Interior and boundary estimates for the kinetic Kolmogorov-Fokker-Planck equations

Eg. $u = u(t,x,v)$. $t \in \mathbb{R}$, $x \in \mathbb{R}^d$, $v \in \mathbb{R}^d$

$u_t + \underline{v \nabla_x} u - \underline{\Delta_v} u = f$ (Kinetic KFP eq.). $(*)$

usual FP equation $\partial_t p + \partial_x (b(t,x) p) - \frac{1}{2} \partial_x^2 (\sigma^2(t,x) p) = 0$ $(\checkmark)$.

p density function for a stochastic process $dX_t = b(t,X_t)\,dt + \sigma(t,X_t)\,dW_t$

W_t – standard Brownian motion

- Scaling and dimension analysis

$u_t - \Delta u = 0$ $(b=0,\ \sigma = \sqrt{2})$ $u(t,x)$. $u_\lambda(t,x) = u(\lambda^2 t, \lambda x)$ still a solution

Natural scaling $t \sim x^2$ (one time variable $\sim$ two space variables)

effective dimension $= \underline{d} + \underline{2}$

parabolic cylinder $Q_r(t_0,x_0) = (t_0 - r^2, t_0) \times B_r(x_0)$

$|Q_r(t_0,x_0)| = C_d\, r^{d+2} \leftarrow$

$\begin{pmatrix} u(t,x,v) \\ f(t,x,v) \end{pmatrix} \rightsquigarrow \begin{pmatrix} u(\lambda^2 t, \lambda^3 x, \lambda v) \\ \lambda^2 f(\lambda^2 t, \lambda^3 x, \lambda v) \end{pmatrix} \leftarrow u_\lambda$ λ^2. $\underline{v \lambda^3}$. λ^2.

$(\lambda v)\ \underline{\lambda^2}$

one time variable $\begin{cases} t \sim v^2 \\ x \sim v^3. \end{cases}$ effective dim $= \underset{t}{2} + \underset{x}{3d} + \underset{v}{d} = 2(1+2d)$

$d=3$, effective dim $= 14$

- $u_t - \Delta u = 0$ nondegenerate in x, degenerate in t.

diffusion in x

$\begin{cases} X_s = x_0 + \sqrt{2} W_s \\ t_s = t_0 - s \end{cases}$ $s \in [0,\infty)$. Diffusion $\xrightarrow{\text{averaging effect}}$ smoothness of solutions

However, u smooth in both t and x? Combined effect of the transport in t and the diffusion in x

Kinetic FP. $u_t + \underline{v \nabla_x} u - \underline{\Delta_v} u = f$ $\begin{cases} V_s = V_0 + \sqrt{2} W_s & (W_s: \text{B.M.}) \\ X_s = x_0 - V_s \\ t_s = t_0 - s \end{cases}$ degenerate in both t and x.

Still at least in the interior of the domain, the solution is smooth in all variables.

u – density function of charged particles

- Hormander's condition $Lu = \sum_{j=1}^d X_j^2 u + \underbrace{\sum_{j=1}^d x_j X_{j+d} u}_{X_0}$ $d=3$, $(v,x) = (\underbrace{x_1,x_2,x_3}_{v}, \underbrace{x_4,x_5,x_6}_{x})$

$\underline{X_j} = \partial_{x_j}$ $\underline{X_0} = \sum_{j=1}^d x_j X_{j+d} = \sum_{j=1}^d x_j \partial_{x_{j+d}}$

Lie Algebra, $[X_i, X_j] = \underbrace{X_i X_j - X_j X_i}_{\text{first order differential operator}}$. Commutator operator.

If the set generated by

If the set generated by $\underbrace{\{X_0, X_1, \dots X_d\}}_{}$ $= \mathbb{R}^{2d}$. $\quad \partial x_j \sim e_j$

Satisfies the Hormander condition.

$X_{d+1}, X_{d+2} \cdots X_{2d}$ $\qquad [X_1, X_0] = \partial_{x_1} \sum_{j=1}^{d} x_j \partial_{x_{j+d}} - \sum_{j=1}^{d} x_j \partial_{x_{j+d}} \partial_{x_1}$.

$= \partial_{x_{1+d}}$.

$[X_j, X_d] = x_{j+d}$.

