Deviation Kernels for One-Dimensional Diffusion Processes

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Abstract It is proven that for the non-explosive and ergodic diffusion on the half line with the transition probability kernel $p(t,x,y)$, the deviation kernel $d(x,y) = \int_0^\infty (p(t,x,y)-1)dt$ exists and is finite if and only if $\int_0^\infty \mathbb{E}^x H_0 \mu(dx) < \infty$, where H_0 is the hitting time of 0 and μ is the speed measure. The explicit formulas are also obtained.

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1 Introduction

Let $X = (X_t : t \geq 0)$ be a non-singular diffusion on the half-line $[0,\infty)$ with reflecting boundary in 0. The infinitesimal generator $(L, C_0^\infty([0,\infty)))$ is

$$L = a(x)\frac{d^2}{dx^2} + b(x)\frac{d}{dx}, \tag{1.1}$$

where $a > 0, a \in C([0,\infty))$ and b is locally integrable. Then, without regard to boundary conditions, L can be rewritten as

$$L = \frac{1}{m(x)}\frac{d}{dx}\frac{1}{s(x)}\frac{d}{dx} \tag{1.2}$$

where $m(x)$ is the density of the speed measure μ of X w.r.t. Lebesgue measure, and $s(x)$ is the derivative of the scale function $S(x)$. More precisely,

$$s(x) = \exp\left(-\int_0^x \frac{b(y)}{a(y)}dy\right) \quad \text{and} \quad m(x) = \frac{1}{a(x)s(x)}. \tag{1.3}$$

See [1, 4] for background and precise definitions.

Let $p(t,x,y)$ be the transition probability function of X relative to the speed measure μ,

$$P(t,x,dy) = p(t,x,y)\mu(dy) = p(t,x,y)m(y)dy. \tag{1.4}$$

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We will assume throughout the paper that X is ergodic, i.e.

$$\int_0^\infty s(x)dx \int_0^x m(y)dy = \infty \quad \text{and} \quad \int_0^\infty m(x)dx < \infty. \tag{1.5}$$

For simplicity, we assume that $\int_0^\infty m(x)dx = 1$. It is well-known that (1.5) is equivalent to that $p(t, x, y) \to 1$ as $t \to \infty$ for any $x, y \in [0, \infty)$.

In this paper, we are interested in the convergence rates in $p(t, x, y) \to 1$ as $t \to \infty$. For this purpose, we introduce first the following deviation kernel.

Definition 1.1. The deviation kernel is defined as

$$d(x, y) = \int_0^\infty (p(t, x, y) - 1)dt, \quad x, y \in [0, \infty), \tag{1.6}$$

and similarly for an integer $n \geq 0$, define n-order deviation kernel as

$$d^{(n)}(x, y) = \int_0^\infty t^n(p(t, x, y) - 1)dt, \quad x, y \in [0, \infty), \tag{1.7}$$

As for Markov chains, an analogue was appeared in [3, 5, 8, 9, 12] among others.

To investigate the existence of the deviation kernel, we make use of the hitting time

$$H_y = \inf \{t > 0 : X_t = y\}. \tag{1.8}$$

Now, we can state the main results in this paper.

Theorem 1.2. *The function $d(x, y)$ exists and is finite for all $x, y \in [0, \infty)$ if and only if*

$$\int_0^\infty m(x)\mathbb{E}^x H_0 dx < \infty. \tag{1.9}$$

More precisely, for $x, y \in [0, \infty)$

$$d(y, y) = \int_0^\infty m(x)\mathbb{E}^x H_y dx, \quad d(x, y) = d(y, y) - E^x H_y.$$

Moreover, we have the following result for any integer $n \geq 0$, which includes Theorem 1.2 as a special case.

