

# Construction of Immigration Superprocesses with Dependent Spatial Motion from One-Dimensional Excursions

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## Abstract

A superprocess with dependent spatial motion and interactive immigration is constructed as the unique strong solution of a stochastic integral equation with Poisson process on the space of excursions of the one-dimensional Feller branching diffusion.

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## 1 Introduction

Let  $M(\mathbb{R})$  denote the space of finite Borel measures on  $\mathbb{R}$  endowed with a metric compatible with its topology of weak convergence. Let  $B(\mathbb{R})^+$  be the set of non-negative Borel functions on  $\mathbb{R}$ . For  $f \in B(\mathbb{R})^+$  and  $\mu \in M(\mathbb{R})$  set  $\langle f, \mu \rangle = \int f d\mu$ . Let  $\{w_t : t \geq 0\}$  denote the coordinate process of  $C([0, \infty), M(\mathbb{R}))$ , which is furnished with the locally uniform convergence. Suppose that  $\sigma > 0$  is a constant and  $h$  is a continuously differentiable function on  $\mathbb{R}$  such that both  $h$  and  $h'$  are square-integrable. Then the function

$$\rho(x) = \int_{\mathbb{R}} h(y-x)h(y)dy, \quad x \in \mathbb{R}, \quad (1.1)$$

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is twice continuously differentiable with bounded derivatives  $\rho'$  and  $\rho''$ . By the general results of Dawson et al (2001) and Wang (1997, 1998), for each  $\mu \in M(\mathbb{R})$  there is a unique Borel probability measure  $\mathbf{Q}_\mu$  on  $C([0, \infty), M(\mathbb{R}))$  such that, for each  $\phi \in C^2(\mathbb{R})$ ,

$$M_t(\phi) = \langle \phi, w_t \rangle - \langle \phi, \mu \rangle - \frac{1}{2} \rho(0) \int_0^t \langle \phi'', w_s \rangle ds, \quad t \geq 0, \quad (1.2)$$

under  $\mathbf{Q}_\mu$  is a continuous martingale with quadratic variation process

$$\langle M(\phi) \rangle_t = \sigma \int_0^t \langle \phi^2, w_s \rangle ds + \int_0^t ds \int_{\mathbb{R}} \langle h(z - \cdot) \phi', w_s \rangle^2 dz. \quad (1.3)$$

The measure  $\mathbf{Q}_\mu$  was given in Dawson et al (2001) and Wang (1997, 1998) as the limit distribution of a sequence of branching particle systems with dependent spatial motion. It was also proved in Dawson et al (2001) and Wang (1997, 1998) using a dual process that  $\mathbf{Q}_\mu$  is uniquely determined by the martingale problem (1.2) and (1.3). Consequently,  $\{\mathbf{Q}_\mu : \mu \in M(\mathbb{R})\}$  defines a diffusion process, which we shall call a *superprocess with dependent spatial motion (SDSM)*. Clearly, the only difference in the martingale characterization of the SDSM and that of a super Brownian motion in  $M(\mathbb{R})$  comes in the second term on the right hand side of (1.3). Because of the dependence of the spatial motion, the SDSM has properties rather different from those of the superprocess with independent spatial motion. It is well-know that the super Brownian motion started with an arbitrary initial state enters immediately the space of absolutely continuous measures and its density process satisfies a stochastic differential equation; see Konno and Shiga (1988) and Reimers (1989). On the other hand, it was proved in Wang (1997) that the SDSM enters immediately the space of pure atomic measures and lives in this space.

The purpose of this paper is to construct a class of immigration diffusion processes associated with the SDSM. Let  $m$  be a  $\sigma$ -finite Borel measure on  $\mathbb{R}$  and let  $q$  be a Borel function on  $M(\mathbb{R}) \times \mathbb{R}$  satisfying certain regularity conditions to be specified. A modification of the SDSM is to replace (1.2) by

$$M_t(\phi) = \langle \phi, w_t \rangle - \langle \phi, \mu \rangle - \frac{1}{2} \rho(0) \int_0^t \langle \phi'', w_s \rangle ds - \int_0^t \langle \phi q(w_s, \cdot), m \rangle ds, \quad t \geq 0. \quad (1.4)$$

A solution of the martingale problem given by (1.3) and (1.4) can be interpreted as an SDSM with interactive immigration determined by  $q(w_s, \cdot)$  and the reference measure  $m$ . Because of the dependence of  $q(w_s, \cdot)$  on  $w_s$ , the duality method of Dawson et al (2001) and Wang (1997, 1998) fails and the uniqueness of the solution becomes a poser. The next paragraph describes our approach to the construction of the immigration SDSM as a *diffusion process*.

Let  $W(ds, dy)$  be a time-space white noise on  $[0, \infty) \times \mathbb{R}$  based on the Lebesgue measure; see e.g. Walsh (1986). We denote by  $W_0$  the totality of paths  $w \in C([0, \infty), \mathbb{R}^+)$  such that  $w(0) = w(t) = 0$  for  $t \geq \tau_0(w)$ , where  $\tau_0(w) = \inf\{s > 0 : w(s) = 0\}$ , and endow  $W_0$  with the topology of uniform convergence. Let  $\mathbf{Q}_\kappa$  be the excursion law of the Feller branching diffusion defined by (2.5). Let  $N_0(dx, dw)$  be a Poisson random measure on  $\mathbb{R} \times W_0$  with intensity  $\mu(ds) \mathbf{Q}_\kappa(dw)$  and let  $N(ds, da, du, dw)$  be a Poisson random measure on  $[0, \infty) \times \mathbb{R} \times [0, \infty) \times W_0$  with intensity  $dsm(da) du \mathbf{Q}_\kappa(dw)$ . We assume that  $\{W(dt, dy)\}$ ,  $\{N_0(dx, dw)\}$  and  $\{N(ds, da, du, dw)\}$  are defined on a complete probability space and are independent of each other. By Dawson et al (2001, Lemma 3.1) or Wang (1997, Lemma 1.3), for any  $r \geq 0$  and  $a \in \mathbb{R}$  the stochastic equation

$$x(r, a, t) = a + \int_r^t \int_{\mathbb{R}} h(y - x(r, a, s)) W(ds, dy), \quad (1.5)$$

has a unique continuous solution  $\{x(r, a, t) : t \geq r\}$ , which determines an isotropic stochastic flow. Let  $Y_0 = \mu$ . Our main result is that the stochastic equation

$$Y_t = \int_{\mathbb{R}} \int_{W_0} w(t) \delta_{x(0, x, t)} N_0(dx, dw) + \int_0^t \int_{\mathbb{R}} \int_0^{q(Y_s, a)} \int_{W_0} w(t-s) \delta_{x(s, a, t)} N(ds, da, du, dw), \quad t > 0, \quad (1.6)$$

has a unique continuous solution  $\{Y_t : t > 0\}$  and the process  $\{Y_t : t \geq 0\}$  is a diffusion. We also show that the distribution of  $\{Y_t : t \geq 0\}$  on  $C([0, \infty), M(\mathbb{R}))$  solves the martingale problem (1.3) and (1.4). The application of (1.6) is crucial in the construction the immigration diffusion process since the *uniqueness of solution* of the martingale problem given by (1.3) and (1.4) still *remains open*. The equation (1.6) also provides useful information on the structures of the sample paths of the immigration diffusion. For instance, from this formulation we have immediately the following interpretations:

- (i) for any initial state  $\mu \in M(\mathbb{R})$ , the immigration process  $\{Y_t : t > 0\}$  consists of at most countably many atoms;
- (ii) the atoms are carried by the flow  $\{x(r, a, t) : t \geq r\}$  defined by (1.5), which determine the spatial motion of the process;
- (iii) the immigration times and locations as well as the mass excursions are selected by the Poisson random measures  $N_0(dx, dw)$  and  $N(ds, da, du, dw)$ ;
- (iv) the mass changes of the atoms are described by excursions of the Feller branching diffusion;
- (v) there are infinitely but countably many immigration times in any non-trivial time interval if  $\langle q(\nu, \cdot), m \rangle > 0$  for all  $\nu \in M(\mathbb{R})$ ;
- (vi) for any constants  $t > r > 0$  there are at most only a finite number of atoms which have lived longer than  $r$  before time  $t$ .

A class of one-dimensional immigration diffusions was constructed in Pitman and Yor (1982) as sums of excursions selected by Poisson point processes. Similar constructions in infinite-dimensional setting were discussed in Fu and Li (2002), Li (1999), Li and Shiga (1995) and Shiga (1990). In particular, Shiga (1990) constructed purely atomic measure-valued immigration branching diffusions *without spatial motion* as the unique strong solution of a stochastic integral equation similar to but simpler than (1.6). An extension of his result to non-trivial *independent spatial motion* was given recently in Fu and Li (2002) by considering measure-valued excursions. In the present situation, we have to put up with the hardship brought about by the *dependent spatial motion*. To resolve the difficulties, we divide the discussion into several parts. In section 2, we prove some useful characterizations for the excursion law of the Feller branching diffusion. In section 3, we treat the case with  $\langle 1, \mu \rangle > 0$  and  $\langle 1, m \rangle = 0$ , i.e., we construct the SDSM without immigration. The results provide useful insights into the sample path structures of the SDSM and serve as preliminaries of the construction of immigration processes. In particular, we prove that the SDSM consists of a countable number of atoms, their spatial motions are determined by a common white noise, their mass processes are excursions of the Feller branching diffusion and, under suitable conditional probabilities, the initial locations of the atoms are i.i.d. random variables with distribution  $\langle 1, \mu \rangle^{-1} \mu(dx)$ . In section 4, we consider the case with  $0 < \langle 1, m \rangle < \infty$

and  $q(\cdot, \cdot) \equiv 1$ , i.e., the deterministic immigration case. In this case, the right hand side of (1.6) is actually independent of  $Y_s$ . What we need to do is to show that the process  $\{Y_t : t \geq 0\}$  defined by this formula is a diffusion process solving the martingale problem (1.4) and (1.3) with  $q(\cdot, \cdot) \equiv 1$ . The difficulty lies in that the immigrants have dependent spatial motion and come infinitely many times at any non-trivial time interval. Because of the dependent spatial motion we cannot compute the Laplace functional of  $\{Y_t : t \geq 0\}$  as in Fu and Li (2002) and Shiga (1990). To get the conclusion, we chop off from every excursion a part with length  $1/n$  and construct a right continuous strong Markov process  $\{X_t^{(n)} : t \geq 0\}$ , which is characterized as a SDSM with positive jumps. Then we obtain the desired martingale characterization of  $\{Y_t : t \geq 0\}$  from that of  $\{X_t^{(n)} : t \geq 0\}$  by letting  $n \rightarrow \infty$ . Based on those constructions and characterizations, we discuss the general equation (1.6) in section 5 using the techniques developed in Fu and Li (2002) and Shiga (1990).

