Lecture 4 (12/28/21)

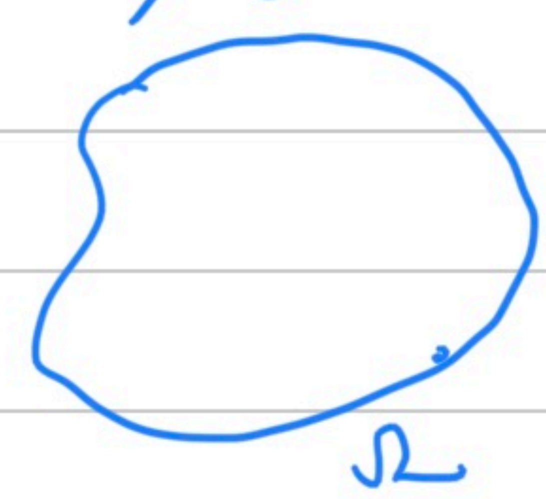
Divergence form: $\underbrace{u_t + v\cdot\nabla_x u}_{} - D_{v_i}(a_{ij} D_{v_j} u) + b_i D_{v_i} u + D_{v_i}(\tilde b_i u) + \lambda u = \operatorname{div} F + g$

When a_{ij}'s satisfy the same VMO condition, then we have

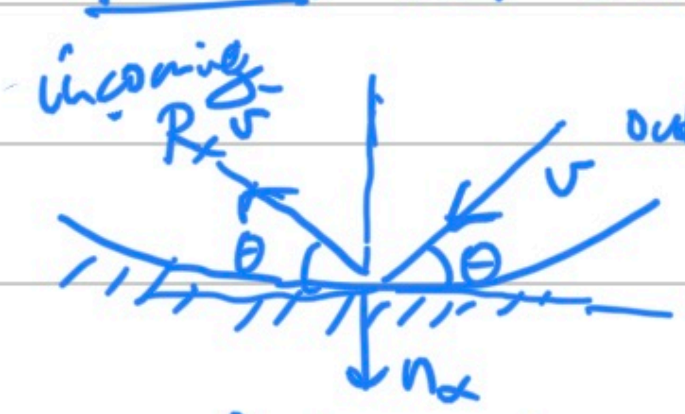
$\|D_v u\|_{L_p} + \sqrt{\lambda}\|u\|_{L_p} + \|(-\Delta_x)^{1/6} u\|_{L_p} \le N(\|F\|_{L_p} + \frac{1}{\sqrt{\lambda}}\|g\|_{L_p})$. + solvability

Remark $u_t + v\cdot\nabla_x u \notin L_p$, it is in $H^{-1}_{p,v}$

Boundary estimates. $x\in\Omega$ smooth bounded domain.



Specular reflection boundary condition



Billiard (Snooker)

flip the normal velocity and keep the tangential velocity

$u \to f$ Boundary condition: $f(t,x,v) = f(t,x,R_x v)$, $x\in\partial\Omega$, $v\in\mathbb{R}^3$

$f \to h$ $R_x v = v_\tau - v_n\cdot n = \underline{v - 2(v\cdot n_x) n_x}$ normal $(v\cdot n_x)n_x$

Other BC: ① Absorbing, $\gamma_+ = \{(x,v): x\in\partial\Omega,\ n_x\cdot v>0\}$ outgoing set.

$\gamma_- = \{\cdots\ x\in\partial\Omega,\ n_x\cdot v<0\}$ incoming set

$\gamma_0 = \{\qquad n_x\cdot v=0\}$ grazing set.

on γ_-, $f(t,x,v) = F$.

② Diffuse BC. $(x,v)\in\gamma_-$, $f(t,x,v) = \sqrt{\mu(v)}\int_{v'\cdot n_x>0} v'\cdot n_x f(t,x,v')\sqrt{\mu(v')}\,dv'$

μ Maxwellian

Notation: $\Sigma^T = (0,T)\times\Omega\times\mathbb{R}^3$.