Hypoelliptic operator. $\quad u_t - Lu = f \quad$ Then. $\sum_{i,j=1}^{d} \|D_{x_i x_j} u\|_{L_p} \leq N \|f\|_{L_p}$

$\rightarrow$ Hormander's theorem. (constant coefficients)

General form of kinetic FP in nondivergence form

$Pu = u_t + v \cdot \nabla_x u - a^{ij}(z) D_{v_i v_j} u, \quad z = (t, x, v) \in \mathbb{R}^{1+2d}, \ \delta|\xi|^2 \leq a^{ij}\xi_i \xi_j \leq \frac{1}{\delta}|\xi|^2$

Nondivergence form eq $\begin{cases} Pu = f. \\ u(0,\cdot,\cdot) = u_0(\cdot,\cdot) \end{cases}$

Plasma Physics : Landau kinetic equation (1936, time evolution for collitional plasma)

$F_t + v \cdot \nabla_x F = Q[F, F] \qquad$ F: density of charged particles $F \geq 0$.

$(0,T) \times \underset{x\in}{\Omega} \times \underset{v\in}{\mathbb{R}^3} \qquad \Omega : \mathbb{R}^3, \mathbb{T}^3$, smooth bounded domain.

collition operator $\quad Q[F_1, F_2](t,x,v) = \mathrm{div}_v \int_{\mathbb{R}^3} \Phi(v-v')[F_1(t,x,v')(\nabla_v F_2)(t,x,v)$

$- F_2(t,x,v)(\nabla_v F_1)(t,x,v')] dv'$

bilinear, not symmetric $\quad \Phi(v) = \left(I_3 - \frac{v}{|v|} \otimes \frac{v}{|v|}\right) \cdot \frac{1}{|v|^\alpha}$ (coulomb interation: $\alpha = 1$)

- Steady state solution (homogeneous) $\mu = \frac{1}{\pi^{3/2}} e^{-|v|^2}$, $\quad Q[\mu, \mu] = 0$

is a solution in the whole space. (Maxwellian)

- Small perturbation near the Maxwellian $F = \mu + \sqrt{\mu} f \quad f$ very small.

f is not required to be nonnegative.

$\sqrt{\mu} f_t + \sqrt{\mu} v \cdot \nabla_x f - Q[\mu, \sqrt{\mu} f] - Q[\sqrt{\mu} f, \mu] - Q[\sqrt{\mu} f, \sqrt{\mu} f] = 0$.

Dividing both sides by $\sqrt{\mu}$

$f_t + v \cdot \nabla_x f - Lf - \Gamma[\overset{\leftarrow g}{f}, f] = 0 \qquad$ Can be solved by using Picard iteration

$Lf = \frac{1}{\sqrt{\mu}} Q[\mu, \sqrt{\mu} f] + \frac{1}{\sqrt{\mu}} Q[\sqrt{\mu} f, \mu] \qquad \Gamma[g, f] = \frac{1}{\sqrt{\mu}} Q[\sqrt{\mu} g, \sqrt{\mu} f]$.

$f(0,\cdot,\cdot) = 0. \quad f_0 = 0, \ \underbrace{\partial_t f_k + v \cdot \nabla_x f_k - L f_k}_{\text{Kinetic FP eq}} - \underbrace{\Gamma[f_{k-1}, f_k]}_{\text{Zeroth order term.}} = 0$. linear eq of f_k

Reduction : $f_t + v \cdot \nabla_x f = \underbrace{\mathrm{div}_v(\sigma_G \nabla_v f)}_{2^{nd}\text{ order}} + \underbrace{a_g \cdot \nabla_v f}_{1^{st}\text{ order}} + \underbrace{k_g f}_{\text{Zeroth order (nonlocal)}}$

σ_G is at least Lipschitz in v. So the equation can be rewritten into nondivergence form equation.

σ_G degenerate as $v \to \infty$, $\frac{1}{|v|^3} I \leq \sigma_G \leq \frac{1}{|v|} I$

can be fixed by considering weighted spaces (weights in v).

- If $\Omega_{x\in}$ is a domain, we need certain boundary conditions

* For now, we focus on equations in $(0,T)\times\mathbb{R}^d\times\mathbb{R}^d$ $(d=3)$.