## 2 Excursions of Feller branching diffusions

Let  $\sigma > 0$  be a constant and  $\{B(t) : t \geq 0\}$  be a standard Brownian motion. For any initial condition  $\xi(0) = \xi$  the stochastic differential equation

$$d\xi(t) = \sqrt{\sigma\xi(t)}dB(t), \quad t \geq 0, \quad (2.1)$$

has a unique solution  $\{\xi(t) : t \geq 0\}$ , which is the so-called *Feller branching diffusion with branching rate  $\sigma$* . In this paper, we simply call it a  $\sigma$ -branching diffusion. The transition semigroup  $(Q_t^\sigma)_{t \geq 0}$  of the process is determined by

$$\int_0^\infty e^{-\lambda y} Q_t^\sigma(x, dy) = \exp\{-x\lambda/(1 + \sigma t\lambda/2)\}, \quad t, x, \lambda \geq 0; \quad (2.2)$$

see e.g. Ikeda and Watanabe (1989, p.236). Let  $(Q_t^\circ)_{t \geq 0}$  be the restriction of  $(Q_t^\sigma)_{t \geq 0}$  to  $(0, \infty)$  and let  $(\kappa_t)_{t > 0}$  the family of finite measures on  $(0, \infty)$  defined by

$$\kappa_t(dy) = 4(\sigma t)^{-2} e^{-2y/\sigma t} dy, \quad t > 0, x > 0. \quad (2.3)$$

It is easy to check that

$$\int_0^\infty (1 - e^{-\lambda y}) \kappa_t(dy) = \lambda/(1 + \sigma t\lambda/2), \quad t > 0, \lambda \geq 0, \quad (2.4)$$

and  $\kappa_r Q_t^\circ = \kappa_{r+t}$  for  $r > 0$  and  $t > 0$ , that is  $(\kappa_t)_{t > 0}$  is an *entrance law* for  $(Q_t^\circ)_{t \geq 0}$ . Then there is a unique  $\sigma$ -finite measure  $\mathbf{Q}_\kappa$  on  $(W_0, \mathcal{B}(W_0))$  such that

$$\mathbf{Q}_\kappa\{w(t_1) \in dy_1, \dots, w(t_n) \in dy_n\} = \kappa_{t_1}(dy_1) Q_{t_2-t_1}^\circ(y_1, dy_2) \cdots Q_{t_n-t_{n-1}}^\circ(y_{n-1}, dy_n) \quad (2.5)$$

for  $0 < t_1 < t_2 < \dots < t_n$  and  $y_1, y_2, \dots, y_n \in (0, \infty)$ . The measure  $\mathbf{Q}_\kappa$  is known as the *excursion law* of the  $\sigma$ -branching diffusion; see Pitman and Yor (1983) for details. Observe that

$$\mathbf{Q}_\kappa\{\tau_0 > t\} = \kappa_t(0, \infty) = 2/\sigma t, \quad t > 0. \quad (2.6)$$

Let  $\mathcal{B}_t = \mathcal{B}_t(W_0)$  denote the  $\sigma$ -algebra on  $W_0$  generated by  $\{w(s) : 0 \leq s \leq t\}$ . Roughly speaking, (2.5) asserts that  $\{w(t) : t > 0\}$  is a  $\sigma$ -branching diffusion with one-dimensional distributions  $\{\kappa_t : t > 0\}$  relative to  $(\mathbf{Q}_\kappa, \mathcal{B}_t)$ . For  $r > 0$ , let  $\tau_r(w) = r \vee \tau_0(w)$ , let  $\mathbf{Q}_\kappa^r$  denote the restriction of  $\mathbf{Q}_\kappa$  to  $\{w \in W_0 : \tau_0(w) > r\}$ .

**Theorem 2.1** For any  $r > 0$ , the coordinate process  $\{w(t) : t \geq r\}$  under  $\mathbf{Q}_\kappa^r\{\cdot | \mathcal{B}_r\}$  is a  $\sigma$ -branching diffusion. Moreover, there is a measurable mapping  $B : W_0 \rightarrow W_0$  such that  $\{B(w, t) : t \geq r\}$  under  $\mathbf{Q}_\kappa^r\{\cdot | \mathcal{B}_r\}$  is a Brownian motion stopped at time  $\tau_r(w)$  and

$$dw(t) = \sqrt{\sigma w(t)} dB(w, t), \quad t \geq r. \quad (2.7)$$

*Proof.* By (2.5),  $\{w(t) : t \geq r\}$  under  $\mathbf{Q}_\kappa^r\{\cdot | \mathcal{B}_r\}$  is a  $\sigma$ -branching diffusion or, equivalently, a continuous martingale with quadratic variation  $\sigma w(t)dt$ . Thus

$$\tilde{B}_r(w, t) := \int_r^{t \wedge \tau_r} \frac{1}{\sqrt{\sigma w(s)}} dw(s), \quad t \geq r, \quad (2.8)$$

under  $\mathbf{Q}_\kappa^r\{\cdot | \mathcal{B}_r\}$  is a Brownian motion stopped at time  $\tau_r(w)$ . We may define another Markov transition semigroup  $(Q_t^h)_{t \geq 0}$  on  $[0, \infty)$  by

$$Q_t^h(x, dy) = \begin{cases} x^{-1}yQ_t^\sigma(x, dy) & \text{for } t > 0, x > 0, y \geq 0, \\ y\kappa_t(dy) & \text{for } t > 0, x = 0, y \geq 0, \end{cases}$$

which is a special form of Doob's  $h$ -transform of  $(Q_t)_{t \geq 0}$ . Let  $\bar{\mathbf{Q}}_\kappa^r(dw) := w(r)\mathbf{Q}_\kappa^r(dw)$ . Based on (2.5), it is easy to check that  $\{w(t) : 0 \leq t \leq r\}$  under  $\bar{\mathbf{Q}}_\kappa^r$  is a diffusion process with transition semigroup  $(Q_t^h)_{t \geq 0}$  and generator  $2^{-1}\sigma x d^2/dx^2 + \sigma d/dx$ , and  $\{2\sqrt{w(t)}/\sigma : 0 \leq t \leq r\}$  is a 4-dimensional Bessel diffusion. It follows that

$$m(t) := w(t) - \sigma t, \quad 0 \leq t \leq r, \quad (2.9)$$

is a continuous martingale with quadratic variation  $\sigma w(t)dt$  and the limits

$$\bar{M}_r(w, t) := \lim_{u \rightarrow 0^+} \int_u^t \frac{1}{\sqrt{\sigma w(s)}} dm(s), \quad 0 < t \leq r, \quad (2.10)$$

exist in  $L^2(\bar{\mathbf{Q}}_\kappa^r)$ -sense. Let  $\bar{M}_r(w, 0) = 0$ . Then  $\{\bar{M}_r(w, t) : 0 \leq t \leq r\}$  under  $\bar{\mathbf{Q}}_\kappa^r$  is a Brownian motion. By Shiga and Watanabe (1973, Theorem 3.3.ii), for any  $0 < \beta < 1$  we have  $\bar{\mathbf{Q}}_\kappa^r\{\sqrt{w(t)} < t^{(1+\beta)/2} \text{ infinitely often as } t \rightarrow 0^+\} = 0$ . By (2.9) and (2.10), the limits

$$\bar{B}_r(w, t) := \lim_{u \rightarrow 0^+} \int_u^t \frac{1}{\sqrt{\sigma w(s)}} dw(s), \quad 0 < t \leq r,$$

also exist in  $L^2(\bar{\mathbf{Q}}_\kappa^r)$ -sense. Indeed, setting  $\bar{B}_r(w, 0) = 0$  we have

$$\bar{B}_r(w, t) = \bar{M}_r(w, t) + \int_{0^+}^t \frac{\sqrt{\sigma}}{\sqrt{w(s)}} ds, \quad 0 \leq t \leq r.$$

Now let  $B(w, t) = \bar{B}_r(w, t)$  for  $0 \leq t \leq r$  and  $B(w, t) = \bar{B}_r(w, r) + \tilde{B}_r(w, t)$  for  $t \geq r$ . Since  $\mathbf{Q}_\kappa^r$  is absolutely continuous relative to both  $\mathbf{Q}_\kappa^r$  and  $\bar{\mathbf{Q}}_\kappa^r$ , we see that  $B(w, t)$  is uniquely defined on  $\{w \in W_0 : \tau_0(w) > r\}$  out of a  $\mathbf{Q}_\kappa$ -null set and it does not depend on the particular choice of  $r > 0$ . Using this property, we can extend the definition of  $B(w, t)$  to the whole space  $W_0$  so that  $\{B(w, t) : t \geq r\}$  under  $\mathbf{Q}_\kappa^r\{\cdot | \mathcal{B}_r\}$  is a Brownian motion stopped at time  $\tau_r(w)$  and satisfies (2.7).  $\square$

### 3 Construction of SDSM by excursions

In this section, some results of Wang (1997) on the pure atomic SDSM are proved under more general assumptions. The results are useful in our construction of the associated immigration processes. Note that the generator  $\mathcal{L}$  of the SDSM is expressed as

$$\begin{aligned}\mathcal{L}F(\mu) &= \frac{1}{2}\rho(0) \int_{\mathbb{R}} \frac{d^2}{dx^2} \frac{\delta F(\mu)}{\delta \mu(x)} \mu(dx) \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^2} \rho(x-y) \frac{d^2}{dx dy} \frac{\delta^2 F(\mu)}{\delta \mu(x) \delta \mu(y)} \mu(dx) \mu(dy) \\ &\quad + \frac{1}{2} \sigma \int_{\mathbb{R}} \frac{\delta^2 F(\mu)}{\delta \mu(x)^2} \mu(dx).\end{aligned}\tag{3.1}$$

The domain of  $\mathcal{L}$  include functions on  $M(\mathbb{R})$  of the form  $F_{n,f}(\nu) := \int f d\nu^n$  with  $f \in C^2(\mathbb{R}^n)$  and functions of the form

$$F_{f,\{\phi_i\}}(\nu) := f(\langle \phi_1, \nu \rangle, \dots, \langle \phi_n, \nu \rangle)\tag{3.2}$$

with  $f \in C^2(\mathbb{R}^n)$  and  $\{\phi_i\} \subset C^2(\mathbb{R})$ . Let  $\mathcal{D}(\mathcal{L})$  denote the collection of all those functions.