$\Sigma^T_\pm = (0,T)\times\gamma_\pm$ ← sum of 2nd order – finite difference

Consider. $\begin{cases} f_t + v\cdot\nabla_x f - \nabla_v\cdot(a\nabla_v f) + b\cdot\nabla_v f + \lambda f = h & (*) \\ f(t,x,v) = f(t,x,R_x v), & x\in\partial\Omega. \end{cases}$

a bounded uniformly elliptic, a Lipschitz in v, b Lipschitz in v.

Weak solutions. $\|f\|_{L_{p,\theta}} = \left(\int |f|^p \underbrace{(1+|v|^2)^{\theta/2}}_{\langle v\rangle^\theta} dz\right)^{1/p}$ $\langle v\rangle = (1+|v|^2)^{1/2}$ weighted L_p norm

$\theta\ge 0$, $T>0$.

Def: Finite energy weak sol (FEWS). We say (f, f_+, f_-, f_T, f_0) is a FEWS to $(*)$ iff

①. $f, \nabla_v f, h\in L_{2,\theta}(\Sigma^T)$, $f_\pm \in L_\infty(\Sigma^T_\pm, |v\cdot n_x|)$ ← weight. $f_T, f_0\in L_{2,\theta}(\Omega\times\mathbb{R}^3)$

② $f_-(t,x,v) = f_+(t,x,R_x v)$ on Σ^T_- a.e.

③ $\forall\phi\in C^1_0(\overline{\Sigma^T})$, $-\int_{\Sigma^T} (Y\phi) f\,dz + \int_{\Omega\times\mathbb{R}^3} (f_T(x,v)\phi(T,x,v) - f_0(x,v)\phi(0,x,v))\,dx\,dv$

$+\int_{\Sigma^T_+} f_+\phi|v\cdot n_x|\,d\sigma\,dt - \int_{\Sigma^T_-} f_-\phi|v\cdot n_x|\,d\sigma\,dt$

$+\int_{\Sigma^T} (\lambda f\phi + a D_v f D_v\phi + b D_v f\phi)\,dz = \int_{\Sigma^T} h\phi$ $\Sigma^T = (0,T)\times\Omega\times\mathbb{R}^3$

Remark: For weak solutions, $f_\pm$, f_T, f_0 are not trace of f.

$\leftarrow f, D_v f, D_v^2 f, \in L_2$

Finite energy strong sol (FESS): first f is a FEWS. $f \in S_2(\Sigma_T)$

(*) is satisfied a.e. in Σ^T.

In fact, when f is a FESS, then f satisfies the Green's identity

$$\int_{\Sigma^T} \underbrace{(Yf)}_{\in L_2}\phi + (Y\phi) f = \int_{\Omega\times\mathbb{R}^3} \big(f(T,\cdot,\cdot)\phi(T,\cdot,\cdot) - f(0,\cdot,\cdot)\phi(0,\cdot,\cdot)\big)\,dx\,dv$$
$$+ \int_{\Sigma^T_+} f_+\phi\,|v\cdot n_x|\,d\sigma\,dt - \int_{\Sigma^T_-} f_-\phi\,|v\cdot n_x|\,d\sigma\,dt$$

by using integration by parts. $f_\pm, f_T, f_0$ are the traces of f

Existence of FEWS: $\Omega\in C^2$, $\theta\ge 0$, $T>0$, $h\in L_{2,\theta}(\Sigma^T)\cap L_\infty(\Sigma^T)$, $f_0\in L_{2,\theta}(\Omega\times\mathbb{R}^3)\cap L_\infty(\cdot)$

Then (*) has a FEWS, and it satisfies

$$\|f_T\|_{L_{2,\theta}(\Omega\times\mathbb{R}^3)} + \sqrt{\lambda}\,\|f\|_{L_{2,\theta}(\Sigma^T)} + \|\nabla_v f\|_{L_{2,\theta}(\Sigma^T)} \le N\Big(\frac{1}{\sqrt{\lambda}}\|h\|_{L_{2,\theta}(\Sigma^T)} + \|f_0\|_{L_{2,\theta}(\Omega\times\mathbb{R}^3)}\Big)$$

$$\max\Big\{\|f_T\|_{L_\infty(\Omega\times\mathbb{R}^3)} + \|f\|_{L_\infty(\Sigma^T)} + \|f_\pm\|_{L_\infty(\Sigma^T_\pm,|v\cdot n_x|)} \le \frac{1}{\lambda}\|h\|_{L_\infty(\Sigma^T)} + \|f_0\|_{L_\infty(\Omega\times\mathbb{R}^3)}\Big\}$$

Ideas: 1 Finite difference approximations (N.V. Krylov).