Recall for the heat equation $\begin{cases} u_t-\Delta u=f & \text{in } (0,\infty)\times\mathbb{R}^d \\ u(0,x)=u_0(x) \end{cases}$

$$u(t,x)=K(t,\cdot)\overset{x}{*}u_0(\cdot)+\int_0^t K(t-s,\cdot)*f(s,\cdot)\,ds\,,\quad K(t,x)=\frac{1}{(4\pi t)^{d/2}}e^{-\frac{|x|^2}{4t}}$$ heat kernel fundamental sol.

Recall. $\begin{cases}\widehat{K}_t=-|\xi|^2\widehat{K} \\ \widehat{K}(0,\cdot)=1.\end{cases}$ $\widehat{K}(t,\xi)=\int_{\mathbb{R}^d}K(t,x)e^{-i\xi\cdot x}\,dx.$ ← F.T.

Inverse F.T. $F^{-1}(f(\xi))=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}f(\xi)e^{i\xi x}\,d\xi.$

$\widehat{K}(t,\xi)=e^{-|\xi|^2t}.$ Now we take the inverse F.T. to get $K(t,x)$.

Now $u_t+v\cdot\nabla_x u-\Delta_v u=0.$ Let $G(t,x,v)$ be the fundamental solution when we put a delta mass at $(0,0)$ when $t=0$. $\widehat{G}(t,\underset{x}{\underset{\uparrow}{k}},\underset{v}{\underset{\uparrow}{\xi}})$

$\widehat{-\Delta_v G}=|\xi|^2\widehat{G}.$ $\begin{cases}\widehat{G}_t-k\nabla_\xi\widehat{G}+|\xi|^2\widehat{G}=0 \\ \widehat{G}(0,k,\xi)=1.\end{cases}$

$\widehat{v\nabla_x G}=-k\nabla_\xi\widehat{G}$

This is a first order PDE, can be solved by using the method of characteristics

$H(t,k,\xi)=\widehat{G}(t,k,\xi-tk).$ $\begin{cases}\partial_t H+|\xi-tk|^2H=0 \\ H(0,\cdot,\cdot)=1.\end{cases}$ $\Rightarrow H(t,k,\xi)\,\underline{=}\,e^{-\int_0^t|\xi-ks|^2ds}.$

$$\widehat{G}(t,k,\xi)=e^{-\int_0^t|\xi+tk-sk|^2ds}\underset{t-s\to s}{\overset{\uparrow}{=}}e^{-\int_0^t|\xi+sk|^2ds}$$

Now we take the inverse F.T. to find G.

$$G(t,x,v)=\frac{1}{(2\pi)^{2d}}\iint_{\mathbb{R}^d\,\mathbb{R}^d}e^{-\int_0^t|\xi+sk|^2ds}e^{ik\cdot x+i\xi\cdot v}\,d\xi\,dk.$$

$$\int_0^t|\xi+sk|^2ds=\int_0^t\left(|\xi|^2+s^2|k|^2+2s\,\xi\cdot k\right)ds=|\xi|^2t+\frac{t^3}{3}|k|^2+t^2\,\xi\cdot k.$$

$$G(t,x,v)=\frac{1}{(2\pi)^{2d}}\iint e^{-\left(|\xi|^2t+\xi\cdot k\,t^2+\frac{|k|^2}{3}t^3\right)}e^{ik\cdot x+i\xi\cdot v}\,d\xi\,dk.$$

$d=1$ $$e^{-\underbrace{(\xi,k)}_{2d}\underbrace{\begin{bmatrix}t & t^2/2\\ t^2/2 & t^3/3\end{bmatrix}}_{A}\begin{pmatrix}\xi\\k\end{pmatrix}}$$ $A\geq0.$ $\begin{bmatrix}tI_d & t^2/2\,I_d\\ t^2/2\,I_d & t^3/3\,I_d\end{bmatrix}$

$$G(t,x,v)=\frac{C_d}{\sqrt{\det A}}e^{-\frac14(x,v)A^{-1}\begin{pmatrix}x\\v\end{pmatrix}}=\frac{C_d}{t^{2d}}e^{-\frac{|v|^2}{4t}-\frac{3|x-tv/2|^2}{t^3}}$$ (More involved than the heat kernel).