**Theorem 3.1** (Dawson et al, 2001; Wang, 1998) *For each  $\mu \in M(\mathbb{R})$  there is a unique probability measure  $\mathbf{Q}_\mu$  on the space  $C([0, \infty), M(\mathbb{R}))$  such that  $\{w_t : t \geq 0\}$  under  $\mathbf{Q}_\mu$  solves the  $(\mathcal{L}, \mathcal{D}(\mathcal{L}), \mu)$ -martingale problem. Therefore, the system  $\{\mathbf{Q}_\mu : \mu \in M(\mathbb{R})\}$  defines a diffusion process generated by the closure of  $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ .*

**Theorem 3.2** *A probability measure  $\mathbf{Q}_\mu$  on  $C([0, \infty), M(\mathbb{R}))$  is a solution of the  $(\mathcal{L}, \mathcal{D}(\mathcal{L}), \mu)$ -martingale problem if and only if for each  $\phi \in C^2(\mathbb{R})$ ,*

$$M_t(\phi) = \langle \phi, w_t \rangle - \langle \phi, \mu \rangle - \frac{1}{2} \rho(0) \int_0^t \langle \phi'', w_s \rangle ds, \quad t \geq 0,\tag{3.3}$$

under  $\mathbf{Q}_\mu$  is a continuous martingale with quadratic variation process

$$\langle M(\phi) \rangle_t = \sigma \int_0^t \langle \phi^2, w_s \rangle ds + \int_0^t ds \int_{\mathbb{R}} \langle h(z - \cdot) \phi', w_s \rangle^2 dz.\tag{3.4}$$

*Proof.* Suppose that  $\mathbf{Q}_\mu$  is a solution of the  $(\mathcal{L}, \mathcal{D}(\mathcal{L}), \mu)$ -martingale problem, that is,

$$F(w_t) - F(w_0) - \int_0^t \mathcal{L}F(w_s) ds, \quad t \geq 0,\tag{3.5}$$

is a continuous martingale for every  $F \in \mathcal{D}(\mathcal{L})$ . Comparing the results of (3.5) applied to the functions  $\mu \mapsto \langle \phi, \mu \rangle$  and  $\mu \mapsto \langle \phi, \mu \rangle^2$  and using Itô's formula we see that (3.3) is a continuous martingale with quadratic variation process (3.4). Conversely, suppose that  $\mathbf{Q}_\mu$  is a probability measure on  $C([0, \infty), M(\mathbb{R}))$  under which (3.3) is a continuous martingale with quadratic variation process (3.4) for each  $\phi \in C^2(\mathbb{R})$ . Observe that for the function  $F_{f,\{\phi_i\}}$  defined by (3.2) we have

$$\begin{aligned}\mathcal{L}F_{f,\{\phi_i\}}(\nu) &= \frac{1}{2} \rho(0) \sum_{i=1}^n f'_i(\langle \phi_1, \nu \rangle, \dots, \langle \phi_n, \nu \rangle) \langle \phi_i'', \nu \rangle \\ &\quad + \frac{1}{2} \sum_{i,j=1}^n f''_{ij}(\langle \phi_1, \nu \rangle, \dots, \langle \phi_n, \nu \rangle) \int_{\mathbb{R}^2} \rho(x-y) \phi'_i(x) \phi'_j(y) \mu(dx) \mu(dy) \\ &\quad + \frac{1}{2} \sigma \sum_{i,j=1}^n f''_{ij}(\langle \phi_1, \nu \rangle, \dots, \langle \phi_n, \nu \rangle) \langle \phi_i \phi_j, \nu \rangle.\end{aligned}$$

By Itô's formula we see that (3.5) is a continuous martingale if  $F = F_{f, \{\phi_i\}}$ . Then the theorem follows by an approximation of an arbitrary  $F \in \mathcal{D}(\mathcal{L})$ .  $\square$

We now give the construction for a SDSM with atomic initial state. Suppose that  $(\Omega, \mathcal{F}, \mathbf{P})$  is a standard probability space on which we have a sequence of  $\sigma$ -branching diffusions  $\{\xi_i(t) : t \geq 0\}$  with  $\xi_i(0) = \xi_i$  ( $i = 1, 2, \dots$ ) and a white noise  $W(ds, dy)$  on  $[0, \infty) \times \mathbb{R}$  based on the Lebesgue measure; see e.g. Walsh (1986). We assume that  $\sum_{i=1}^{\infty} \xi_i < \infty$  and that  $\{\xi_i(t)\}$  and  $\{W(ds, dy)\}$  are independent of each other. For  $t \geq 0$  let  $\mathcal{G}_t$  be the  $\sigma$ -algebra generated by the family of  $\mathbf{P}$ -null sets in  $\mathcal{F}$  and the family of random variables

$$\{W([0, s] \times B), \xi_i(s) : 0 \leq s \leq t, B \in \mathcal{B}(\mathbb{R}), i = 1, 2, \dots\}. \quad (3.6)$$

By Dawson et al (2001, Lemma 3.1) or Wang (1997, Lemma 1.3), for any  $a \in \mathbb{R}$  the equation

$$x(t) = a + \int_0^t \int_{\mathbb{R}} h(y - x(s)) W(ds, dy), \quad t \geq 0; i = 1, 2, \dots, \quad (3.7)$$

has a unique solution  $\{x(a, t) : t \geq 0\}$ .

**Theorem 3.3** *The process  $\{X_t : t \geq 0\}$  defined by*

$$X_t = \sum_{i=1}^{\infty} \xi_i(t) \delta_{x_i(t)}, \quad t \geq 0, \quad (3.8)$$

relative to  $(\mathcal{G}_t)_{t \geq 0}$  is a diffusion process generated by the closure of  $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ .

*Proof.* Clearly,  $\{X_t : t \geq 0\}$  defined by (3.8) is a continuous  $M(\mathbb{R})$ -valued process. By Itô's formula, for  $\phi \in C^2(\mathbb{R})$  we have

$$\begin{aligned} \xi_i(t) \phi(x_i(t)) - \xi_i(0) \phi(x_i(0)) &= \int_0^t \int_{\mathbb{R}} \xi_i(s) \phi'(x_i(s)) h(y - x_i(s)) W(ds, dy) \\ &\quad + \frac{1}{2} \int_0^t ds \int_{\mathbb{R}} \xi_i(s) \phi''(x_i(s)) h(y - x_i(s))^2 dy \\ &\quad + \int_0^t \phi(x_i(s)) d\xi_i(s). \end{aligned}$$

Taking the summation we get

$$\langle \phi, X_t \rangle - \langle \phi, X_0 \rangle = M_t(\phi) + \frac{1}{2} \rho(0) \int_0^t \langle \phi'', X_s \rangle ds, \quad t \geq 0,$$

where

$$M_t(\phi) := \int_0^t \int_{\mathbb{R}} \langle h(y - \cdot) \phi', X_s \rangle W(ds, dy) + \sum_{i=1}^{\infty} \int_0^t \phi(x_i(s)) d\xi_i(s), \quad t \geq 0,$$

is a continuous martingale relative to  $(\mathcal{G}_t)_{t \geq 0}$  with quadratic variation process (1.3). Then we have the desired result by Theorem 3.2.  $\square$

By Theorem 3.3, if the SDSM is started from an initial state in  $M_a(\mathbb{R})$ , it lives in  $M_a(\mathbb{R})$  forever. More precisely, the position of its  $i$ th atom is described by  $\{x_i(t) : t \geq 0\}$  and its mass by  $\{\xi_i(t) : t \geq 0\}$ . As observed in Wang (1998, p.756), each  $\{x_i(t) : t \geq 0\}$  is a Brownian motion

with quadratic variation process  $\rho(0)dt$ . If  $x_i = x_j$ , we have a.s.  $x_i(t) = x_j(t)$  for all  $t \geq 0$  by the uniqueness of solution of (3.7). On the contrary, if  $x_i \neq x_j$ , then  $\{x_i(t) : t \geq 0\}$  and  $\{x_j(t) : t \geq 0\}$  a.s. never hit each other.

To consider a more general initial state, we need a lemma for Poisson random measures. Suppose that  $E$  and  $F$  are metrizable topological spaces. Let  $\mu \in M(E)$  and let  $q(x, dy)$  be a Borel probability kernel from  $E$  to  $F$ . Then

$$\int_E \int_F h(x, y) \lambda(dx, dy) = \int_E \mu(dx) \int_F h(x, y) q(x, dy), \quad h \in C(E \times F), \quad (3.9)$$

defines a measure  $\lambda \in M(E \times F)$ . For a Borel function  $g$  on  $E$ , we can define  $\mu_g \in M(\mathbb{R})$  by

$$\mu_g(f) = \int_E f(g(x)) \mu(dx), \quad f \in C(\mathbb{R}). \quad (3.10)$$

**Lemma 3.1** *In the situation described above, suppose  $\mu_g$  is a diffuse measure on  $\mathbb{R}$  and let  $Y$  be a Poisson random measure on  $E \times F$  with intensity  $\lambda$  and let  $\eta = Y(E \times F)$ . Then we can arrange the atoms of  $Y$  into an ordered random set  $\{(x_i, y_i) : i = 1, \dots, \eta\}$  so that a.s.  $g(x_1) < \dots < g(x_\eta)$ . Moreover,  $X := \sum_{i=1}^\eta \delta_{x_i}$  is a Poisson random measure on  $E$  with intensity  $\mu$  and, given  $\eta = k$  and  $x_i = c_i$  ( $i = 1, \dots, k$ ), the sequence  $\{y_i : i = 1, \dots, k\}$  is formed of independent random variables with distributions  $\{q(c_i, dy) : i = 1, \dots, k\}$ .*

*Proof.* We first consider a special version of the Poisson random measure  $Y$  constructed as follows. Let  $\eta$  be a Poisson random variable with parameter  $\mu(E)$  and  $\{(a_i, b_i) : i = 1, 2, \dots\}$  a sequence of i.i.d. random variables in  $E \times F$  with distribution  $\mu(E)^{-1} \lambda$  which are independent of  $\eta$ . Then

$$Y(dx, dy) := \sum_{i=1}^\eta \delta_{(a_i, b_i)}(dx, dy), \quad x \in E, y \in F,$$

is a Poisson random measure  $Y$  with intensity  $\lambda$  and

$$X(dx) := \sum_{i=1}^\eta \delta_{a_i}(dx), \quad x \in E,$$

is a Poisson random measure on  $E$  with intensity  $\mu(dx)$ . Observe that, for any integer  $k \geq 1$ , any permutation  $\{i_1, \dots, i_k\}$  of  $\{1, \dots, k\}$  and any sequence of real numbers  $\{c_j : j = 1, \dots, k\}$ , under the regular conditional probability  $\mathbf{P}\{\cdot | a_{i_j} = c_j : j = 1, \dots, k\}$ , the sequence  $\{b_{i_j} : j = 1, \dots, k\}$  consists of independent random variables with distributions  $\{q(c_j, dy_j) : j = 1, \dots, k\}$ . On the other hand, since  $\mu(E)^{-1} \mu_g$  is a diffuse distribution on  $\mathbb{R}$ , the points  $\{g(a_i) : i = 1, 2, \dots\}$  are a.s. distinct. Then, given  $\eta = k$ , we can rearrange  $\{a_i : i = 1, \dots, k\}$  into  $\{a_{i_j} : j = 1, \dots, k\}$  so that a.s.  $g(a_{i_1}) < \dots < g(a_{i_k})$ . This proves the lemma for the special version of  $Y$ . The results for an arbitrary realization of the Poisson random measure hold by the uniqueness of distribution.  $\square$