2 Beals-Protopopescu (1987) Existence of weak solutions of Vlasov type equations

Uniqueness. (need energy identity). f_1, f_2 both of them satisfies the energy inequality

$f_1 - f_2$ may not satisfy the energy inequality.

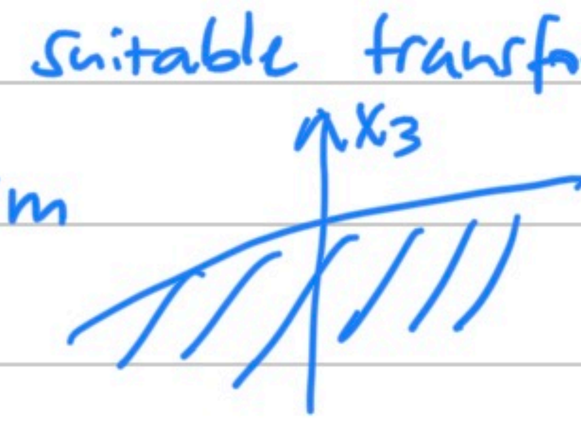
Need higher regularity of solutions (weak sol $\Rightarrow$ strong sol).

Thm: Let $\Omega\in C^3$, $\theta\ge 2$, $T>0$, $a=I$, $h\in L_{2,\theta-2}$, f_0 in a suitable space.
then any FEWS f is also a FESS $f\in S_{2,\theta-2}(\Sigma^T)$ $[f, D_v f, D_v^2 f \in L_{2,\theta-2}(\Sigma^T)]$.
$f, D_v f \in L_{7/3,\theta-2}(\Sigma^T)$. Moreover, any two FEWS must be the same.
Finally, $h\in L_{p,\theta-4}$, $(p>14,\ \theta\ge 16)$. then $f\in S_{p,\theta-16}\cap C(\overline{\Sigma^T})$
$f, f_v \in L_\infty((0,T); C^{\alpha/3,\alpha}_{x,v})$, $\alpha = 1-\frac{14}{p}$ $\quad 2(1+2d)$ $\quad\uparrow$ not bounded

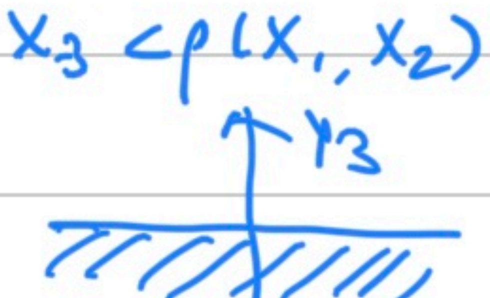
$= 2\cdot 7 = 14$ effective dim

Idea: Find a suitable transformation to flatten the boundary, then use reflection