Theorem 1.3. *The function $d^{(n)}(x, y)$ exists and is finite for all $x, y \in E$ if and only if*

$$\int_0^\infty m(x)\mathbb{E}^x[(H_0)^{n+1}]dx < \infty. \tag{1.10}$$

By aid of an expression obtained along the proof of Theorem 1.2, we can derive the following convergent rates for $p(t, x, y) - 1$.

Theorem 1.4. *Suppose that $\int_0^\infty m(x)E^x[(H_0)^n]dx < \infty$ for some $n \geq 0$. Then*

$$p(t, x, y) - 1 = o(t^{-n}) \quad \text{as} \quad t \to \infty. \tag{1.11}$$

If we assume additionally that the drift $b(x)$ is continuous in $(0, \infty)$, then we can calculate the explicit formulas for any order moment $\mathbb{E}^x H_y$. And so we can compare them with the explicit formulas for the existence of spectral gap ([2]) and the strong ergodicity ([7]).

Corollary 1.5. *Let $a(x), b(x)$ be continuous functions in $(0, \infty)$, then*

$$d(0, 0) = \int_0^\infty s(x)dx \left[\int_x^\infty m(y)dy\right]^2.$$

Corollary 1.6. *If the process is either such that* gap(L) > 0 *or strongly ergodic, then $d(x, y)$ exists and is finite for any $n \geq 0$.*

2 Preliminaries

To relate the transition function to the hitting times, we need to recall some known facts. For more details, see [1, 4, 10, 11].

Let the local times

$$L_t^y = a(x) \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t I_{[|X_s - y| \leq \epsilon]} ds \tag{2.1}$$

be the occupation densities. It is well-known that the limit exists and defines a continuous increasing process $(L_t^y, t \geq 0)$. And let the inverse local times

$$\tau_\ell^y = \inf \{t : L_t^y > \ell\}, \tag{2.2}$$

then the process $(\tau_\ell^y, \ell \geq 0)$ is a subordinator, with Laplace exponent $\psi^y(\lambda)$ defined by the formula

$$\mathbb{E}^y[\exp(-\lambda \tau_\ell^y)] = \exp[-\ell \psi^y(\lambda)]. \tag{2.3}$$

Note in particular that

$$\mathbb{E}^y[L_\infty^y > \ell] = \mathbb{E}_y[\tau_\ell^y < \infty] = \exp[-\ell \psi^y(0)]. \tag{2.4}$$

It is well known [1, 4] that X is recurrent whenever $\psi^y(0) = 0$, or $\mathbb{E}_y[\tau_\ell^y < \infty] = 1$ for all ℓ.

The Laplace exponent $\psi^y(\lambda)$ is related to the transition kernel through the Green function. For $\lambda > 0$, the Green function for the transition kernel is

$$G_\lambda(x, y) := \int_0^\infty e^{-\lambda t} p(t, x, y) dt, \tag{2.5}$$

then we have that

$$G_\lambda(y, y) = \frac{s(y)}{2\psi^y(\lambda)}. \tag{2.6}$$

Recall that the process X is said to be ergodic if $\lim_{t\to\infty} p(t, y, y) = 1$. Thus by (2.6) and

$$\lim_{t\to\infty} p(t, y, y) = \lim_{\lambda \to 0} \lambda G_\lambda(y, y) = s(y) \left(\frac{d}{d\lambda}\bigg|_{\lambda=0} \psi^y(\lambda) \right)^{-1},$$

we can deduce that

$$\frac{d}{d\lambda}\bigg|_{\lambda=0} \psi^y(\lambda) = \frac{2}{s(y)}. \tag{2.7}$$

3 Proofs

It is well known [1, No. II.12] that the speed measure $m(x)dx$ serves as an invariant measure for the diffusion process X, and there are global formulae

$$\psi^y(\lambda) = \lambda s(y) \int_0^\infty m(x)\mathbb{E}^x[e^{-\lambda H_y}]dx, \quad \lambda > 0. \tag{3.1}$$