Now we discuss the construction of the sample paths of the SDSM with an arbitrary initial state  $\mu \in M(\mathbb{R})$  with  $\langle 1, \mu \rangle > 0$ . Suppose we are given on some standard probability space  $(\Omega, \mathcal{F}, \mathbf{P})$  a time-space white noise  $W(ds, dy)$  on  $[0, \infty) \times \mathbb{R}$  based on the Lebesgue measure and a Poisson random measure  $N(dx, dw)$  on  $\mathbb{R} \times W_0$  with intensity  $\mu(dx) \mathbf{Q}_\kappa(dw)$ . Assume that

$\{W(ds, dy)\}$  and  $\{N(dx, dw)\}$  are independent. For  $t \geq 0$  let  $\mathcal{G}_t$  be the  $\sigma$ -algebra generated by the families

$$\{W(B) : B \in \mathcal{B}([0, t] \times \mathbb{R})\} \quad \text{and} \quad \{N(A) : A \in \mathcal{B}(\mathbb{R}) \times \mathcal{B}_t(W_0)\}. \quad (3.11)$$

In view of (2.6), we can a.s. enumerate the atoms of  $N(dx, dw)$  into a sequence  $\{(x_i, w_i) : i = 1, 2, \dots\}$  such that  $\tau_0(w_{i+1}) < \tau_0(w_i)$  for all  $i \geq 1$  and  $\tau_0(w_i) \rightarrow 0$  as  $i \rightarrow \infty$ . Let  $m(t)$  denote the number of elements of  $\{(x_i, w_i) : i = 1, 2, \dots\}$  satisfying  $w_i(t) > 0$ . Then  $m(t) = i$  for  $\tau_0(w_{i+1}) \leq t < \tau_0(w_i)$ , and  $m(t) = 0$  for  $t \geq \tau_0(w_1)$ .

For any  $r > 0$  the finite measure  $\kappa_r(dy)$  on  $(0, \infty)$  is absolutely continuous. Thus the values  $w_1(r), \dots, w_{m(r)}(r)$  are a.s. distinct. Let  $\{w_{i_j} : j = 1, \dots, m(r)\}$  denote the permutation of  $\{w_i : i = 1, \dots, m(r)\}$  such that  $w_{i_1}(r) < \dots < w_{i_{m(r)}}(r)$  holds. Then we have

**Lemma 3.2** *Under  $\mathbf{P}\{\cdot | \mathcal{G}_r\}$ , the family  $\{w_{i_j}(t) : t \geq r; j = 1, \dots, m(r)\}$  consists of independent  $\sigma$ -branching diffusions which are independent of  $\{W(dt, dy) : t \geq r, y \in \mathbb{R}\}$ .*

*Proof.* Let  $\mathcal{G}'_t$  be the  $\sigma$ -algebra generated by the second family in (3.11). By Theorem 2.1 and Lemma 3.1,  $\{w_{i_j}(t) : t \geq r; j = 1, \dots, m(r)\}$  under  $\mathbf{P}\{\cdot | \mathcal{G}'_r\}$  are independent  $\sigma$ -branching diffusions. Then the result follows from the independence of  $\{W(ds, dy)\}$  and  $\{N(da, dw)\}$  and the property of independent increments.  $\square$

**Theorem 3.4** *Let  $\{X_t : t \geq 0\}$  be defined by  $X_0 = \mu$  and*

$$X_t = \sum_{i=1}^{m(t)} w_i(t) \delta_{x(x_i, t)}, \quad t > 0. \quad (3.12)$$

*Then  $\{X_t : t \geq 0\}$  relative to  $(\mathcal{G}_t)_{t \geq 0}$  is a diffusion process generated by the closure of  $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ .*

*Proof.* For any  $r > 0$ , Lemma 3.2 implies that  $\{w_{i_j}(t) : t \geq r; j = 1, \dots, m(r)\}$  under  $\mathbf{P}\{\cdot | \mathcal{G}_r\}$  are independent  $\sigma$ -branching diffusions which are independent of  $\{W(dt, dy) : t \geq r, y \in \mathbb{R}\}$ . By Theorem 3.3,  $\{X_t : t \geq r\}$  under  $\mathbf{P}\{\cdot | \mathcal{G}_r\}$  is a Markov process generated by  $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$ , which clearly implies the desired result.  $\square$

Thus the SDSM started with an arbitrary initial measure enters the space  $M_a(\mathbb{R})$  of pure atomic measures immediately and lives in this space forever. From Lemma 3.1 we know that for any  $r > 0$  the family  $\{x_i : i = 1, \dots, m(r)\}$  under the regular conditional probability  $\mathbf{P}\{\cdot | m(r)\}$  are i.i.d. random variables with distribution  $\langle 1, \mu \rangle^{-1} \mu(dx)$ . This gives an intuitive description of the locations  $\{x_i : i = 1, \dots, m(r)\}$  of the ‘‘ancestors’’ at the initial time of  $X_r$ . By Lemma 3.2, under  $\mathbf{P}\{\cdot | \mathcal{G}_r\}$ , we have that  $\{w_{i_j}(t+r) : t \geq 0; j = 1, \dots, m(r)\}$  are independent  $\sigma$ -branching diffusions and  $\{B(w_{i_j}, t+r) : t \geq 0; j = 1, \dots, m(r)\}$  are independent stopped Brownian motions. Of course, we also have

$$dw_{i_j}(t+r) = \sqrt{\sigma w_{i_j}(t+r)} dB(w_{i_j}, t+r), \quad t \geq 0. \quad (3.13)$$

Note that  $\{w_i(t) : t \geq 0; i = 1, 2, \dots\}$  under the non-conditional probability  $\mathbf{P}$  are not  $\sigma$ -branching diffusions, otherwise the initial condition  $w_i(0) = 0$  would imply  $w_i(t) = 0$  for all  $t \geq 0$ . In this sense, the constructions (3.8) and (3.12) of the SDSM are essentially different.

## 4 SDSM with independent immigration

In this section, we construct a class of immigration processes associated with the SDSM. Let  $\mathcal{D}(\mathcal{J}) = \mathcal{D}(\mathcal{L})$  and for  $F \in \mathcal{D}(\mathcal{J})$  define

$$\mathcal{J}F(\nu) = \mathcal{L}F(\nu) + \int_{\mathbb{R}} \frac{\delta F(\nu)}{\delta \nu(x)} m(dx), \quad \nu \in M(\mathbb{R}). \quad (4.1)$$

We shall characterize the process as a solution of the  $(\mathcal{J}, \mathcal{A}(\mathcal{J}))$ -martingale problem.

Let us consider first an atomic initial state. Suppose that  $m \in M(\mathbb{R})$  and  $(\Omega, \mathcal{F}, \mathbf{P})$  is a standard probability space on which we have: (i) a white noise  $W(ds, dy)$  on  $[0, \infty) \times \mathbb{R}$  based on the Lebesgue measure; (ii) a sequence of  $\sigma$ -branching diffusions  $\{\xi_i(t) : t \geq 0\}$  with  $\xi_i(0) = \xi_i$  ( $i = 1, 2, \dots$ ); and (iii) a Poisson random measure  $N(ds, da, dw)$  on  $[0, \infty) \times \mathbb{R} \times W_0$  with intensity  $ds m(da) \mathbf{Q}_\kappa(dw)$ . We assume that  $\sum_{i=1}^{\infty} \xi_i < \infty$  and that  $\{W(ds, dy)\}$ ,  $\{\xi_i(t)\}$  and  $\{N(ds, da, dw)\}$  are independent of each other. For  $t \geq 0$  let  $\mathcal{G}_t$  be the  $\sigma$ -algebra generated by the family of  $\mathbf{P}$ -null sets in  $\mathcal{F}$  and the families of random variables

$$\{W([0, s] \times B), \xi_i(s) : 0 \leq s \leq t, B \in \mathcal{B}(\mathbb{R}), i = 1, 2, \dots\} \quad (4.2)$$

and

$$\{N(J \times A) : J \in \mathcal{B}([0, s] \times \mathbb{R}), A \in \mathcal{B}_{t-s}(W_0), 0 \leq s \leq t\}. \quad (4.3)$$

Given  $(r, a) \in [0, \infty) \times \mathbb{R}$ , let  $\{x(r, a, t) : t \geq r\}$  denote the unique solution of

$$x(t) = a + \int_r^t \int_{\mathbb{R}} h(y - x_i(s)) W(ds, dy), \quad t \geq r. \quad (4.4)$$

For any sequence  $\{x_i\} \subset \mathbb{R}$  let

$$Y_t = \sum_{i=1}^{\infty} \xi_i(t) \delta_{x(0, x_i, t)} + \int_0^t \int_{\mathbb{R}} \int_{W_0} w(t-s) \delta_{x(s, a, t)} N(ds, da, dw), \quad t \geq 0. \quad (4.5)$$

(Here and in the sequel we make the convention that  $\int_0^t = \int_{(0, t]}$ .)