We can use this to prove the Hormander's theorem.

Q: variable coefficients (non-smooth coefficients)

Kernel free method.

- Translation of solutions $\begin{pmatrix} u(t,x) \\ f(t,x) \end{pmatrix} \longrightarrow \begin{pmatrix} u(t-t_0,\, x-x_0) \\ f(t-t_0,\, x-x_0) \end{pmatrix}$

Kinetic F.P. $u_t + v\cdot\nabla_x u - \Delta_v u = f$

$u(t,x,v) \to u(t-t_0, x-x_0, v-v_0)$.

$f(t,x,v) \to f(t-t_0, x-x_0, v-v_0)$

Not a solution!

Can we used the same translation

$u_t(t-t_0, x-x_0, v-v_0) + v\cdot(\nabla_x u)(t-t_0, x-x_0, v-v_0)$
$-(\Delta_v u)(t-t_0, x-x_0, v-v_0) \overset{?}{=} f(t-t_0, x-x_0, v-v_0)$

$\begin{cases} t-t_0 \to s \\ x-x_0 \to y \\ v-v_0 \to v' \end{cases}$

$u_t(s,y,v') + (v_0+v')(\nabla_x u)(s,y,v')$
$-(\Delta_v u)(s,y,v') \overset{?}{=} f(s,y,v')$

$\begin{pmatrix} u(t,x,v) \\ f(t,x,v) \end{pmatrix} \to \begin{pmatrix} u(t-t_0,\, x-x_0-v_0(t-t_0),\, v-v_0) \\ f(t-t_0,\, x-x_0-v_0(t-t_0),\, v-v_0) \end{pmatrix}$ is still a solution

Center of x is transported by v_0.

$G(t,x,v;t',x',v') = \underbrace{G(t-t', x-x'-(t-t')v, v-v')}$

Cylinder: $Q_r(z_0) = \{z : t_0 - r^2 < t < t_0,\ |v-v_0|<r,\ |x-x_0-v_0(t-t_0)|<r^3\}$

$\tilde{Q}_r(z_0) = \{z : t_0 - r^2 < t < t_0 + r^2 \ \dots\}$ double cylinder

Recall the parabolic distance $\rho((t,x),(s,y)) = \max\{|x-y|, \sqrt{|t-s|}\}$.

$\rho(z,z_0) = \max\{\sqrt{|t-t_0|},\ |v-v_0|,\ |x-x_0-(t-t_0)v_0|^{1/3}\}$ Not symmetric.

$\tilde{\rho}(z,z_0) = \rho(z,z_0) + \rho(z_0,z)$ (symmetric)

Kinetic Hölder Space: $0<\alpha\le 1$. $[f]_{C^\alpha_{kin}} = \sup_{z\ne z_0} \frac{|f(z)-f(z_0)|}{\rho^\alpha(z,z_0)}$. $\rho \sim \tilde{\rho}$.

Recall Def of quasidistance $\tilde{\rho}$: ① $\tilde{\rho}(x,y)\ge 0$, $\tilde{\rho}(x,y)=0$ iff $x=y$

② $\tilde{\rho}(x,y) = \tilde{\rho}(y,x)$

③ $\tilde{\rho}(x,z) \le K(\tilde{\rho}(x,y)+\tilde{\rho}(y,z))$

Lem: ① $\rho(z,z_0) \le 2\rho(z_0,z)$.

② $\rho(z,z_0) \le 2(\rho(z,z_1)+\rho(z_1,z_0))$

③ $\hat{\rho}(z,z_0) := \rho(z,z_0)+\rho(z_0,z)$ is a quasidistance. by ① and ②

④ $\hat{Q}_r = \{z : \hat{\rho}(z,z_0)<r\}$ $\hat{Q}_r \subset \tilde{Q}_r \subset \hat{Q}_{3r}$

Consequently, $(\mathbb{R}^{1+2d}_T, \tilde{\rho}, dz)$ is a space of homogeneous type.