Then by (2.6), we have

$$G_\lambda(y,y) = \left\{ 2\lambda \int_0^\infty m(x)\mathbb{E}^x[e^{-\lambda H_y}]dx \right\}^{-1}. \tag{3.2}$$

Lemma 3.1. *For $\lambda > 0$, let*

$$\xi^y(\lambda) = \frac{\psi^y(\lambda)}{\lambda} \quad \text{and} \quad \eta^y(\lambda) = \frac{\xi^y(0) - \xi^y(\lambda)}{\lambda}, \tag{3.3}$$

where $\xi^y(0) := \lim_{\lambda\to 0} \xi^y(\lambda) = s(y)/2$. Set

$$D_\lambda(x,y) := \int_0^\infty e^{-\lambda t}\left(p(t,x,y) - 1\right) dt,$$

then

$$D_\lambda(y,y) = \frac{\eta^y(\lambda)}{\xi^y(\lambda)} \tag{3.4}$$

and

$$D_\lambda(y,y) - D_\lambda(x,y) = \frac{s(y)(1 - \mathbb{E}^x[e^{-\lambda H_y}])}{2\psi^y(\lambda)}. \tag{3.5}$$

Proof. a) $\xi^y(0) = s(y)/2$ follows from (2.7) or (3.3) by letting $\lambda \to 0$.

b) By (2.6), we have

$$D_\lambda(y,y) = G_\lambda(y,y) - 1/\lambda = \frac{\lambda s(y) - 2\psi^y(\lambda)}{2\lambda\psi^y(\lambda)}$$
$$= \frac{s(y)/2 - \xi^y(\lambda)}{\psi^y(\lambda)} = \frac{\lambda\eta^y(\lambda)}{\psi^y(\lambda)} = \frac{\eta^y(\lambda)}{\xi^y(\lambda)}.$$

c) Note that

$$D_\lambda(y,y) - D_\lambda(x,y) = G_\lambda(y,y) - G_\lambda(x,y),$$

which, combining [11, Eq. (9) and (39)], implies (3.5).

□

Proof of Theorem 1.2 First note that for any $y \in [0,\infty)$, $H_y \le H_0$ and $\mathbb{E}^x H_y < \infty$ provided that (1.9) holds. To prove the assertions, we need only to prove that

$$\eta^y(0) := \lim_{\lambda\to 0} \eta^y(\lambda) = \frac{s(y)}{2} \int_0^\infty m(x)\mathbb{E}^x H_y dx. \tag{3.6}$$

In fact, by (3.1) and (3.3), we have

$$\eta^y(0) = \lim_{\lambda\to 0} \frac{s(y)}{2\lambda} \int_0^\infty m(x)[1 - \mathbb{E}^x e^{-\lambda H_y}]dx$$
$$= \frac{s(y)}{2} \int_0^\infty m(x)\mathbb{E}^x H_y dx.$$

while (3.6) follows from (3.3) by letting $\lambda \to 0$ and the L'Höspital rules.

As for $x, y \in [0, \infty)$, let $\lambda \to 0$ to get from (2.7) that

$$d(x, y) = d(y, y) - E^x H_y.$$

□

Now we are ready to prove Theorem 1.3. The idea is to differentiate n-times the both sides of (3.5) and the let $\lambda \to 0$ to get $d^{(n)}(x, y)$ in the left hand side. As for the right hand side, we need the following calculations.