We shall prove that  $\{Y_t : t \geq 0\}$  is a.s. continuous and solves the  $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$ -martingale problem relative to the filtration  $(\mathcal{G}_t)_{t \geq 0}$ . Let  $W_n = \{w \in W_0 : \tau_0(w) > 1/n\}$  and recall from (2.6) that  $\mathbf{Q}_\kappa(W_n) = 2n/\sigma$ . To prove the continuity of  $\{Y_t : t \geq 0\}$  we consider the following approximating sequence:

$$Y_t^{(n)} = \sum_{i=1}^{\infty} \xi_i(t) \delta_{x(0, x_i, t)} + \int_0^t \int_{\mathbb{R}} \int_{W_n} w(t-s) \delta_{x(s, a, t)} N(ds, da, dw), \quad t \geq 0. \quad (4.6)$$

**Lemma 4.1** *Both  $\{Y_t : t \geq 0\}$  and  $\{Y_t^{(n)} : t \geq 0\}$  are a.s. continuous, and for any  $T > 0$  and  $\phi \in C(\mathbb{R})^+$  we have a.s.  $\{\langle \phi, Y_t^{(n)} \rangle : 0 \leq t \leq T\}$  converges to  $\{\langle \phi, Y_t \rangle : 0 \leq t \leq T\}$  increasingly and uniformly as  $n \rightarrow \infty$ .*

*Proof.* Let  $N_0(ds, dw)$  denote the image of  $N(ds, da, dw)$  under the mapping  $(s, a, w) \mapsto (s, w)$ . Then  $N_0(ds, dw)$  is a Poisson random measure on  $[0, \infty) \times W_0$  with intensity  $\langle m, 1 \rangle ds \mathbf{Q}_\kappa(dw)$  and is independent of  $\{\xi_i(t) : t \geq 0, i = 1, 2, \dots\}$ . Note that

$$\langle 1, Y_t \rangle = \sum_{i=1}^{\infty} \xi_i(t) + \int_0^t \int_{W_0} w(t-s) N_0(ds, dw), \quad t \geq 0.$$

By Pitman and Yor (1982, Theorem 4.1),  $\{\langle 1, Y_t \rangle : t \geq 0\}$  is a diffusion process with generator  $2^{-1}\sigma x d^2/dx^2 + \langle 1, m \rangle d/dx$ . Let  $\Omega_1 \in \mathcal{F}$  be a set with full  $\mathbf{P}$ -measure such that  $\{\langle 1, Y_t(\omega) \rangle : t \geq 0\}$  is continuous and  $N(\omega, [0, n] \times \mathbb{R} \times W_n) < \infty$  for all  $n \geq 1$  and  $\omega \in \Omega_1$ . For  $\omega \in \Omega_1$  and  $\phi \in C(\mathbb{R})^+$ , we have that  $\{\langle \phi, Y_t^{(n)}(\omega) \rangle : t \geq 0\}$  is continuous and converges to  $\{\langle \phi, Y_t(\omega) \rangle : t \geq 0\}$  increasingly as  $n \rightarrow \infty$ . Then  $\{\langle \phi, Y_t(\omega) \rangle : t \geq 0\}$  is lower semi-continuous. The same reasoning shows that

$$\|\phi\| - \langle \phi, Y_t(\omega) \rangle = \|\phi\| \langle 1, Y_t(\omega) \rangle - \langle \phi, Y_t(\omega) \rangle, \quad t \geq 0,$$

is also lower semi-continuous. Since  $\{\langle 1, Y_t(\omega) \rangle : t \geq 0\}$  is continuous, we conclude that  $\{\langle \phi, Y_t(\omega) \rangle : t \geq 0\}$  is continuous, giving the desired result.  $\square$

To show  $\{Y_t : t \geq 0\}$  is a solution of the  $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$ -martingale problem relative to the filtration  $(\mathcal{G}_t)_{t \geq 0}$ , we consider another approximating sequence  $\{X_t^{(n)} : t \geq 0\}$  defined by

$$X_t^{(n)} = \sum_{i=1}^{\infty} \xi_i(t) \delta_{x(0, x_i, t)} + \int_0^t \int_{\mathbb{R}} \int_{W_n} w(t-s+1/n) \delta_{x(s, a, t)} N(ds, da, dw). \quad (4.7)$$

Note that the first parts of the excursions were chopped off in taking the second summation in (4.7); see also Li (2001).

**Lemma 4.2** *The process  $\{X_t^{(n)} : t \geq 0\}$  is a.s. cádlág and for any  $T > 0$  and  $\phi \in C(\mathbb{R})$  we have a.s.  $\{\langle \phi, X_t^{(n)} \rangle : 0 \leq t \leq T\}$  converges to  $\{\langle \phi, Y_t \rangle : 0 \leq t \leq T\}$  uniformly as  $n \rightarrow \infty$ .*

*Proof.* The first assertion holds since  $N([0, n] \times \mathbb{R} \times W_n) < \infty$  a.s. for all  $n \geq 1$ . We only need to prove the second assertion for  $X_t = X_t^{(n)} = 0$  so the first terms in both (4.5) and (4.6) vanish. In this case, we have a.s.  $\langle 1, X_t^{(n)} \rangle \leq \langle 1, Y_{t+1/n} \rangle$  simultaneously for all  $t \geq 0$ . By Lemma 4.1, there is a set  $\Omega_2 \in \mathcal{F}$  with full  $\mathbf{P}$ -measure such that  $\{\langle 1, Y_t^{(n)} \rangle : 0 \leq t \leq T\}$  converges to  $\{\langle 1, Y_t \rangle : 0 \leq t \leq T\}$  uniformly as  $n \rightarrow \infty$  for all  $T > 0$  and  $\omega \in \Omega_2$ . Fix  $T > 0$  and  $\omega \in \Omega_2$ . For  $\varepsilon > 0$  let  $m(\omega) \geq 1$  be an integer such that

$$\langle 1, Y_t(\omega) \rangle - \langle 1, Y_t^{(n)}(\omega) \rangle < \varepsilon, \quad 0 \leq t \leq T+1,$$

or equivalently

$$\int_0^t \int_{\mathbb{R}} \int_{W_0 \setminus W_{m(\omega)}} w(t-s) N(\omega, ds, da, dw) < \varepsilon, \quad 0 \leq t \leq T+1.$$

Since  $N(\omega, [0, T] \times \mathbb{R} \times W_{m(\omega)}) < \infty$ , there is an integer  $M(\omega) \geq m(\omega)$  such that, for  $n \geq M(\omega)$ ,

$$\int_0^t \int_{\mathbb{R}} \int_{W_{m(\omega)}} |w(t-s+1/n) - w(t-s)| N(ds, da, dw, \omega) < \varepsilon, \quad 0 \leq t \leq T.$$

Then for  $\phi \in C(\mathbb{R})$  and  $n \geq M(\omega)$  we have

$$\begin{aligned} & |\langle \phi, X_t^{(n)}(\omega) \rangle - \langle \phi, Y_t(\omega) \rangle| \\ & \leq \int_0^t \int_{\mathbb{R}} \int_{W_{m(\omega)}} \|\phi\| |w(t-s+1/n) - w(t-s)| N(ds, da, dw, \omega) \\ & \quad + \int_0^t \int_{\mathbb{R}} \int_{W_0 \setminus W_{m(\omega)}} \|\phi\| |w(t-s+1/n) + w(t-s)| N(ds, da, dw, \omega) \\ & < 3\|\phi\|\varepsilon \end{aligned}$$

for  $0 \leq t \leq T$ . That is,  $\{\langle \phi, X_t^{(n)}(\omega) \rangle : 0 \leq t \leq T\}$  converges to  $\{\langle \phi, Y_t(\omega) \rangle : 0 \leq t \leq T\}$  uniformly as  $n \rightarrow \infty$ .  $\square$

Let  $N_n(ds, da, dw)$  denote the restriction of  $N(ds, da, dw)$  to  $[0, \infty) \times \mathbb{R} \times W_n$ . For  $t \geq 0$  let  $\eta_n(t) = N([0, t] \times \mathbb{R} \times W_n)$  and let  $\mathcal{G}_{nt}$  be the  $\sigma$ -algebra generated by the families (4.2) and

$$\{N(J \times A) : J \in \mathcal{B}([0, s] \times \mathbb{R}), A \in \mathcal{B}_{t-s+1/n}(W_0), 0 \leq s \leq t\}. \quad (4.8)$$

Clearly, we can a.s. arrange the atoms of  $N_n(ds, da, dw)$  into a sequence  $\{(r_j, a_j, w_j) : j = 1, 2, \dots, \}$  so that  $0 < r_1 < r_2 < \dots$ . With this enumeration we have

**Lemma 4.3** *For any  $r \geq 0$ ,  $\{\xi_i(t+r), w_j(t+r-r_j+1/n) : t \geq 0; i = 1, 2, \dots; j = 1, \dots, \eta_n(r)\}$  under  $\mathbf{P}\{\cdot | \mathcal{G}_{nr}\}$  are independent  $\sigma$ -branching diffusions relative to  $(\mathcal{G}_{n,r+t})_{t \geq 0}$ .*

*Proof.* For  $r \geq 0$  let  $\mathcal{F}_{nr}$  be the  $\sigma$ -algebra generated by

$$\{N_n(J \times A) : J \in \mathcal{B}([0, r] \times \mathbb{R}), A \in \mathcal{B}_{1/n}(W_0) \cap W_n\}.$$

By Theorem 2.1 and Lemma 3.1,  $\{w_j(t+1/n) : t \geq 0; j = 1, \dots, \eta_n(r)\}$  under  $\mathbf{P}\{\cdot | \mathcal{F}_{nr}\}$  are independent  $\sigma$ -branching diffusions. Clearly, the same assertion is true for  $\{w_j(t+r-r_j+1/n) : t \geq 0; j = 1, \dots, \eta_n(r)\}$  under  $\mathbf{P}\{\cdot | \mathcal{F}'_{nr}\}$ , where  $\mathcal{F}'_{nr}$  is the  $\sigma$ -algebra generated by

$$\{N_n(J \times A) : J \in \mathcal{B}([0, s] \times \mathbb{R}), A \in \mathcal{B}_{r-s+1/n}(W_0) \cap W_n, 0 \leq s \leq r\}.$$

Then the result follows from the independence of  $\{\xi_n(t)\}$ ,  $\{W(ds, dy)\}$  and  $\{N(ds, da, dw)\}$ .  $\square$

For any fixed  $r \geq 0$  we can a.s. enumerate the atoms  $(s, a, w)$  of  $N(ds, da, dw)$  satisfying  $0 < s \leq r$  and  $w(r-s) > 0$  into a sequence  $\{(r_j, a_j, w_j) : j = 1, 2, \dots\}$  so that  $r_j < r_{j+1}$  for all  $j \geq 1$ . The following theorem extends the result of Lemma 4.3.

