

BROWNIAN MOTION AND DIRICHLET PROBLEMS AT INFINITY

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1. INTRODUCTION

A Cartan-Hadamard manifold is a complete, simply connected Riemannian manifold with nonpositive sectional curvature. We fix a reference point $o \in M$ once and for all. It is well known that the exponential map $\exp : T_oM \rightarrow M$ from the tangent space T_oM based at o is a diffeomorphism. This defines a polar coordinate system (r, θ) on M . Two geodesic rays γ_1 and γ_2 on M are called equivalent if there is a constant C such that $d(\gamma_1(t), \gamma_2(t)) \leq C$ for all $t \geq 0$. It can be shown that this is an equivalence relation on the set of geodesic rays. The set of equivalence classes is the sphere at infinity $S_\infty(M)$. A basic fact of Cartan-Hadamard manifolds is that $\widehat{M} = M \cup S_\infty(M)$ with a properly defined topology (called the cone topology) is a compactification of M . For each $o \in M$, the sphere at infinity $S_\infty(M)$ can be identified homeomorphically with the unit sphere in the tangent space T_oM . If (r, θ) are the polar coordinates based at o , then a sequence of points $z_n \in M$ converges to a boundary point $\theta_0 \in S_\infty(M)$ if and only if $r(z_n) \rightarrow \infty$ and $\theta(z_n) \rightarrow \theta_0$ (see Bishop and O'Neill[6]).

Given a continuous function f on $S_\infty(M)$, the Dirichlet problem at infinity is to find a function $u_f \in C^\infty(M) \cap C(\widehat{M})$ which is harmonic on M and is equal to f on $S_\infty(M)$. We say that the Dirichlet problem at infinity is solvable for M if for every $f \in C(S_\infty(M))$ there is a unique solution u_f . This property of a Cartan-Hadamard manifold can be obtained under certain conditions on the curvature of M and can be approached analytically or probabilistically. For analytic methods see Anderson[3], Choi[7], Anderson and Schoen[4], Ancona[1], Borbely[5]; for probabilistic methods see Pratt[16], Kifer[12], Pinsky[15], Sullivan[18], Kendall[10], Hsu and March[9], Hsu and Kendall[10], Leclercq[14]. The more difficult problem of identifying the Martin boundary with the boundary at infinity was discussed in Anderson and Schoen[4] and Kifer[13]. We are mainly concerned with a probabilistic approach to the problem, which involves basically proving the angular convergence of transient Brownian motion.

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In this paper we will combine an improved version of the method used in Hsu and March[9] and an idea from Lerclercq[14] to prove the solvability of the Dirichlet problem at infinity under certain curvature growth conditions more generous than previously known. We consider two typical situations. In the first case, the sectional curvature is assumed to be bounded by a negative constant: $\text{Sect}_x \leq -a^2$. In the second case, we assume that $\text{Sect}_x \leq -c/r^2$ ($r = r(x) = d(x, o)$). This second case is significant because it vanishes as $r \rightarrow \infty$. Let's now state our main theorems.

Theorem 1.1. *Let M be a Cartan-Hadamard manifold. Suppose that there exists a positive constant a and a positive and nonincreasing function h with $\int_0^\infty rh(r)dr < \infty$ such that*

$$-h(r)^2 e^{2ar} \leq \text{Ric}_x \quad \text{and} \quad \text{Sect}_x \leq -a^2.$$

Then the Dirichlet problem at infinity for M is solvable.

Early lower bounds of the form $Ce^{\lambda ar}$ were obtained in Bordely[5] with $\lambda < 1/3$ and Lerclercq[14] with $\lambda < 1/2$. Our result represents a significant improvement in this respect.

Theorem 1.2. *Let M be a Cartan-Hadamard manifold. Suppose that there exist positive constants r_0 , $\alpha > 2$, and $\beta < \alpha - 2$ such that*

$$-r^{2\beta} \leq \text{Ric}_x \quad \text{and} \quad \text{Sect}_x \leq -\frac{\alpha(\alpha - 1)}{r^2}$$

for all $r = r(x) \geq r_0$. Then the Dirichlet problem at infinity for M is solvable.

Hsu and March[9] proved a lower bound of the form $-r^{2\beta}$ with $\beta < 1 - 2/\alpha < 1$. Our new result opens the possibility of $\beta \geq 1$.

The rest of this paper has three sections. In SECTION 2, we state some preliminary results needed for the proof of our main theorems. In SECTIONS 3 and 4 we deal with the constant upper bound case and the vanishing upper bound case, respectively.