$\mathbb{R}^{1+2d}_T = (-\infty,T)\times\mathbb{R}^{2d}$

(space with a quasi-distance and a doubling measure)

$|\hat{Q}_{2r}| \le C|\hat{Q}_r| \leftarrow$ doubling property

Pf of part ①: We only need to show that $|x-x_0-(t-t_0)v_0|^{1/3} \le 2\rho(z_0,z)$

LHS $= |x-x_0-(t-t_0)v|^{1/3} + |(t-t_0)(v-v_0)|^{1/3}$ $(a+b)^{1/3} \le a^{1/3}+b^{1/3}$, $a,b\ge 0$

$|x-x_0-(t-t_0)v|^{1/3} \le \rho(z_0,z)$

$\uparrow$ by the Δ-inlq.

$|(t-t_0)(v-v_0)|^{1/3} = |t-t_0|^{1/3}|v-v_0|^{1/3} \le \frac{2}{3}|t-t_0|^{1/2} + \frac{1}{3}|v-v_0| \le \rho(z_0,z)$

$C \geq 1$ $\quad\quad \leq \rho(z_0, z) \quad\quad \rho(z_0, z)$.

<u>Ex</u>. $\rho_C(z, z_0) = \max\left\{ |t-t_0|^{1/2}, |v-v_0|, \frac{1}{C}|x-x_0-(t-t_0)v_0|^{1/3} \right\}$

$\hat{\rho}_C(z, z_0) = \rho_C(z, z_0) + \rho_C(z_0, z)$. Then the lemma still holds.

S_p-space: $(1<p<\infty)$ $\left\{ u \in L_p : D_v u,\ D_v^2 u,\ u_t + v\cdot\nabla_x u \in L_p \right\}$. Banach space.

$\|u\|_{S_p} = \|u\|_{L_p} + \|D_v u\|_{L_p} + \|D_v^2 u\|_{L_p} + \|u_t + v\cdot\nabla_x u\|_{L_p}$ $\quad$ $u_t \Leftarrow$ No estimate

$\nabla_x u \Leftarrow$ No estimate.

<u>Thm</u> (L_2-estimate) $P_0 u = u_t + v\cdot\nabla_x u - a_{ij}(t) D_{v_i v_j} u$. $\lambda \geq 0$,

$\delta|\xi|^2 \leq a_{ij}(t)\xi_i\xi_j \leq \frac{1}{\delta}|\xi|^2$

① Suppose $u \in S_2(\mathbb{R}^{1+2d}_T)$. $\quad (\mathbb{R}^{1+2d}_T = (-\infty, T)\times\mathbb{R}^d)$.

$P_0 u + \lambda u = f \in L_2(\mathbb{R}^{1+2d}_T)$. Then we have

$\|D_v^2 u\|_{L_2} + \sqrt{\lambda}\|D_v u\|_{L_2} + \lambda\|u\|_{L_2} + \|(-\Delta_x)^{1/3} u\|_{L_2} + \|D_v(-\Delta_x)^{1/6} u\|_{L_2} \leq N(d,\delta)\|f\|_{L_2}$

② Suppose $f \in L_2(\mathbb{R}^{1+2d}_T)$. $\lambda > 0$.

Then $\exists!$ solution $u \in S_2(\mathbb{R}^{1+2d}_T)$ to this equation

③ Cauchy problem (e.g. zero initial condition. by taking the zero extension for $t<0$).

<u>Lem</u>: $\lambda > 0$, $\overset{=h(t,k,\xi)}{h} \in C_b(\mathbb{R}^{1+2d}_T)$, $D_\xi h \in C_b(\mathbb{R}^{1+2d}_T)$ $\partial_t h, f \in L_\infty(C_b \cap L_2(\mathbb{R}^{2d}))$.

$\partial_t h + a^{ij}(t)\xi_i\xi_j h - k D_\xi h + \lambda h = f$. $\quad (**)$

Then $\lambda\|h\|_{L_2} + \||\xi|^2 h\|_{L_2} + \||k|^{2/3} h\|_{L_2} + \||k|^{1/3}\xi h\|_{L_2} \leq N(d,\delta)\|f\|_{L_2}$.

<u>Pf</u>: We can solve the eq $(**)$ by using the method of characteristics

$h(t,k,\xi) = \int_{-\infty}^{t} e^{-\lambda(t-t') - \int_{t'}^{t} a^{ij}(s)(\xi_i - k_i(s-t))(\xi_j - k_j(s-t))} f(t', k, \xi - k(t'-t))\,dt'$

Young's inequality and Minkowski's ineq.