**Theorem 4.1** *Under  $\mathbf{P}\{\cdot | \mathcal{G}_r\}$ , the sequence  $\{\xi_i(t+r), w_j(t+r-r_j) : t \geq 0; i = 1, 2, \dots; j = 1, 2, \dots\}$  are independent  $\sigma$ -branching diffusions relative to  $(\mathcal{G}_{r+t})_{t \geq 0}$ .*

*Proof.* It suffices to prove the result for  $r > 0$ . Clearly, for any integer  $n > 1/r$ , we have  $r_j < r - 1/n$  if and only if  $j \leq \eta_n(r - 1/n)$ . As in the proof of Lemma 4.3 we see that, the sequence  $\{w_j(t+r-r_j) : t \geq 0; j = 1, \dots, \eta_n(r - 1/n)\}$  under  $\mathbf{P}\{\cdot | \mathcal{F}''_{nr}\}$  are independent  $\sigma$ -branching diffusions, where  $\mathcal{F}''_{nr}$  is the  $\sigma$ -algebra generated by

$$\{N_n(J \times A) : J \in \mathcal{B}([0, s] \times \mathbb{R}), A \in \mathcal{B}_{r-s}(W_0) \cap W_n, 0 \leq s \leq r\}.$$

By the independence of  $\{\xi_n(t)\}$ ,  $\{W(ds, dy)\}$  and  $\{N(ds, da, dw)\}$  we have that  $\{\xi_i(t+r), w_j(t+r-r_j) : t \geq 0; i = 1, 2, \dots; j = 1, \dots, \eta_n(r - 1/n)\}$  under  $\mathbf{P}\{\cdot | \mathcal{G}_r\}$  are independent  $\sigma$ -branching diffusions. Since  $\eta_n(r - 1/n) \rightarrow \infty$  as  $n \rightarrow \infty$ , we have the desired result.  $\square$

**Lemma 4.4** *The process  $\{X_t^{(n)} : t \geq 0\}$  relative to  $(\mathcal{G}_{nt})_{t \geq 0}$  is a strong Markov process generated by the closure of  $(\mathcal{J}_n, \mathcal{D}(\mathcal{J}_n))$ , where*

$$\mathcal{J}_n F(\nu) = \mathcal{L}F(\nu) + \int_{\mathbb{R}} m(dx) \int_0^\infty [F(\nu + y\delta_x) - F(\nu)] \kappa_{1/n}(dy), \quad \nu \in M(\mathbb{R}), \quad (4.9)$$

for  $F \in \mathcal{D}(\mathcal{J}_n) = \mathcal{D}(\mathcal{L})$ .

*Proof.* We may assume  $\langle 1, m \rangle > 0$ , otherwise the result reduces to Theorem 3.3. For any  $k \geq \eta_n(0)$  we have

$$X_{t+r_k}^{(n)} = \sum_{i=1}^{\infty} \xi_i(t+r_k) \delta_{x_i(0, a_i, r_k+t)} + \sum_{j=1}^k w_j(t+r_k-r_j+1/n) \delta_{x(r_j, a_j, r_k+t)}, \quad 0 \leq t < r_{k+1} - r_k,$$

Note that  $\sum_{j=\eta_n(0)+1}^{\infty} \delta_{r_j}$  is a Poisson random measure on  $(0, \infty)$  with intensity  $2n\sigma^{-1}\langle 1, m \rangle ds$ . Since  $r \geq 0$  in Lemma 4.3 was arbitrary, for any inter  $k \geq 1$  the family  $\{\xi_i(t+r_k), w_j(t+r_k-r_j+1/n) : t \geq 0; i = 1, 2, \dots; j = 1, \dots, k\}$  under  $\mathbf{P}\{\cdot | \mathcal{G}_{n, r_k}\}$  are independent  $\sigma$ -branching diffusions relative to  $(\mathcal{G}_{n, r_k+t})_{t \geq 0}$ . By the independence of  $\{\xi_n(t)\}$ ,  $\{W(ds, dy)\}$  and  $\{N(ds, da, dw)\}$  we may appeal Theorem 3.3 to get that  $\{X_{t+r_k}^{(n)} : 0 \leq t < r_{k+1} - r_k\}$  under  $\mathbf{P}\{\cdot | \mathcal{G}_{n, r_k}\}$  is a (killed) diffusion process generated by  $\mathcal{L} - 2n\langle 1, m \rangle / \sigma$  with respect to  $(\mathcal{G}_{n, r_k+t})_{t \geq 0}$ . Observe also that

$$\mathbf{P}\{F(X_{r_k}^{(n)}) | \mathcal{G}_{n, r_k-}\} = \frac{\sigma}{2n\langle 1, m \rangle} \int_{\mathbb{R}} m(dx) \int_0^{\infty} F(X_{r_k-}^{(n)} + y\delta_x) \kappa_{1/n}(dy)$$

for any  $F \in C(M(\mathbb{R}))$  and integer  $k \geq \eta_n(r) + 1$ . Then  $\{X_t^{(n)} : t \geq 0\}$  relative to  $(\mathcal{G}_{n, t})_{t \geq 0}$  is a strong Markov process generated by the closure of  $(\mathcal{J}_n, \mathcal{D}(\mathcal{J}_n))$ .  $\square$

**Theorem 4.2** *The process  $\{Y_t : t \geq 0\}$  constructed by (4.5) is a.s. continuous and solves the  $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$ -martingale problem relative to the filtration  $(\mathcal{G}_t)_{t \geq 0}$ .*

*Proof.* As observed in the proof of lemma 4.1,  $\{\langle 1, Y_t(\omega) \rangle : t \geq 0\}$  is a diffusion process with generator  $2^{-1}\sigma x d^2/dx^2 + \langle 1, m \rangle d/dx$  and initial value  $\langle 1, \mu \rangle$ . Then  $\mathbf{P}\{\langle 1, Y_t \rangle^n\}$  is a locally bounded function of  $t \geq 0$  for every  $n \geq 1$ . For  $F = F_{f, \{\phi_i\}}$  given by (3.2), Lemma 4.4 asserts that

$$F(X_t^{(n)}) - F(X_0^{(n)}) - \int_0^t \mathcal{J}_n F(X_s^{(n)}) ds, \quad t \geq 0,$$

is a martingale relative to  $(\mathcal{G}_{n, t})_{t \geq 0}$ . Since  $(\mathcal{G}_t)_{t \geq 0}$  is smaller than  $(\mathcal{G}_{n, t})_{t \geq 0}$ , for any  $t \geq r \geq 0$  and any  $G \in \text{bp}\mathcal{G}_r$  we have

$$\mathbf{E} \left\{ G \left[ F(X_t^{(n)}) - F(X_r^{(n)}) - \int_r^t \mathcal{J}_n F(X_s^{(n)}) ds \right] \right\} = 0. \quad (4.10)$$

By (2.3) and (4.9) it is not hard to check that  $\mathcal{J}_n F(\nu) \rightarrow \mathcal{J}F(\nu)$  uniformly on the set  $\{\nu \in M(\mathbb{R}) : \langle 1, \nu \rangle \leq a\}$  as  $n \rightarrow \infty$  for each  $a \geq 0$ . By Lemmas 4.2, letting  $n \rightarrow \infty$  in (4.10) we get

$$\mathbf{E} \left\{ G \left[ F(Y_t) - F(Y_r) - \int_r^t \mathcal{J}F(Y_s) ds \right] \right\} = 0. \quad (4.11)$$

That is,

$$F(Y_t) - F(Y_0) - \int_0^t \mathcal{J}F(Y_s) ds, \quad t \geq 0,$$

is a martingale relative to  $(\mathcal{G}_t)_{t \geq 0}$ . Then the desired result follows by an approximation of an arbitrary  $F \in \mathcal{D}(\mathcal{J})$ .  $\square$

Now let us consider an arbitrary initial state  $\mu \in M(\mathbb{R})$ . Suppose on the standard probability space  $(\Omega, \mathcal{F}, \mathbf{P})$  have: (i) a white noise  $W(ds, dy)$  on  $[0, \infty) \times \mathbb{R}$  based on the Lebesgue measure;

(ii) a Poisson random measure  $N_0(dx, dw)$  on  $\mathbb{R} \times W_0$  with intensity  $\mu(dx)\mathbf{Q}_\kappa(dw)$ ; and (iii) a Poisson random measure  $N(ds, da, dw)$  on  $[0, \infty) \times \mathbb{R} \times W_0$  with intensity  $dsm(da)\mathbf{Q}_\kappa(dw)$ . We assume that  $\{W(ds, dy)\}$ ,  $\{N_0(dx, dw)\}$  and  $\{N(ds, da, dw)\}$  are independent of each other. For  $t \geq 0$  let  $\mathcal{G}_t$  be the  $\sigma$ -algebra generated by the family of  $\mathbf{P}$ -null sets in  $\mathcal{F}$  and the families of random variables

$$\{W([0, s] \times B), N_0(F \times A) : F \in \mathcal{B}(\mathbb{R}), A \in \mathcal{B}_t(W_0), B \in \mathcal{B}(\mathbb{R}), 0 \leq s \leq t\} \quad (4.12)$$

and

$$\{N(I \times B \times A) : I \in \mathcal{B}([0, s]), B \in \mathcal{B}(\mathbb{R}), A \in \mathcal{B}_{t-s}(W_0), 0 \leq s \leq t\}. \quad (4.13)$$

Given  $(r, a) \in [0, \infty) \times \mathbb{R}$ , let  $\{x(r, a, t) : t \geq r\}$  denote the unique solution of (4.4). Let  $Y_0 = \mu$  and for  $t > 0$  let

$$Y_t = \int_{\mathbb{R}} \int_{W_0} w(t) \delta_{x(0, x, t)} N_0(dx, dw) + \int_0^t \int_{\mathbb{R}} \int_{W_0} w(t-s) \delta_{x(s, a, t)} N(ds, da, dw). \quad (4.14)$$

**Theorem 4.3** *The process  $\{Y_t : t \geq 0\}$  defined above has is a.s. continuous and solves the  $(\mathcal{J}, \mathcal{D}(\mathcal{J}), \mu)$ -martingale problem relative to the filtration  $(\mathcal{G}_t)_{t \geq 0}$ .*

*Proof.* By Theorem 3.4, the first term on the right hand side of (4.14) is a.s. continuous and converges to  $\mu$  as  $t \rightarrow 0$ . By Lemma 4.1 the second term is also a.s. continuous. Thus  $\{Y_t : t \geq 0\}$  has a continuous modification. For  $r > 0$  let  $\{(x_i, w_i) : i = 1, \dots, m(r)\}$  be set of atoms of  $N_0(dx, dw)$  satisfying  $u_i(r) > 0$  and be arranged so that  $u_1(r) < \dots < u_{m(r)}(r)$ . Let  $\{(r_j, a_j, w_j) : j = 1, 2, \dots\}$  be the set of atoms of  $N(ds, da, dw)$  satisfying  $0 < r_j \leq r$  and  $w_j(r - r_j) > 0$  and be arranged so that  $r_j < r_{j+1}$  for all  $j \geq 1$ . By Lemma 3.2, Theorem 4.1 and the independence assumption,  $\{u_i(t+r), w_j(t+r-r_i) : t \geq 0; i = 1, \dots, m(r); j = 1, 2, \dots\}$  under  $\mathbf{P}\{\cdot | \mathcal{G}_r\}$  are independent  $\sigma$ -branching diffusions relative to  $(\mathcal{G}_{r+t})_{t \geq 0}$ . By Theorem 4.3 and the property of independent increments of  $W(ds, dy)$  and  $N(ds, da, dw)$ , the continuous process  $\{Y_{t+r} : t \geq 0\}$  under  $\mathbf{P}\{\cdot | \mathcal{G}_r\}$  is a solution of the  $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$ -martingale problem relative to the filtration  $(\mathcal{G}_{t+r})_{t \geq 0}$ , yielding the desired result.  $\square$