2. PRELIMINARY RESULTS

Let M be a Riemannian manifold and $\widetilde{M} = M \cup \{\Delta\}$ its one-point compactification. The path space $W(M)$ based on M is the space of continuous maps $X \in C([0, \infty); \widetilde{M})$ with the following property: if $X_t = \Delta$ for some t then $X_s = \Delta$ for all $s \geq t$. The lifetime $e(X)$ is defined by $e(X) = \inf \{t : X_t = \Delta\}$. The path space $W(M)$ is equipped with the standard filtration $\mathcal{B}_* = \{\mathcal{B}_t\}$ and the lifetime $e : W(M) \rightarrow \mathbb{R}_+$ is a \mathcal{B}_* -stopping time. We use \mathbb{P}_x to denote the law of Brownian motion on M starting from x . It is a probability measure on $W(M)$.

Now let M be a Cartan-Hadamard manifold and $\widehat{M} = M \cup S_\infty(M)$ its compactification by the sphere at infinity. A Brownian motion X can be decomposed into the radial process $r_t = r(X_t)$ and the angular process

$\theta_t = \theta(X_t)$. The probabilistic approach to the Dirichlet problem is based on the following well known fact.

Theorem 2.1. *Let M be a Cartan-Hadamard manifold. Suppose that for any $x \in M$ we have*

$$\mathbb{P}_x \left\{ X_e = \lim_{t \uparrow e} X_t \text{ exists} \right\} = 1$$

(in the topology of \widehat{M}) and for any $\theta_0 \in S_\infty(M)$ and any neighborhood U of θ_0 in $S_\infty(M)$

$$\lim_{x \rightarrow \theta_0} \mathbb{P}_x \{X_e \in U\} = 1.$$

Then the Dirichlet problem at infinity for M is solvable. For any $f \in C(S_\infty(M))$, the function $u_f(x) = \mathbb{E}_x f(X_e)$ is the unique solution of the Dirichlet problem with boundary function f .

Proof. Since $u_f(x) = \mathbb{E}_x u_f(X_{\tau_D})$ for any relative compact open set D containing x , where τ_D is the first exit time of D , we see that u is harmonic on M . For any $\epsilon > 0$ and $\theta_0 \in S_\infty(M)$ choose a neighborhood U of θ_0 such that $|f(\theta) - f(\theta_0)| \leq \epsilon$ for $\theta \in U$. Then

$$\begin{aligned} |u_f(x) - f(\theta_0)| &\leq \mathbb{E}_x |f(X_e) - f(\theta_0)| \\ &\leq \epsilon \mathbb{P}_x \{X_e \in U\} + 2\|f\|_\infty \mathbb{P}_x \{X_e \notin U\}. \end{aligned}$$

Letting $x \rightarrow \theta_0$ we have $\limsup_{x \rightarrow \theta_0} |u_f(x) - f(\theta_0)| \leq \epsilon$. This shows that $\lim_{x \rightarrow \theta_0} u_f(x) = f(\theta_0)$, as desired.

To prove the uniqueness, let $\{D_n\}$ be an exhaustion of M and u a solution of the Dirichlet problem at infinity with boundary function f . Then $\{u_f(X_{t \wedge \tau_{D_n}}), t \geq 0\}$ is a uniformly bounded martingale under \mathbb{P}_x , hence $u(x) = \mathbb{E}_x u(X_{t \wedge \tau_{D_n}})$. Letting $t \uparrow \infty$ and then $n \uparrow \infty$ we have

$$u(x) = \mathbb{E}_x u(X_e) = \mathbb{E}_x f(X_e) = u_f(x).$$

□

Remark 2.2. Ancona[2] constructed a Cartan-Hadamard manifold such that Brownian motion converges to a single point on the boundary at infinity. For such manifolds, the Dirichlet problem at infinity is clearly not solvable.

We end this section with a description of the general method for proving angular convergence of Brownian motion. Define a sequence of stopping times $\{\tau_n\}$ by $\tau_0 = 0$ and

$$\tau_n = \inf \{t \geq \tau_{n-1} : d(X_t, X_{\tau_{n-1}}) = 1\}.$$

Let $\Delta\tau_n = \tau_n - \tau_{n-1}$ be the amount of time for the n th step. The angular oscillation during the time interval $[\tau_{n-1}, \tau_n]$ is

$$\Delta\theta_n = \max_{\tau_{n-1} \leq t \leq \tau_n} \angle(\theta(X_{\tau_{n-1}}), \theta(X_t)).$$

Proposition 2.3. *Let M be a Cartan-Hadamard manifold on which Brownian motion is transient, i.e.,*

$$\mathbb{P}_x \{r_t \rightarrow \infty \text{ as } t \uparrow e\} = 1.$$

The Dirichlet problem at infinity is solvable if for any positive ϵ and δ , there is an R such that for all $z \in M$ with $r(z) \geq R$,

$$(2.1) \quad \mathbb{P}_z \left\{ \sum_{n=1}^{\infty} \Delta\theta_n \leq \delta \right\} \geq 1 - \epsilon.$$