Usual transform

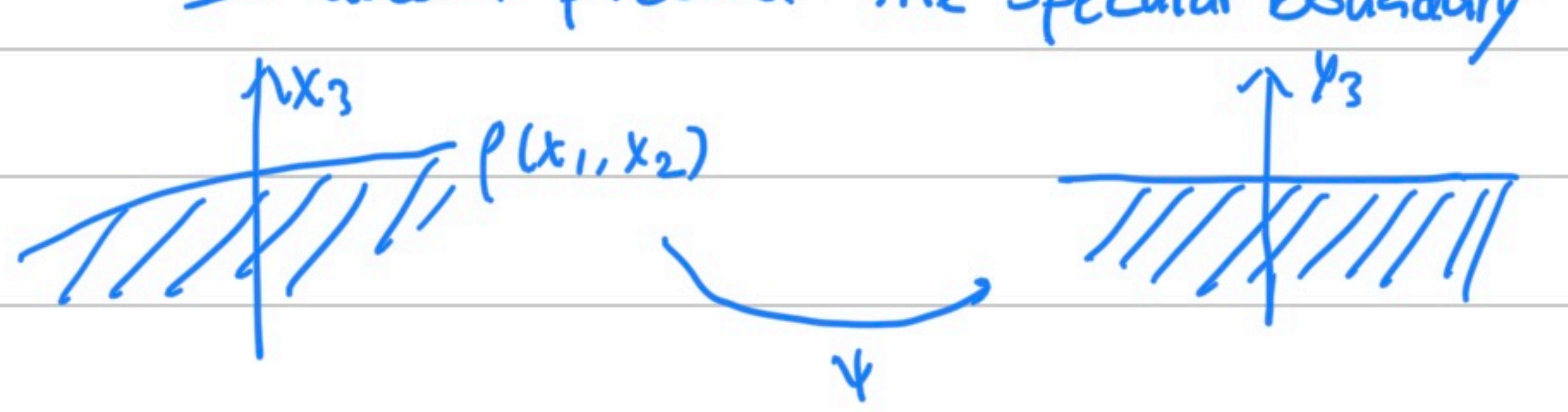


$x_3 < \rho(x_1,x_2)$ $\quad\rho\in C^3$ function



$$\begin{cases} y_1 = x_1 \\ y_2 = x_2 \\ y_3 = x_3 - \rho(x_1,x_2) \end{cases}$$

It doesn't preserve the specular boundary condition.



$$\psi^{-1}(y) = \begin{pmatrix} y_1 - \rho_1 y_3 \\ y_2 - \rho_2 y_3 \\ y_3 + \rho \end{pmatrix} \qquad \rho_i = D_i\rho \quad i=1,2.$$

This transformation maps n_x to $c(0,0,1)$, specular BC is preserved.

$$D\psi^{-1}\big|_{y_3=0} = \begin{bmatrix} 1 & 0 & -\rho_1 \\ 0 & 1 & -\rho_2 \end{bmatrix} \qquad D\psi^{-1}\big|_{y_3=0}\cdot\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}-\rho_1\\-\rho_2\end{pmatrix} \parallel n_x.$$

$$D\Psi^{-1}\Big|_{y_3=0} = \underbrace{\begin{bmatrix} 1 & 0 & -\rho_1 \\ 0 & 1 & -\rho_2 \\ \rho_1 & \rho_2 & 1 \end{bmatrix}}. \qquad D\Psi^{-1}\Big|_{y_3=0} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\rho_1 \\ -\rho_2 \\ 1 \end{pmatrix} \,/\!/\, n_x.$$

$y = \Psi(x)$. $\quad w = \underbrace{D\Psi(x)}_{3\times 3}\, \underset{\sim}{v} \qquad (t, x, v) \to (t, y, w). \Rightarrow \frac{\partial w}{\partial v} = D\Psi = \frac{\partial y}{\partial x}$

$$\begin{bmatrix} 1 & 0 & -\rho_1 \\ 0 & 1 & -\rho_2 \\ \rho_1 & \rho_2 & 1 \end{bmatrix} \cdot \underbrace{\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix}} = \underbrace{\begin{bmatrix} w_1 - \rho_1 w_3 \\ w_2 - \rho_2 w_3 \\ \rho_1 w_1 + \rho_2 w_2 + w_3 \end{bmatrix}}_{v}. \qquad \begin{bmatrix} 1 & 0 & -\rho_1 \\ 0 & 1 & -\rho_2 \\ \rho_1 & \rho_2 & 1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ -w_3 \end{bmatrix} = \underbrace{\begin{bmatrix} w_1 + \rho_1 w_3 \\ w_2 + \rho_2 w_3 \\ \rho_1 w_1 + \rho_2 w_2 - w_3 \end{bmatrix}}_{R_x v \quad (\text{for } x)}.$$

Jacobia: $\det \frac{\partial x}{\partial y} \cdot \det \frac{\partial v}{\partial w} = \left(\det \frac{\partial x}{\partial y}\right)^2$.