Lemma 3.2. *For $n \geq 1$, we have that*

(1)

$$\frac{d^n}{d\lambda^n}\eta^y(0) = \frac{(-1)^{n+1}}{n+1} s(y) \int_0^\infty m(x)\mathbb{E}^x(H_y^{n+1})dx. \tag{3.7}$$

(2)

$$\frac{d^n}{d\lambda^n}\xi^y(0) = (-1)^n s(y) \int_0^\infty m(x)\mathbb{E}^x(H_y^n)dx. \tag{3.8}$$

Proof. (1) By (3.1) and (3.3), we have

$$\lambda\eta^y(\lambda) = \frac{s(y)}{2} \int_0^\infty m(x)[1 - \mathbb{E}^x(e^{-\lambda H_y})]dx.$$

Since $\frac{d^n}{d\lambda^n}[\lambda\eta^y(\lambda)] = \lambda\frac{d^n}{d\lambda^n}\eta^y(\lambda) + n\frac{d^{n-1}}{d\lambda^{n-1}}\eta^y(\lambda)$, then for $n \geq 1$, we have

$$\frac{d^n}{d\lambda^n}\eta^y(\lambda) = \frac{1}{\lambda}\left[(-1)^{n+1}s(y)\int_0^\infty m(x)\mathbb{E}^x(H_y^n e^{-\lambda H_y})dx - n\frac{d^{n-1}}{d\lambda^{n-1}}\eta^y(\lambda)\right].$$

By induction, we obtain that

$$n\frac{d^{n-1}}{d\lambda^{n-1}}\eta^y(0) = (-1)^{n+1}s(y)\int_0^\infty m(x)\mathbb{E}^x(H_y^n)dx.$$

Thus, by the L'Höspital rule, we have

$$\frac{d^n}{d\lambda^n}\eta^y(0) = (-1)^n s(y)\int_0^\infty m(x)\mathbb{E}^x(H_y^{n+1})dx - n\frac{d^n}{d\lambda^n}\eta^y(0),$$

which implies (3.7) by a little rearrangement.

(2) Note by (3.1) and (3.3) that

$$\xi^y(\lambda) = s(y)\int_0^\infty m(x)\mathbb{E}^x[e^{-\lambda H_y}]dx,$$

from which we can deduce (3.8) by differentiation. □

Proof of Thoerem 1.3 Note that for $n \geq 1$,

$$\frac{d^n}{d\lambda^n} D_\lambda(x,y) = (-1)^n \int_0^\infty t^n e^{-\lambda t}(p(t,x,y)-1)dt,$$

thus $d^{(n)}(x,y) = \int_0^\infty t^n(p(t,x,y)-1)dt$ exists if and only if $\lim_{\lambda\to 0} \frac{d^n}{d\lambda^n} D_\lambda(x,y)$ exists. On the other hand, it follows from (3.4) that

$$\frac{d^n}{d\lambda^n} D_\lambda(y,y) = \sum_{\alpha+\beta=n} \frac{d^\alpha}{d\lambda^\alpha}\eta^y(\lambda)\frac{d^\beta}{d\lambda^\beta}\left[\xi^y(\lambda)\right]^{-1}. \tag{3.9}$$

By a direct computation, the leading term (which contains the highest derivative of $G_j(\lambda)$) in $\lim_{\lambda\to 0} \frac{d^\beta}{d\lambda^\beta}\left[\xi^y(\lambda)\right]^{-1}$ is

$$-(\xi^y(0))^{-2}\frac{d^\beta}{d\lambda^\beta}\xi^y(0) = (-1)^{\beta+1}\frac{4}{s(y)}\int_0^\infty m(x)\mathbb{E}^x(H_y^\beta)dx. \tag{3.10}$$

Combining (3.9),(3.7) and (3.10), we obtain that $d^{(n)}(y,y)$ exists and is finite if and only if $\int_0^\infty m(x)\mathbb{E}^x(H_y^{n+1})dx < \infty$.

For $x, y \in [0,\infty)$, it follows from (3.5) that

$$\frac{d^n}{d\lambda^n} D_\lambda(x,y) = \frac{d^n}{d\lambda^n} D_\lambda(y,y) - \frac{s(y)}{2}\frac{d^n}{d\lambda^n}\frac{1-\mathbb{E}^x(e^{-\lambda H_y})}{\xi^y(\lambda)}.$$

A same argument as above shows that

$$\lim_{\lambda\to 0}\frac{d^n}{d\lambda^n}\frac{1-\mathbb{E}^x(e^{-\lambda H_y})}{\xi^y(\lambda)} \quad \text{exists provided} \quad \int_0^\infty m(x)\mathbb{E}^x(H_y^n)dx < \infty. \tag{3.11}$$

This completes the proof of the theorem.