Proof. First we note that $\sum_{n=1}^{\infty} \Delta\theta_n < \infty$ implies that $\lim_{t \uparrow e} X_t = X_e$ exists. Let $x \in M$ and $\epsilon > 0$. Choose $R \geq r(x)$ such that (2.1) holds (for $\delta = 1$, say). Let $\tau_R = \inf \{t : r_t = R\}$. Then

$$\begin{aligned} \mathbb{P}_x \left\{ X_e = \lim_{t \uparrow e} X_t \text{ exists} \right\} &\geq \mathbb{P}_x \left\{ \sum_{n=1}^{\infty} \Delta\theta_n < \infty \right\} \\ &= \mathbb{E}_x \mathbb{P}_{X_{\tau_R}} \left\{ \sum_{n=1}^{\infty} \Delta\theta_n < \infty \right\} \\ &\geq 1 - \epsilon. \end{aligned}$$

Since ϵ is arbitrary, this shows that $\mathbb{P}_x \{X_e = \lim_{t \uparrow e} X_t \text{ exists}\} = 1$.

Let $\theta_0 \in S_{\infty}(M)$ and U a neighborhood of θ_0 on $S_{\infty}(M)$ containing θ_0 . There is a $\delta > 0$ such that

$$\{\theta \in S_{\infty}(M) : \angle(\theta, \theta_0) \leq 2\delta\} \subset U.$$

We have

$$\angle(\theta_0, \theta(X_e)) \leq \angle(\theta_0, \theta(X_0)) + \sum_{n=0}^{\infty} \Delta\theta_n.$$

For any $\epsilon > 0$ choose $R > 0$ such that (2.1) holds. Then for all $x \in M$ such that $r(x) \geq R$ and $\angle(\theta(x), \theta_0) \leq \delta$, we have

$$\mathbb{P}_x \{X_e \in U\} \geq \mathbb{P}_x \{\angle(\theta_0, \theta(X_e)) \leq 2\delta\} \geq \mathbb{P}_x \left\{ \sum_{n=0}^{\infty} \Delta\theta_n \leq \delta \right\}.$$

This shows that

$$\lim_{x \rightarrow \theta_0} \mathbb{P}_x \{X_e \in U\} = 1.$$

By THEOREM 2.1, the Dirichlet problem at infinity for M is solvable. \square

We use the following result to estimate the time the Brownian motion spends for each step. Let

$$\tau_1 = \inf \{t > 0 : d(X_t, X_0) = 1\}.$$

Proposition 2.4. *There are positive constants C_1, C_2 such that if the Ricci curvature on the geodesic ball $B(x; 1)$ of radius 1 centered at x is bounded from below by a negative constant $-L^2 \leq -1$, then*

$$\mathbb{P}_x \left\{ \tau_1 \leq \frac{C_1}{L} \right\} \leq e^{-C_2 L}.$$

In fact we can take $C_1 = 1/8d$ and $C_2 = 1/2$.

Proof. This is LEMMA 4 in Hsu and March[9]. We give a simpler proof here. Let $r_t = d(X_t, x)$ be the radial process. According to Kendall[11], there is a Brownian motion β such that

$$r_t = \beta_t + \frac{1}{2} \int_0^t \Delta r(X_s) ds - L_t,$$

where L is nondecreasing and increases only when X_t is on the cut locus of o . By Itô's formula we have

$$r_t^2 = 2 \int_0^t r_s dr_s + \langle r \rangle_t.$$

Hence

$$(2.2) \quad r_t^2 \leq 2 \int_0^t r_s d\beta_s + \int_0^t r_s \Delta r(X_s) ds + t.$$

By the Laplacian comparison theorem, we have for all $z \in B(x; 1)$

$$\Delta r(z) \leq (d-1)L \coth Lr(z)$$

On the other hand, $l \coth l \leq 1 + l$ for all $l \geq 0$. Hence if $s \leq \tau_1$, we have

$$r_s \Delta r(X_s) \leq (d-1)Lr_s \coth Lr_s \leq (d-1)(1+L).$$

We now let $t = \tau_1$ in (2.2) and obtain

$$1 \leq 2 \int_0^{\tau_1} r_s d\beta_s + 2dL\tau_1.$$

From the above inequality we see that the event $\tau_1 \leq 1/8dL$ implies

$$\int_0^{\tau_1} r_s d\beta_s \geq \frac{3}{8}.$$

By Lévy's criterion, there is a Brownian motion W such that

$$\int_0^{\tau_1} r_s d\beta_s = W_\eta, \quad \eta = \int_0^{\tau_1} r_s^2 ds \leq \frac{1}{8dL}.$$

Hence $\tau_1 \leq 1/8dL$ implies

$$\max_{0 \leq s \leq 1/8dL} W_s \geq W_\eta \geq \frac{3}{8}.$$

The random variable on the left side is distributed as $\sqrt{1/8dL}|W_1|$. It follows that

$$\mathbb{P}_x \left[\tau_1 \leq \frac{1}{8dL} \right] \leq \mathbb{P}_x \left[|W_1| \geq \sqrt{\frac{9L}{8}} \right] \leq e^{-L/2}.$$

□

We will use the following geometric result to estimate the angle in a Cartan-Hadamard manifold. It is essentially LEMMA 2 in Hsu and March[9], but we include a complete proof to clarify a few points.