Diffusion: $D_v(a D_v f) \rightsquigarrow D_w(A D_w \tilde f)$

$\tilde f(t, y, w) = f(t, x(y), v(y,w)) \left(\det \frac{\partial x}{\partial y}\right)^2$, $\quad \underline{A} = \underbrace{\frac{\partial y}{\partial x}\, a(t, x(y), v(y,w)) \left(\frac{\partial y}{\partial x}\right)^T}$

$b \cdot \nabla_v f \rightsquigarrow B \nabla_w \tilde f$, $\quad B = \frac{\partial y}{\partial x} b.$

$Yf = f_t + v \cdot \underline{\nabla_x} f \rightsquigarrow (\partial_t + w \cdot \nabla_y)\tilde f + \boxed{\mathrm{div}_w (X \tilde f)}$ geometric term.

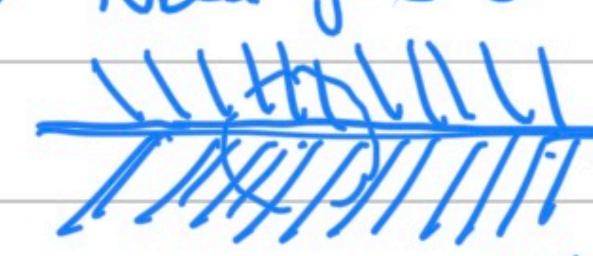
$\uparrow\ \tilde f(t, y(x), \underline{w(x,v)})$

$\underbrace{v \cdot \frac{\partial y}{\partial x}}_{w} \nabla_y \tilde f$

$X = \frac{\partial y}{\partial x} \cdot \underline{\frac{\partial}{\partial y}\left(\frac{\partial x}{\partial y} w\right)} \cdot w$ $\quad$ quadratic in w

① Need to introduce weights.

② Need $\rho \in C^3$



$$\begin{cases} (\partial_t + w \cdot \nabla_y)\tilde f - D_w(A D_w \tilde f) \\ \qquad + B \nabla_w \tilde f + \mathrm{div}_w(X \tilde f) = \tilde h \\ \tilde f(t, y_1, y_2, 0, \underbrace{w_1, w_2, w_3}_{w}) = \tilde f(t, y_1, y_2, 0, \underbrace{w_1, w_2, -w_3}_{Rw}) \end{cases}$$

$\tilde f$, $y_3 < 0$

Specular boundary condition

Extension $\quad \tilde f(t, y, w) := \tilde f(t, Ry, Rw)$ $\quad$ is well defined because when $y_3 = 0$ $\quad y = Ry$

$y_3 > 0 \qquad Ry = (y_1, y_2, -y_3) \qquad Rw = (w_1, w_2, -w_3)$. $\qquad \tilde f(t, y_1, y_2, 0, w) = \tilde f(t, y_1, y_2, 0, Rw)$

$A = [A_{ij}]_{i,j=1}^3$ $\quad$ when $y_3 > 0$ $\quad A_{ij}(t,y,w) = \begin{cases} A_{ij}(t, Ry, Rw) & \text{when } i,j < 3, \text{ or } (i,j) = (3,3) \\ -A_{ij}(t, Ry, Rw) & \text{when } \begin{cases} i = 1,2 \\ j = 3 \end{cases} \text{ or } \begin{cases} i = 3 \\ j = 1,2. \end{cases} \end{cases}$

To apply the L^p estimates, need A to be $\underline{L^\infty((0,T), VMO_{x,v})}$.

Need to check that the coefficients are continuous across the boundary $y_3 = 0$

$\underline{VMO_{t,x,v}}$. with respect to the quasimetric, but the transformation is not compatible with the quasi metric.

This is equivalent to $\quad \underline{A_{ij}(t, y, Rw) = -A_{ij}(t, y, w)}^{(**)}$ $\quad$ when $y_3 = 0$, $\begin{cases} i = 1,2 \\ j = 3 \end{cases}$ or $\begin{cases} i = 3 \\ j = 1,2 \end{cases}$

$\Rightarrow$ continuity.