**Theorem 4.4** *The  $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$ -martingale problem has a unique solution.*

*Proof.* By Theorem 4.4, for each  $\mu \in M(\mathbb{R})$  the  $(\mathcal{J}, \mathcal{D}(\mathcal{J}), \mu)$ -martingale problem has a solution on  $C([0, \infty), M(\mathbb{R}))$ . The approach to the uniqueness is similar to that in Dawson et al (2001), so we only provide an outline. For  $f \in C^2(\mathbb{R}^n)$  let

$$G^n f(x) = \frac{1}{2} \rho(0) \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2} f(x) + \frac{1}{2} \sum_{i,j=1, i \neq j}^n \rho(x_i - x_j) \frac{\partial^2}{\partial x_i \partial x_j} f(x), \quad x \in \mathbb{R}^n.$$

Then we have

$$\mathcal{J}F_{n,f}(\nu) = F_{n,G^n f}(\nu) + \frac{1}{2} \sum_{i,j=1, i \neq j}^n F_{n-1, \Phi_{ij} f}(\nu) + \sum_{i=1}^n F_{n-1, \Psi_i f}(\nu), \quad \nu \in M(\mathbb{R}),$$

with  $\Psi_i f \in C^2(\mathbb{R}^n)$  defined by

$$\Psi_i f(x_1, \dots, x_{n-1}) = \int_{\mathbb{R}} f(x_1, \dots, x_{i-1}, x, x_i, \dots, x_{n-1}) m(dx), \quad x_j \in \mathbb{R},$$

where  $x \in \mathbb{R}$  is the  $i$ th variable of  $f$  on the right hand side. Write  $F_\nu(n, f) = F_{n,f}(\nu)$  and let

$$\begin{aligned} \mathcal{J}^* F_\nu(n, f) &= F_\mu(n, G^n f) + \frac{1}{2} \sum_{i,j=1, i \neq j}^n [F_\nu(n-1, \Phi_{ij} f) - F_\nu(n, f)] \\ &\quad + \sum_{i=1}^n [F_\nu(n-1, \Psi_i f) - F_\nu(n, f)]. \end{aligned} \quad (4.15)$$

Then we have

$$\mathcal{J} F_{n,f}(\nu) = \mathcal{J}^* F_\nu(n, f) + \frac{1}{2} n(n+1) F_\nu(n, f). \quad (4.16)$$

Guided by (4.15) we can construct a Markov process  $\{(M_t, Y_t) : t \geq 0\}$  with initial value  $(M_0, Y_0) = (n, f)$  and generator  $\mathcal{J}^*$ . From (4.16) we have

$$\mathbf{P}\{\langle f, X_t^n \rangle\} = \mathbf{E}_{(n,f)} \left[ \langle Y_t, \mu^{M_t} \rangle \exp \left\{ \frac{1}{2} \int_0^t M_s(M_s + 1) ds \right\} \right], \quad t \geq 0.$$

This duality determines the one-dimensional distributions of  $\{Y_t : t \geq 0\}$  uniquely, and hence the conclusion follows by Ethier and Kurtz (1986, p.184).  $\square$

By the uniqueness of solution of the  $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$ -martingale problem, the processes mentioned in Theorems 4.3 and 4.4 are diffusions generated by the closure of  $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$ , which give constructions of immigration processes associated with the SDSM. By the construction (4.14), the immigration SDSM started any initial state enters the space of pure atomic measures immediately and lives in this space forever. The next theorem, which can be proved similarly as Theorem 3.2, gives a useful alternate characterization of the immigration SDSM.

**Theorem 4.5** *A continuous  $M(\mathbb{R})$ -valued process  $\{Y_t : t \geq 0\}$  is a solution of the  $(\mathcal{J}, \mathcal{D}(\mathcal{J}))$ -martingale problem if and only if for each  $\phi \in C^2(\mathbb{R})$ ,*

$$M_t(\phi) := \langle \phi, Y_t \rangle - \langle \phi, Y_0 \rangle - \langle \phi, m \rangle t - \frac{1}{2} \rho(0) \int_0^t \langle \phi'', Y_s \rangle ds, \quad t \geq 0, \quad (4.17)$$

is a martingale with quadratic variation process

$$\langle M(\phi) \rangle_t = \int_0^t \langle \sigma \phi^2, Y_s \rangle ds + \int_0^t ds \int_{\mathbb{R}} \langle h(z - \cdot) \phi', Y_s \rangle^2 dz. \quad (4.18)$$

Under the condition of Theorem 4.5, the martingales  $\{M_t(\phi) : t \geq 0\}$  defined by (4.17) and (4.18) form a system which is linear in  $\phi \in C^2(\mathbb{R})$ . Following the method of Walsh (1986), we can define the stochastic integral

$$\int_0^t \int_{\mathbb{R}} \phi(s, x) M(ds, dx), \quad t \geq 0,$$

if both  $\phi(s, x)$  and  $\phi'_x(s, x)$  are continuous on  $[0, \infty) \times \mathbb{R}$ . By a standard argument we get the following

**Theorem 4.6** *In the situation described above, for any  $t \geq 0$  and  $\phi \in C^1(\mathbb{R})$  we have a.s.*

$$\langle \phi, Y_t \rangle = \langle P_t \phi, Y_0 \rangle + \int_0^t \langle P_{t-s} \phi, m \rangle ds + \int_0^t \int_{\mathbb{R}} P_{t-s} \phi(x) M(ds, dx),$$

where  $(P_t)_{t \geq 0}$  is the semigroup of the Brownian motion generated by  $2^{-1} \rho(0) d^2/dx^2$ .

## 5 Stochastic equations for immigration processes

In this section, we construct interactive immigration processes associated with the SDSM as solutions of stochastic equations. Let  $m$  be a  $\sigma$ -finite Borel measure on  $\mathbb{R}$ . Suppose we are given on a standard probability space  $(\Omega, \mathcal{F}, \mathbf{P})$  the following: (i) a white noise  $W(ds, dy)$  on  $[0, \infty) \times \mathbb{R}$  based on the Lebesgue measure; (ii) a sequence of  $\sigma$ -branching diffusions  $\{\xi_i(t) : t \geq 0\}$  with  $\xi_i(0) = \xi_i$  ( $i = 1, 2, \dots$ ); and (iii) a Poisson random measure  $N(ds, da, du, dw)$  on  $[0, \infty) \times \mathbb{R} \times [0, \infty) \times W_0$  with intensity  $dsm(da)du\mathbf{Q}_\kappa(dw)$ . We assume that  $\sum_{i=1}^{\infty} \xi_i < \infty$  and that  $\{W(ds, dy)\}$ ,  $\{\xi_i(t)\}$  and  $\{N(ds, da, du, dw)\}$  are independent of each other. For  $t \geq 0$  let  $\mathcal{G}_t$  be the  $\sigma$ -algebra generated by the family of  $\mathbf{P}$ -null sets in  $\mathcal{F}$  and the families of random variables (4.2) and

$$\{N(J \times A) : J \in \mathcal{B}([0, s] \times \mathbb{R} \times [0, \infty)), A \in \mathcal{B}_{t-s}(W_0), 0 \leq s \leq t\}. \quad (5.1)$$

Let  $\mathcal{P}$  be the  $\sigma$ -algebra on  $[0, \infty) \times \mathbb{R} \times \Omega$  generated by functions of the form

$$g(s, x, \omega) = \eta_0(x, \omega)1_{\{0\}}(s) + \sum_{i=0}^{\infty} \eta_i(x, \omega)1_{(r_i, r_{i+1}]}(s), \quad (5.2)$$

where  $0 = r_0 < r_1 < r_2 < \dots$  and  $\eta_i(\cdot, \cdot)$  is  $\mathcal{B}(\mathbb{R}) \times \mathcal{G}_{r_i}$ -measurable. We say a function on  $[0, \infty) \times \mathbb{R} \times \Omega$  is *predictable* if it is  $\mathcal{P}$ -measurable.

Suppose that  $q(\cdot, \cdot, \cdot)$  is a non-negative predictable function on  $[0, \infty) \times \mathbb{R} \times \Omega$  such that  $\mathbf{E}\{m(q(t, \cdot))^2\}$  is locally bounded in  $t \geq 0$ . For a sequence  $\{x_i\} \subset \mathbb{R}$  let

$$Y_t = \sum_{i=1}^{\infty} \xi_i(t) \delta_{x(0, x_i, t)} + \int_0^t \int_{\mathbb{R}} \int_0^{q(s, a)} \int_{W_0} w(t-s) \delta_{x(s, a, t)} N(ds, da, du, dw), \quad t \geq 0. \quad (5.3)$$

**Theorem 5.1** *The process  $\{Y_t : t \geq 0\}$  defined by (5.3) has a continuous modification. For this modification and each  $\phi \in C^2(\mathbb{R})$ ,*

$$M_t(\phi) := \langle \phi, Y_t \rangle - \langle \phi, Y_0 \rangle - \frac{1}{2} \rho(0) \int_0^t \langle \phi'', Y_s \rangle ds - \int_0^t \langle q(s, \cdot) \phi, m \rangle ds, \quad t \geq 0, \quad (5.4)$$

is a continuous martingale relative to the filtration  $(\mathcal{G}_t)_{t \geq 0}$  with quadratic variation process

$$\langle M(\phi) \rangle_t = \int_0^t \langle \sigma \phi^2, Y_s \rangle ds + \int_0^t ds \int_{\mathbb{R}} \langle h(z - \cdot) \phi', Y_s \rangle^2 dz. \quad (5.5)$$

*Proof. Step 1)* Suppose that  $q(s, x, \omega) \equiv q(x)$  for a function  $q \in L^1(\mathbb{R}, m)$ . Let  $N_q(ds, da, du, dw)$  denote the restriction of  $N(ds, da, du, dw)$  to the set  $\{(s, a, u, w) : s \geq 0, a \in \mathbb{R}, 0 \leq u \leq q(x), w \in W_0\}$  and let  $N_q(ds, da, dw)$  be the image of  $N_q(ds, da, du, dw)$  under the mapping  $(s, a, u, w) \mapsto (s, a, w)$ . Clearly,  $N_q(ds, da, dw)$  is a Poisson measure on  $[0, \infty) \times \mathbb{R} \times W_0$  with intensity  $dsq(a)m(da)\mathbf{Q}_\kappa(dw)$  and (5.3) can be rewritten as

$$Y_t = \sum_{i=1}^{\infty} \xi_i(t) \delta_{x(0, x_i, t)} + \int_0^t \int_{\mathbb{R}} \int_{W_0} w(t-s) \delta_{x(s, a, t)} N_q(ds, da, dw), \quad t > 0.$$