Lemma 2.5. *Let M be a Cartan-Hadamard manifold. Suppose that there are positive constants $\alpha \geq 1$ and $r_0 \geq 1$ such that*

$$\text{Sect}_x \leq -\frac{\alpha(\alpha-1)}{r(x)^2}, \quad r(x) \geq r_0.$$

Let $x, y \in M$ be such that

$$r(x) \geq 2r_0, \quad r(y) \geq 2r_0, \quad d(x, y) \leq 1.$$

Then there is a constant C independent of x and y such that the angle between the geodesic rays to x and y satisfies

$$\angle(\theta(x), \theta(y)) \leq \frac{C}{r(x)^\alpha}.$$

Proof. Without loss of generality we assume $r(x) \leq r(y)$. Let

$$K(r) = \min \left\{ -\sup_{r(x) \leq r} \text{Sect}_x, \frac{\alpha(\alpha-1)}{r^2} \right\}.$$

Let G be the unique solution of the Jacobi equation

$$G''(r) - K(r)G(r) = 0, \quad G(0) = 0, \quad G'(0) = 1.$$

Since $K(r) = \alpha(\alpha-1)/r^2$ for $r \geq r_0$, we have $G(r) = c_1 r^\alpha + c_2 r^{1-\alpha}$, hence

$$(2.3) \quad G(r) \sim c_1 r^\alpha, \quad \frac{G'(r)}{G(r)} \sim \frac{\alpha}{r}, \quad \text{as } r \uparrow \infty$$

In particular, $G(r) \geq C^{-1}r^\alpha$ for some C and all $r \geq r_0$. Now let N be the rotationally symmetric manifold with the metric $ds_N^2 = dr^2 + G(r)^2 d\theta^2$. In N consider the geodesic triangle AOB such that

$$d(O, A) = r(x), \quad d(O, B) = r(y), \quad \angle(\theta(A), \theta(B)) = \angle(\theta(x), \theta(y)).$$

By the Rauch comparison theorem, we have $d_N(A, B) \leq d(x, y)$. Hence

$$1 \geq d_N(A, B) \geq G(r(x))\angle(\theta(A), \theta(B)) = G(r(x))\angle(\theta(x), \theta(y)).$$

This implies that $\angle(\theta(x), \theta(y)) \leq C/r(x)^\alpha$. □

When the sectional curvature is bounded from above by a negative constant we have the following analogue of the above lemma.

Lemma 2.6. *Let M be a Cartan-Hadamard manifold. Suppose that there is a positive constant a such that $\text{Sect}_x \leq -a^2$. Let $x, y \in M$ be such that $r(x) \leq r(y)$ and $d(x, y) \leq 1$. Then*

$$\angle(\theta(x), \theta(y)) \leq \frac{a}{\sinh ar(x)} \leq \left[\frac{1}{r(x)} + 2a \right] e^{-ar(x)}.$$

Proof. Let $G(r) = \sinh ar/a$ and follow the proof of the preceding lemma. \square

3. CONSTANT UPPER BOUND

In this section we consider the case of a constant upper bound on the sectional curvature of M . We first give an estimate on the probability that Brownian motion starting at $r(x) = R$ will ever return to $r = R \leq r(x)$.

Lemma 3.1. *Suppose that $\text{Sect}_x \leq -a^2$. For any $R \geq 0$ we have for $r(x) \geq R$,*

$$(3.1) \quad \mathbb{P}_x \{r_t \leq R \text{ for some } t \geq 0\} \leq \cosh^{1-d} a(r - R).$$

Proof. There is a Brownian motion β such that

$$r_t = r_0 + \beta_t + \frac{1}{2} \int_0^t \Delta r(X_s) ds.$$

By the Laplacian comparison theorem we have $\Delta r \geq (d-1)a \coth ar$. If we define r^* by

$$r_t^* = r_0 + \beta_t + \frac{d-1}{2} \int_0^t a \coth ar_s^* ds,$$

then a comparison theorem for stochastic differential equations shows that $r_t \geq r_t^*$. Thus it is enough to prove the estimate for r^* .