When $a = I$. $\quad A = \frac{\partial y}{\partial x}\left(\frac{\partial y}{\partial x}\right)^T$. $\qquad A^{-1} = \left(\frac{\partial x}{\partial y}\right)\left(\frac{\partial x}{\partial y}\right)^T = \begin{bmatrix} 1 & 0 & -\rho_1 \\ 0 & 1 & -\rho_2 \\ \rho_1 & \rho_2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \rho_1 \\ 0 & 1 & \rho_2 \\ -\rho_1 & -\rho_2 & 1 \end{bmatrix}$

$$A = \begin{array}{c} \left[\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right] \begin{array}{c} 2 \\ 1 \end{array} \end{array} \Rightarrow (**) . \qquad = \begin{bmatrix} 1+\rho_1^2 & \rho_1\rho_2 & 0 \\ \rho_1\rho_2 & 1+\rho_2^2 & 0 \\ 0 & 0 & 1+\rho_1^2+\rho_2^2 \end{bmatrix} \quad \text{block diagonal matrix}$$

general a, there is a quite implicit condition. test by f

* Linear Landau equation $F=\mu+\sqrt{\mu}f$. $\partial_t f + v\cdot\nabla_x f - \underbrace{\nabla_v(\sigma_G \nabla_v f)}_{2^{nd}} \underset{+\varepsilon I}{} - \underbrace{a_g\cdot\nabla_v f}_{1^{st}} - \underbrace{K_g f}_{\text{nonlocal } 0^{th} \text{ order.}} = h.$

$G=\mu+\sqrt{\mu}g$

$\sigma_G = \Phi * G = \underbrace{\Phi*\mu}_{\sigma} + \Phi*(\sqrt{\mu}g)$. $\Phi = \left(I_3 - \frac{v}{|v|}\otimes\frac{v}{|v|}\right)\frac{1}{|v|}$ $\quad \begin{cases} g=f_k \\ f=f_{k+1}\end{cases}$ Picard

If g is sufficiently small, then $\frac{c_1}{1+|v|^3}I_3 \le \sigma_G \le \frac{c_2}{1+|v|}I_3$, degenerate in v as $|v|\to\infty$

σ_G: $\lambda_1 \sim \frac{1}{1+|v|^3}$. Simple. eigenvector is $\parallel v$.

$\lambda_2 \sim \frac{1}{1+|v|}$ multiplicity 2. eigenvectors are $\perp v$.

Impose the specular BC.

$H_{\sigma,\theta}$ space $\|f\|_{\sigma,\theta} = \left(\int_{\Sigma^T}(\sigma^{ij}D_{v_i}f\,D_{v_j}f + \sigma^{ij}v_iv_jf)\langle v\rangle^\theta dz\right)^{1/2}$

$\|f\|_{\sigma,\theta} \le N\left(\|f\|_{L_{2,\theta-1}} + \|D_vf\|_{L_{2,\theta-1}}\right)$ $\quad \|f\|_{L_{2,\theta-1}} \le N\|f\|_{\sigma,\theta}$.

We define FESS, $f\in S_2(\Sigma^T)\cap L_{2,\theta}(\Sigma^T)\cap H_{\sigma,\theta}(\Sigma^T)$.

For g we assume $\begin{cases} g(t,x,v) = g(t,x,R_xv) & x\in\partial\Omega,\ v\in\mathbb{R}^3 \\ g\in L_\infty((0,T), C^{\alpha/3,\alpha}_{x,v}(\Omega\times\mathbb{R}^3)) \end{cases}$

Thm. $\Omega\in C^3$. $T>0$, $p>14$, f_0 in a suitable space. $\|g\|_{L_\infty}\le\varepsilon$.