Then the results are reduced to those of Theorems 4.2 and 4.5. *Step 2)* Suppose that  $q(\omega, s, x)$  is of the form (5.2). Note that  $\eta_i(x)$  is actually deterministic under the conditional probability  $\mathbf{P}\{\cdot | \mathcal{G}_{r_i}\}$ . By Theorems 4.2 and 4.5, the results hold on each interval  $[r_i, r_{i+1}]$  and hence on

$[0, \infty)$ . *Step 3*) The case of a general non-negative predictable function  $q(\cdot, \cdot, \cdot)$  can be proved by approximating arguments similarly as in Fu and Li (2002) and Shiga (1990).  $\square$

Then we consider an equation for measure-valued processes. Suppose that  $q(\cdot, \cdot)$  is a Borel function on  $M(\mathbb{R}) \times \mathbb{R}$  such that there is a constant  $K$  such that

$$m(q(\nu, \cdot)) \leq K(1 + \|\nu\|), \quad \nu \in M(\mathbb{R}), \quad (5.6)$$

and for each  $R > 0$  there is a constant  $K_R > 0$  such that

$$m(|q(\nu, \cdot) - q(\gamma, \cdot)|) \leq K_R \|\nu - \gamma\| \quad (5.7)$$

for  $\nu$  and  $\gamma \in M(\mathbb{R})$  satisfying  $\nu(1) \leq R$  and  $\gamma(1) \leq R$ , where  $\|\cdot\|$  denotes the total variation. For any sequence  $\{x_i\} \subset \mathbb{R}$ , consider the stochastic equation:

$$Y_t = \sum_{i=1}^{\infty} \xi_i(t) \delta_{x(0, x_i, t)} + \int_0^t \int_{\mathbb{R}} \int_0^{q(Y_s, a)} \int_{W_0} w(t-s) \delta_{x(s, a, t)} N(ds, da, du, dw), \quad t \geq 0. \quad (5.8)$$

**Theorem 5.2** *Under the above conditions, there is a unique continuous solution  $\{Y_t : t \geq 0\}$  of (5.8), which is a diffusion process. Moreover, for each  $\phi \in C^2(\mathbb{R})$ ,*

$$M_t(\phi) = \langle \phi, Y_t \rangle - \langle \phi, Y_0 \rangle - \frac{1}{2} \int_0^t \langle \phi'', Y_s \rangle ds - \int_0^t \langle q(Y_s, \cdot) \phi, m \rangle ds, \quad t \geq 0, \quad (5.9)$$

is a continuous martingale relative to the filtration  $(\mathcal{G}_t)_{t \geq 0}$  with quadratic variation process

$$\langle M(\phi) \rangle_t = \int_0^t \langle \sigma \phi^2, Y_s \rangle ds + \int_0^t ds \int_{\mathbb{R}} \langle h(z - \cdot) \phi', Y_s \rangle^2 dz, \quad t \geq 0. \quad (5.10)$$

*Proof.* Based on Theorem 5.1 and the results in the last section, it can be proved by techniques similar to those in Fu and Li (2002) and Shiga (1990) that (5.8) has a unique solution and (5.9) is a continuous martingale with quadratic variation process (5.10). Let  $\mu = \sum_{i=1}^{\infty} \xi_i \delta_{x_i}$  and let  $Q_t^q(\mu, \cdot)$  denote the distribution of  $Y_t$  defined by (5.8). For any bounded  $(\mathcal{G}_t)$ -stopping time  $\tau \geq 0$ , one can a.s. enumerate the atoms  $(s, a, w)$  of  $N(ds, da, dw)$  satisfying  $0 < s \leq \tau$  and  $w(r-s) > 0$  into a sequence  $\{(r_j, a_j, w_j) : j = 1, 2, \dots\}$  so that  $r_j < r_{j+1}$  for all  $j \geq 1$ . By the strong Markov property of  $\sigma$ -branching diffusions,  $\{\xi_i(t + \tau), w_j(t + \tau - r_j) : t \geq 0; i = 1, 2, \dots; j = 1, 2, \dots\}$  under  $\mathbf{P}\{\cdot | \mathcal{G}_\tau\}$  are independent  $\sigma$ -branching diffusions relative to  $(\mathcal{G}_{\tau+t})_{t \geq 0}$ . By the property of independent increments,  $W_\tau(ds, dy) := W(ds + \tau, dy)$  under  $\mathbf{P}\{\cdot | \mathcal{G}_\tau\}$  is a white noise on  $[0, \infty) \times \mathbb{R}$  based on the Lebesgue measure. Similarly,  $N_\tau(ds, da, du, dw) := N(ds + \tau, da, du, dw)$  under  $\mathbf{P}\{\cdot | \mathcal{G}_\tau\}$  is a Poisson random measure on  $[0, \infty) \times \mathbb{R} \times \mathbb{R} \times W_0$  with intensity  $dsm(da)du\mathbf{Q}_\kappa(dw)$ . Moreover, the families  $\{\xi_i(t + \tau), w_j(t + \tau - r_j)\}$ ,  $\{W_\tau(ds, dy)\}$  and  $\{N_\tau(ds, da, du, dw)\}$  under  $\mathbf{P}\{\cdot | \mathcal{G}_\tau\}$  are independent of each other. Observe that

$$Y_\tau = \sum_{i=1}^{\infty} \xi_i(\tau) \delta_{x(0, x_i, \tau)} + \sum_{i=1}^{\infty} w_j(\tau - r_j) \delta_{x(r_j, a_j, \tau)}$$

and

$$\begin{aligned} Y_{t+\tau} &= \sum_{i=1}^{\infty} \xi_i(t + \tau) \delta_{x(0, x_i, t+\tau)} + \sum_{i=1}^{\infty} w_j(t + \tau - r_j) \delta_{x(r_j, a_j, t+\tau)} \\ &\quad + \int_0^t \int_{\mathbb{R}} \int_0^{q(Y_{s+\tau}, a)} \int_{W_0} w(t-s) \delta_{x(s+\tau, a, t+\tau)} N_\tau(ds, da, du, dw) \end{aligned}$$

for  $t \geq 0$ . By the uniqueness of solution of (5.8),  $Y_{t+\tau}$  under  $\mathbf{P}\{\cdot | \mathcal{G}_\tau\}$  has distribution  $Q_t^q(Y_\tau, \cdot)$ , giving the strong Markov property of  $\{Y_t : t \geq 0\}$ .  $\square$

We can also extend the constructions to a general initial state  $\mu \in M(\mathbb{R})$ . Suppose we have on the complete probability space  $(\Omega, \mathcal{F}, \mathbf{P})$  we have the following: (i) a white noise  $W(ds, dy)$  on  $[0, \infty) \times \mathbb{R}$  based on the Lebesgue measure; (ii) a Poisson random measure  $N_0(dx, dw)$  on  $\mathbb{R} \times W_0$  with intensity  $\mu(dx)\mathbf{Q}_\kappa(dw)$ ; and (iii) a Poisson random measure  $N(ds, da, du, dw)$  on  $[0, \infty) \times \mathbb{R} \times [0, \infty) \times W_0$  with intensity  $dsm(da)du\mathbf{Q}_\kappa(dw)$ . We assume that  $\{W(ds, dy)\}$ ,  $\{N_0(dx, dw)\}$  and  $\{N(ds, da, du, dw)\}$  are independent of each other. For  $t \geq 0$  let  $\mathcal{G}_t$  be the  $\sigma$ -algebra generated by the family of  $\mathbf{P}$ -null sets in  $\mathcal{F}$  and the families of random variables (4.12) and (5.1).

**Theorem 5.3** *Suppose that  $q(\cdot, \cdot)$  is a Borel function on  $M(\mathbb{R}) \times \mathbb{R}$  satisfying (5.6) and (5.7). Then the stochastic equation:*

$$\begin{aligned} Y_t &= \int_{\mathbb{R}} \int_{W_0} w(t) \delta_{x(0,x,t)} N_0(dx, dw) \\ &\quad + \int_0^t \int_{\mathbb{R}} \int_0^{q(Y_s, a)} \int_{W_0} w(t-s) \delta_{x(s,a,t)} N(ds, da, du, dw), \quad t > 0, \end{aligned} \quad (5.11)$$

has a unique continuous solution  $\{Y_t : t > 0\}$ . If we set  $Y_0 = \mu$ , then  $\{Y_t : t \geq 0\}$  is a diffusion process and the martingale characterization of Theorem 5.2 holds.

*Proof.* If  $q(\cdot, \cdot, \cdot)$  is a non-negative predictable function on  $[0, \infty) \times \mathbb{R} \times \Omega$  such that  $\mathbf{E}\{m(q(t, \cdot))^2\}$  is locally bounded in  $t \geq 0$ , it can be proved as in the proof of Theorem 5.1 that the process  $\{Y_t : t \geq 0\}$  defined by  $Y_0 = \mu$  and

$$\begin{aligned} Y_t &= \int_{\mathbb{R}} \int_{W_0} w(t) \delta_{x(0,x,t)} N_0(dx, dw) \\ &\quad + \int_0^t \int_{\mathbb{R}} \int_0^{q(s,a)} \int_{W_0} w(t-s) \delta_{x(s,a,t)} N(ds, da, du, dw), \quad t > 0, \end{aligned}$$

has a continuous modification and the conclusions of Theorem 5.1 also hold for this  $\{Y_t : t \geq 0\}$ . Then one can show as in Fu and Li (2002) and Shiga (1990) that (5.11) has unique solution and the martingale characterization of Theorem 5.2 holds. Based on the uniqueness of solution of (5.8), the strong Markov property of  $\{Y_t : t \geq 0\}$  can be proved as in the proof of Theorem 5.2; see Shiga (1990).  $\square$

The solutions of (5.8) and (5.11) can be regarded as immigration processes associated with the SDSM with interactive immigration determined by  $m$  and  $q(\cdot, \cdot)$ . Note that in the proofs of Theorems 5.2 and 5.3 the Markov property of  $\{Y_t : t \geq 0\}$  was obtained from the uniqueness of solution of (5.8). This application of the stochastic equation is crucial in the construction the immigration diffusion process since the uniqueness of solution of the martingale problem given by (5.9) and (5.10) still remains open; see also Fu and Li (2002).

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