The following argument is well known. Let

$$l(r) = \int_r^\infty (\sinh au)^{1-d} du$$

and $\sigma_R = \inf \{t : r_t^* = R\}$. If $r(x) \geq R$, then $\{l(r_{t \wedge \sigma_R}^*)\}$ is a uniformly bounded martingale. Letting $t \uparrow \infty$ we have

$$l(r) = \mathbb{E}_x l(r_{t \wedge \sigma_R}^*) = l(R) \mathbb{P}_x \{\sigma_R < \infty\}.$$

Hence

$$\mathbb{P}_x \{r_t^* \leq R \text{ for some } t \geq 0\} = \mathbb{P}_x \{\sigma_R < \infty\} = \frac{l(r)}{l(R)}.$$

On the other hand,

$$\begin{aligned} \frac{l(r(x))}{l(R)} &= \frac{\int_r^\infty (\sinh au)^{1-d} du}{\int_R^\infty (\sinh au)^{d-1} du} \\ &\leq \sup_{u \geq R} \left[\frac{\sinh a(u+r-R)}{\sinh au} \right]^{1-d} \\ &\leq \cosh^{1-d} a(r-R). \end{aligned}$$

In the last step we have used

$$\frac{\sinh(x+y)}{\sinh x} = \frac{\sinh x \cosh y + \cosh x \sinh y}{\sinh x} \geq \cosh y.$$

The result follows. \square

Next we consider the rate of escape for Brownian motion.

Lemma 3.2. *Suppose that $\text{Sect}_x \leq -a^2$. For any $\lambda < (d-1)a/2$ we have*

$$\lim_{r(x) \rightarrow \infty} \mathbb{P}_x \{r_t \geq \max\{\lambda t, r(x)/2\}, \forall t \geq 0\} = 1.$$

Proof. Again it is enough to show the result for the r_t^* in the proof of the preceding lemma. Fix a $\lambda_1 \in (\lambda, (d-1)a/2)$ and take R such that

$$[(d-1)a/2] \coth ar \geq \lambda_1, \quad r \geq R/2.$$

Suppose that $\epsilon > 0$. By LEMMA 3.1, we can take R even larger such that for all $x \in M$ with $r(x) \geq R$,

$$(3.2) \quad \mathbb{P}_x \{r_t^* \geq r(x)/2, \forall t \geq 0\} \geq 1 - \epsilon.$$

By the law of iterated logarithm,

$$\liminf_{t \uparrow \infty} \frac{\beta_t}{\sqrt{2t \log \log t}} = -1.$$

Hence there is an even larger R (independent of x) such that

$$(3.3) \quad \mathbb{P}_x \{\beta_t \geq -(\lambda - \lambda_1)t - R, \forall t \geq 0\} \geq 1 - \epsilon.$$

If the events in (3.2) and (3.3) happen simultaneously, then

$$\begin{aligned} r_t^* &= r_0^* + \beta_t + \frac{d-1}{2} \int_0^t a \coth ar_s^* ds \\ &\geq R - (\lambda_1 - \lambda)t - R + \lambda_1 t \\ &= \lambda t. \end{aligned}$$

It follows that for all $x \in M$ with that $r(x) \geq R$ we have

$$\mathbb{P}_x \{r_t^* \geq \max\{\lambda_1 t, r(x)/2\}, \forall t \geq 0\} \geq 1 - 2\epsilon.$$

This proves the lemma. \square

We now estimate the total angular variation. Suppose that $r_t \geq r(x)/2$ for all $t \geq 0$ with large $r(x)$. Recall that in SECTION 2 we have defined

$$\begin{aligned}\tau_n &= \inf \{t \geq \tau_{n-1} : d(X_t, X_{\tau_{n-1}}) = 1\}, \quad (\tau_0 = 0) \\ \Delta\tau_n &= \tau_n - \tau_{n-1}, \\ \Delta\theta_n &= \max_{\tau_{n-1} \leq t \leq \tau_n} \angle(\theta(X_{\tau_{n-1}}), \theta(X_t)).\end{aligned}$$

From LEMMA 2.6 we have $\Delta\theta_n \leq Ce^{-ar\tau_n}$, hence

$$\sum_{n=1}^{\infty} \Delta\theta_n \leq C \sum_{n=1}^{\infty} e^{-ar\tau_n}.$$

Next, let J_k be the total number of steps in the geodesic ball of radius k , i.e.,

$$J_k = \# \{n : r_{\tau_n} \leq k\}.$$

We have

$$(3.4) \quad \sum_{n=1}^{\infty} \Delta\theta_n \leq C_0 \sum_{k=1}^{\infty} J_k e^{-ak}.$$

Thus the problem is reduced to finding a good estimate for J_k .

Remark 3.3. The idea of studying J_k is due to Lerclerq[14].

Theorem 3.4. *Let M be a Cartan-Hadamard manifold whose sectional curvature is bounded from above by $-a^2$. Suppose that the Ricci curvature satisfies the lower bound*

$$\text{Ric}_x \geq -h(r)^2 e^{2ar},$$

where h is a positive and nonincreasing function such that $\int_0^\infty rh(r)dr < \infty$. Then the Dirichlet problem at infinity for M is solvable.