Then $\exists\,\theta_0>4$. $\theta>4$ $\theta\ge\theta_0$. $\varepsilon(\theta)$. θ', θ'' s.t. the equation has a unique FESS f

$f\in C(\overline{\Sigma^T})\cap S_{2,\theta'}\cap S_{p,\theta''}\cap H_{\sigma,\theta}\cap \underbrace{L_\infty(C^{\alpha/3,\alpha}_{x,v})}$ and $\nabla_v f\in L_\infty((0,T), C^{\alpha/3,\alpha}_{x,v})$

$\alpha = 1-14/p$

- For the existence of a FEWS, use the method of vanishing viscosity.
- For the uniqueness, need to show the weak sol is also strong. We again use the argument of flattening the boundary and the mirror extension argument
- Lp estimate (take a partition of v, and keep track of the dependence of the ellipticity constant. $v\sim 2^k$, weight in v is important.
  + Embedding

Key σ_G satisfies $A_{ij}(t,y,Rw) = -A_{ij}(t,y,w)$. (**) when $y_3=0$ $\begin{cases} i=1,2 \\ j=3\end{cases}$ $\begin{cases} i=3 \\ j=1,2.\end{cases}$

$A = \frac{\partial y}{\partial x}\sigma_G(t,x(y),v(y,w))\left(\frac{\partial y}{\partial x}\right)^T$ $\quad (G=\mu+\sqrt{\mu}g)$ $\quad \sigma_G = \underbrace{\Phi*\mu}_{\text{function of } x.} + \underbrace{\Phi*(\sqrt{\mu}g)}_{g}$.

Lem. If g satisfies the specular BC. then (**) is satisfied.

pf: $A(t,y,w) = \underset{\substack{\uparrow\\ \text{Jacobian.}}}{c(y)}\int_{\mathbb{R}^3}\frac{\partial y}{\partial x}\tilde{\Phi}(y,w)\left(\frac{\partial y}{\partial x}\right)^T\tilde{g}(t,y,w-w')dw'$, $\tilde{\Phi}(y,w) = \Phi(v(y,w))$

$\underline{y_3=0}$. $A(t,y,Rw) = c(y)\int_{\mathbb{R}^3}\frac{\partial y}{\partial x}\tilde{\Phi}(y,Rw)\left(\frac{\partial y}{\partial x}\right)^T\tilde{g}(t,y,Rw-w')dw'$ $\quad w'\to Rw'$

$= c(y)\int_{\mathbb{R}^3}\frac{\partial y}{\partial x}\tilde{\Phi}(y,Rw)\left(\frac{\partial y}{\partial x}\right)^T\underbrace{\tilde{g}(t,y,R(w-w'))}_{\overset{\shortparallel}{\tilde{g}(t,y,w-w')}}dw'$. because of the specular BC.

Need to check (**)

$$\frac{\partial y}{\partial x}\tilde{\Phi}(y,w)\left(\frac{\partial y}{\partial x}\right)^T = J = \underbrace{\left|\frac{\partial x}{\partial y}w\right|^{-1}}_{v}\underbrace{\frac{\partial y}{\partial x}\left(\frac{\partial y}{\partial x}\right)^T} - \left|\frac{\partial x}{\partial y}w\right|^{-3}\cdot ww^T$$

Need to check J satisfies $(**)$.

$$ww^T = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}(w_1\ w_2\ w_3) =$$

$+ \to -$

$+ \to -$

$$\left|\frac{\partial x}{\partial y}w\right|^2 = w^T\underbrace{\left(\frac{\partial x}{\partial y}\right)^T\frac{\partial x}{\partial y}}w.$$

$$\left[\begin{array}{cc|c} 1+p_1^2 & p_1p_2 & 0 \\ p_1p_2 & 1+p_2^2 & 0 \\ \hline 0 & 0 & 1+p_1^2+p_2^2 \end{array}\right]$$

Then $(**)$ is satisfies for A.

Open problems: $f\in S_2$ across the boundary. $f, \nabla_v f \in C_v^{1-\varepsilon}$ (specular)

Regularity in x. $D_x^{2/3} f \in L_p \Rightarrow f\in C_x^{2/3-\varepsilon}$, Is f Lip in x? open.

$f\in C^\infty$ if a is very nice.

Other type of boundary conditions. (Absorbing). in Ω smooth $f\in C_v^{1-\varepsilon}$ $C_x^{1/3-\varepsilon}$ optimal?

Diffuse boundary condition (also open) best regularity of solutions near the boundary

uniqueness of weak solutions?