $\mathbb{R}^n \setminus \bar{D}$ satisfying

$$v = \varphi, \quad \text{on } \partial D, \quad (3.12)$$

and

$$\underline{u} \leq v \leq \bar{u}, \quad \text{in } \mathbb{R}^n \setminus D. \quad (3.13)$$

We know that $\underline{u} \in \mathcal{S}_c$. Let

$$u(x) := \sup \{v(x) \mid v \in \mathcal{S}_c\}, \quad x \in \mathbb{R}^n \setminus D.$$

By (3.6), and the definitions of \underline{u} and \bar{u} ,

$$u(x) \geq \underline{u}(x) = \omega_{\alpha(c)}(x) = \frac{1}{2}x^T Ax + c + O(|x|^{\theta(2-n)}), \quad \text{as } x \rightarrow \infty. \quad (3.14)$$

and

$$u(x) \leq \bar{u}(x) = \frac{1}{2}x^T Ax + c.$$

The estimate (1.4) follows.

Next, we prove that u satisfies the boundary condition. It is obvious from (3.10) that

$$\liminf_{x \rightarrow \xi} u(x) \geq \lim_{x \rightarrow \xi} \underline{u}(x) = \varphi(\xi), \quad \forall \xi \in \partial D.$$

So we only need to prove that

$$\limsup_{x \rightarrow \xi} u(x) \leq \varphi(\xi), \quad \forall \xi \in \partial D.$$

Let $\omega_c^+ \in C^2(\overline{E(\bar{s})} \setminus \bar{D})$ be defined by

$$\begin{cases} \Delta \omega_c^+ = 0, & \text{in } E(\bar{s}) \setminus \bar{D}, \\ \omega_c^+ = \varphi, & \text{on } \partial D, \\ \omega_c^+ = \max_{\partial E(\bar{s})} \bar{u} = \bar{s} + c, & \text{on } \partial E(\bar{s}). \end{cases}$$

It is easy to see that a viscosity subsolution v of (1.1) satisfies $\Delta v \geq 0$ in viscosity sense. Therefore, for every $v \in \mathcal{S}_c$, by $v \leq \omega_c^+$ on $\partial(E(\bar{s}) \setminus D)$, we have

$$v \leq \omega_c^+ \quad \text{in } E(\bar{s}) \setminus \bar{D}.$$

It follows that

$$u \leq \omega_c^+ \quad \text{in } E(\bar{s}) \setminus \bar{D},$$

and then

$$\limsup_{x \rightarrow \xi} u(x) \leq \lim_{x \rightarrow \xi} \omega_c^+(x) = \varphi(\xi), \quad \forall \xi \in \partial D.$$

Finally, we prove that u is a viscosity solution of (1.1). The following ingredients for the viscosity adaptation of Perron's method (see [14]) are available.

Lemma 3.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, $u \in \text{LSC}(\bar{\Omega})$ and $v \in \text{USC}(\bar{\Omega})$ are respectively viscosity supersolutions and subsolutions of (1.1) in Ω satisfying $u \geq v$ on $\partial\Omega$. Then $u \geq v$ in Ω .*

Under the assumptions $u, v \in C^0(\bar{\Omega})$, the lemma was proved in [25], based on Jensen approximations (see [15]). The proof remains valid under the weaker regularity assumptions on u and v .

Lemma 3.3. *Let $\Omega \subset \mathbb{R}^n$ be an open set, and let \mathcal{S} be a non-empty family of viscosity subsolutions (supersolutions) of (1.1) in Ω . Set*

$$u(x) = \sup (\inf) \{v(x) \mid v \in \mathcal{S}\},$$

and

$$u^* (u_*) (x) = \limsup_{r \rightarrow 0} (\inf)_{B_r} u$$

be the upper (lower) semicontinuous envelope of u . Then, if $u^* < \infty$ ($u_* > -\infty$) in Ω , $u^* (u_*)$ is a viscosity subsolution (supersolution) of (1.1) in Ω .

Lemma 3.3 can be proved by standard arguments, see e.g. [8]. With these ingredients, an application of the Perron process, see e.g. Lemma 4.4 in [8], gives that $u \in C^0(\mathbb{R}^n \setminus D)$ is a viscosity solution of (1.3). Theorem 1.1 is established. \square

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