Proof. Fix a constant $\lambda < (d-1)a/2$ and let

$$A = \{r_t \geq \max\{\lambda t, r(x)/2\}, \forall t \geq 0\}.$$

By LEMMA 3.2, there is an R such that for $r(x) \geq R$,

$$\mathbb{P}_x \{A\} \geq 1 - \frac{\epsilon}{2}.$$

Let τ_{n_l} be the l th time such that $r_{\tau_{n_l}} \leq k-1$. Then

$$\{\tau_{n_l} \leq t\} = \left\{ \sum_{n=1}^{\infty} I_{\{r_{\tau_n} \leq k-1, \tau_n \leq t\}} \geq l \right\},$$

from which it is clear that τ_{n_l} a stopping time.

For a fixed k , denote for the time being

$$L_k = C_1 h(k) e^{ak}, \quad N_k = \frac{(k+1)L_k}{\lambda C_1}.$$

Without loss of generality we may assume that $h(k) \geq e^{-ak/2}$ (otherwise just add $e^{-ar/2}$ to $h(r)$) and $L_k \geq 1$. Consider the length of time $\Delta\tau_{n_l}$ for the next step. Let

$$B_l = \left\{ \Delta\tau_{n_l} \leq \frac{C_1}{L_k}, \tau_{n_l} < \infty \right\}, \quad C_{N_k} = B_1 \cup B_2 \cdots \cup B_{N_k}.$$

By PROPOSITION 2.4 and the fact that τ_{n_l} is a stopping time,

$$(3.5) \quad \mathbb{P}_x B_l = \mathbb{E}_x \left\{ \mathbb{P}_{X_{\tau_{n_l}}} \left[\tau_1 \leq \frac{C_1}{L_k} \right], \tau_{n_l} < \infty \right\} \leq e^{-C_2 L_k}.$$

Recall that J_{k-1} is the total number of steps such that $r_{\tau_n} \leq k-1$. We have $\{J_{k-1} \geq N_k\} = \{\tau_{n_{N_k}} < \infty\}$. Now

$$(3.6) \quad \{J_{k-1} \geq N_k\} \cap A = \{\tau_{n_{N_k}} < \infty\} \cap A \cap C_{N_k} + \{\tau_{n_{N_k}} < \infty\} \cap A \cap C_{N_k}^c.$$

On A we have $r_t \geq \lambda t$ for all $t \geq 0$. This means that

$$|\{t : r_t \leq k\}| \leq \frac{k}{\lambda}.$$

But on $\{\tau_{n_{N_k}} < \infty\} \cap C_{N_k}^c$,

$$|\{t : r_t \leq k\}| \geq \sum_{l=1}^{N_k} \Delta\tau_{n_l} \geq N_k \cdot \frac{C_1}{L_k} = \frac{k+1}{\lambda}.$$

This shows that $\{\tau_{n_{N_k}} < \infty\} \cap A \cap C_{N_k}^c = \emptyset$ and we have from (3.6)

$$\{J_{k-1} \geq N_k\} \cap A \subseteq C_{N_k} = B_1 \cup B_2 \cdots \cup B_{N_k}.$$

By (3.5),

$$\mathbb{P}_x \{J_{k-1} \geq N_k, A\} \leq N_k e^{-C_2 L_k} \leq C_3 k e^{ak - C_2 e^{ak/2}}.$$

Using the definition of L_k we see that from the above inequality that for any $\epsilon > 0$, there is a sufficiently large R such that for $r(x) \geq R$,

$$\sum_{k \geq r(x)/2}^{\infty} \mathbb{P}_x \left\{ J_k \geq C_4 k h(k) e^{ak}, A \right\} \leq \frac{\epsilon}{2}.$$

On A we have $r_t \geq r(x)/2$ for all t . This means that $J_k = 0$ for $k \leq r(x)/2$. It follows that for $r(x) \geq R$,

$$\mathbb{P}_x \left\{ J_k = 0, k \leq \frac{r(x)}{2}; J_k \leq C_4 k h(k) e^{ak}, k \geq \frac{r(x)}{2} \right\} \geq \mathbb{P}_x A - \frac{\epsilon}{2} \geq 1 - \epsilon.$$

If the event in the above inequality holds, then by (3.4)

$$\sum_{n=1}^{\infty} \Delta\theta_n \leq C_4 \sum_{k \geq \frac{r(x)}{2}} k h(k).$$

This can be made arbitrarily small because the $\sum_{k=1}^{\infty} kh(k)$ converges by hypothesis. Therefore we have shown that for any positive ϵ and δ , there is an R such that for all $x \in M$ with $r(x) \geq R$,

$$\mathbb{P}_x \left\{ \sum_{n=1}^{\infty} \Delta\theta_n \leq \delta \right\} \geq 1 - \epsilon.$$

By PROPOSITION 2.3, this implies the solvability of the Dirichlet problem at infinity for M . \square

4. VANISHING UPPER BOUND

In this section we assume that M is a Cartan-Hadamard manifold whose curvature satisfies the following condition: there are positive constant r_0 , $\alpha > 2$ and $\beta < \alpha - 2$ such that for all $r(x) \geq r_0$,

$$-r(x)^{2\beta} \leq \text{Ric}_x \quad \text{and} \quad \text{Sect}_x \leq -\frac{\alpha(\alpha-1)}{r(x)^2}.$$

The proof for this case is completely parallel to that in the previous section, so we will be brief.

Lemma 4.1. *There is a constant C such that for all $R \geq 1$ and $x \in M$ with $r(x) \geq R$,*

$$\mathbb{P}_x \{r_t \leq R \text{ for some } t \geq 0\} \leq C \left[\frac{R}{r(x)} \right]^{(d-1)\alpha-1}.$$

Proof. Define the function G as in the proof of LEMMA 2.5. As before we may assume that M is rotationally symmetric with metric $ds^2 = dr^2 + G(r)^2 d\theta^2$. In this case, by the same argument as in LEMMA 3.1 we have

$$\mathbb{P}_x \{r_t \leq R \text{ for some } t \geq 0\} = \frac{\int_{r(x)}^{\infty} G(s)^{1-d} ds}{\int_R^{\infty} G(s)^{1-d} ds}.$$

The result follows immediately from the fact that $G(r) \sim c_1 r^\alpha$ as $r \uparrow \infty$. \square

In the proof of the next lemma, we need the following fact (see Shiga and Watanabe[17]): let Y^a be the Bessel process of index $q > 1$ from $a \geq 0$:

$$(4.1) \quad Y_t^a = a + \beta t + \frac{q}{2} \int_0^t \frac{ds}{Y_s^a},$$

where β is a one-dimensional Brownian motion. Then for any $\lambda > 0$ we have

$$(4.2) \quad \mathbb{P} \left\{ \lim_{t \uparrow \infty} \frac{Y_t^a}{t^{1/2-\lambda}} = \infty \right\} = 1.$$

Note that $Y_t^a \leq Y_t^b$ if $a \leq b$.

Lemma 4.2. *For any $\lambda > 0$ we have*

$$\lim_{r(x) \rightarrow \infty} \mathbb{P}_x \left\{ r_t \geq \max \{t, r(x)\}^{1/2-\lambda}, \forall t \geq 0 \right\} = 1.$$

Proof. Again it is enough to assume that M is rotationally symmetric, as in LEMMA 4.1. The radial process is given by

$$r_t = r_0 + \beta_t + \frac{d-1}{2} \int_0^t \frac{G'(r_s)}{G(r_s)} ds.$$

Now take a $q \in (1, (d-1)\alpha)$. By (2.3) there is an $r_1 \geq 1$ such that

$$(d-1) \cdot \frac{G'(r)}{G(r)} \geq \frac{q}{r}, \quad r \geq r_1.$$

Let Y^a be the Bessel process of index q defined by (4.1). If $r(x) \geq r_1$, then we have

$$r_t \geq Y_t^{r(x)} \geq Y_t^{r_1} \geq Y_t^1, \quad t \leq \sigma_{r_1},$$

where σ_{r_1} is the first time r_t reaches r_1 . For any $\epsilon > 0$, there is a $R \geq r_1$ (independent of x) such that

$$\mathbb{P}_x \left\{ Y_t^1 \geq t^{1/2-\lambda}, \forall t \geq R \right\} \geq 1 - \epsilon.$$

Hence, using LEMMA 4.1 we have for $r(x) \geq R \geq 1$

$$\begin{aligned} & \mathbb{P}_x \left\{ r_t \geq \max \{t, r(x)\}^{1/2-\lambda}, \forall t \geq 0 \right\} \\ & \geq \mathbb{P}_x \left\{ r_t \geq t^{1/2-\lambda}, \forall t \geq r(x) \right\} - \mathbb{P}_x \left\{ r_t \leq r(x)^{1/2-\lambda}, \text{ for some } t \geq 0 \right\} \\ & \geq \mathbb{P}_x \left\{ Y_t^1 \geq t^{1/2-\lambda} \quad \forall t \geq R \right\} - Cr(x)^{-(\lambda+1/2)[(d-1)\alpha-1]} \\ & \geq 1 - \epsilon - Cr(x)^{-(\lambda+1/2)[(d-1)\alpha-1]}. \end{aligned}$$

It follows that for all sufficiently large $r(x)$ we have

$$\mathbb{P}_x \left\{ r_t \geq \max \{t, r(x)\}^{1/2-\lambda}, \forall t \geq 0 \right\} \geq 1 - 2\epsilon.$$

□

Theorem 4.3. *Suppose that M is a Cartan-Hadamard manifold. Suppose that there exist positive constants r_0 , $\alpha > 2$, and $\beta < \alpha - 2$ such that*

$$-r(x)^{2\beta} \leq \text{Ric}_x \quad \text{and} \quad \text{Sect}_x \leq -\frac{\alpha(\alpha-1)}{r(x)^2} \quad \text{for } r \geq r_0.$$

Then the Dirichlet problem at infinity is solvable for M .

Proof. We define $\tau_n, \Delta\tau_n, \Delta\theta_n, \tau_{n_l}$, and J_k as in the previous section. Under the current upper bound of the sectional curvature, we have $\Delta\theta_n \leq C/r_{\tau_n}^\alpha$

by LEMMA 2.5. Hence

$$(4.3) \quad \begin{aligned} \sum_{n=1}^{\infty} \Delta\theta_n &\leq C_0 J_1 + C_0 \sum_{k=1}^{\infty} \frac{J_{k+1} - J_k}{k^\alpha} \\ &\leq C_0 J_1 + C_1 \sum_{k=1}^{\infty} \frac{J_k}{k^{\alpha+1}} + C_0 \liminf_{k \uparrow \infty} \frac{J_k}{k^\alpha}. \end{aligned}$$

We will now estimate the size of J_k . By PROPOSITION 2.4 we have

$$\mathbb{P}_x \left\{ \Delta\tau_{n_l} \leq C_1 k^{-\beta}, \tau_{n_l} < \infty \right\} \leq e^{-C_1 k^\beta}.$$

Choose a positive λ such that $\beta + 2/(1 - 2\lambda) < \alpha$. Let

$$A = \left\{ r_t \geq \max\{t, r(x)\}^{1/2-\lambda}, \forall t \geq 0 \right\}.$$

Fix an arbitrary $\epsilon > 0$. By LEMMA 4.2 $\mathbb{P}_x A \geq 1 - \epsilon/2$ for sufficiently large $r(x)$. By the same argument as in THEOREM 3.4 we have

$$\mathbb{P}_x \left\{ J_k \geq (C_1 + 1)k^{\beta+2/(1-2\lambda)}, A \right\} \leq C_3 k^{\beta+2/(1-2\lambda)} e^{-C_2 k^\beta}.$$

On A , we have $|\{t : r_t \leq k\}| \leq k^{2/(1-2\lambda)}$ and $J_k = 0$ for $k \leq r(x)^{1/2-\lambda}$. Hence as in the proof THEOREM 3.4 we have for sufficiently large $r(x)$,

$$\begin{aligned} &\mathbb{P}_x \left\{ J_k = 0, k \leq r(x)^{1/2-\lambda}; J_k \leq C_4 k^{\beta+2/(1-2\lambda)}, k \geq r(x)^{1/2-\lambda} \right\} \\ &\geq \mathbb{P}_x A - C_3 \sum_{k \geq r(x)^{1/2-\lambda}} k^{\beta+2/(1-2\lambda)} e^{-C_2 k^\beta} \\ &\geq 1 - \epsilon. \end{aligned}$$

If the event in the above inequality is true, then $J_k/k^\alpha \rightarrow 0$ as $k \uparrow \infty$ and by (4.3),

$$\begin{aligned} \sum_{n=1}^{\infty} \Delta\theta_n &\leq C_4 \sum_{k \geq r(x)^{1/2-\lambda}} k^{-(\alpha+1)+\beta+2/(1-2\lambda)} \\ &\leq C_5 r(x)^{-(\alpha-\beta)(1-2\lambda)/2+1}. \end{aligned}$$

By our choice of λ , the exponent is negative. Hence we have shown that for any positive ϵ and δ , there is an R such that for $r(x) \geq R$,

$$\mathbb{P}_x \left\{ \sum_{n=1}^{\infty} \Delta\theta_n \leq \delta \right\} \geq 1 - \epsilon.$$

The theorem now follows from PROPOSITION 2.3. \square

Remark 4.4. For the Bessel process Y^a in (4.1) we have

$$\mathbb{P} \left\{ \liminf_{t \rightarrow \infty} \frac{Y_t^a}{\sqrt{t}\psi(t)} \geq 1 \right\} = 1$$

if ψ is a positive nonincreasing function such that $\int_0^\infty \psi(t)^{q-1} dt < \infty$. Using this rate instead of $t^{1/2-\lambda}$ in (4.2), we can improve the lower bound in the above theorem. For example, it can be shown that the Dirichlet problem is solvable if the Ricci curvature is bounded from below by $-r^{2(\alpha-2)}/(\ln r)^{2l}$ for $l > (d\alpha - \alpha + 1)/(d\alpha - \alpha - 1)$.

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