□

To prove Theorem 1.4, we need the following results. For simplicity, let $\partial^0 f = f$ and $\partial^{k+1} f = \frac{d}{d\lambda}\partial^k f$ for a function $f(\lambda)$ and an integer $k \geq 0$.

Lemma 3.3. *Assume that (1.10) holds for some $n \geq 0$. Then*

(1)

$$\lim_{\lambda\to 0}\frac{\partial^n D_\lambda(x,y)}{G_\lambda(x,y)} = 0, \tag{3.12}$$

(2) *The limit*

$$\lim_{\lambda\to 0}\partial\left(\frac{\partial^n D_\lambda(x,y)}{G_\lambda(x,y)}\right) \tag{3.13}$$

exists and is finite.

Proof. (1) By Theorem 1.3, the limit $\lim_{\lambda\to 0}\partial^n D_\lambda(x,y) = (-1)^n d^{(n)}(x,y)$ exists and is finite. Noting that $\lim_{\lambda\to 0} G_\lambda(x,y) = \int_0^\infty p(t,x,y)dt = \infty$, we obtain (3.12) immediately.

(2) By (1) and noting that $\lim_{\lambda\to 0}\lambda G_\lambda(x,y) = \lim_{t\to\infty} p(t,x,y) = 1$, we have

$$\begin{aligned}\lim_{\lambda\to 0}\partial\left(\frac{\partial^n D_\lambda(x,y)}{G_\lambda(x,y)}\right) &= \lim_{\lambda\to 0}\frac{\partial^n D_\lambda(x,y)/G_\lambda(x,y)}{\lambda} \\ &= \lim_{\lambda\to 0}\frac{\partial^n D_\lambda(x,y)}{\lambda G_\lambda(x,y)} = (-1)^{\ell-2}d^{(n)}(x,y)\end{aligned}$$

which exists by the assumption.

□

Proof of Theorem 1.4 Note first that

$$\frac{\partial^n D_\lambda}{G_\lambda} = \partial\left(\frac{\partial^n D_\lambda}{G_\lambda}\right) + \frac{\partial^n D_\lambda}{G_\lambda}(\log G_\lambda)' \tag{3.14}$$

and by (2.6)

$$(\log G_\lambda)' = [\log(s(y)/2) - \log\psi^y(\lambda)]' = -\frac{\psi^y(\lambda)'}{\psi^y(\lambda)}. \tag{3.15}$$

Since as $\lambda \to 0$, $\partial^n(D_\lambda/G_\lambda) \to 0$ by Lemma **??** and $\psi^y(0) = 0$ and $[\psi^y]'(0) > 0$ by ergodicity, then by L'Höspital theorem,

$$\lim_{\lambda\to 0}\frac{\partial^n D_\lambda/G_\lambda}{\psi^y(\lambda)}\psi^y(\lambda)' = \lim_{\lambda\to 0}\partial\left(\frac{\partial^n D_\lambda}{G_\lambda}\right). \tag{3.16}$$

Thus by (3.14),(3.15) and (3.16), we obtain that

$$\begin{aligned}
&\lim_{t\to\infty} t^n(p(t,y,y)-1) = \lim_{\lambda\to 0}(-1)^{n-1}\lambda\partial^{n-1}D_\lambda(y,y)\\
&= \lim_{\lambda\to 0}(-1)^{n-1}\lambda G_\lambda(y,y)\left[\partial\left(\frac{\partial^n D_\lambda}{G_\lambda}\right) - \frac{\partial^n D_\lambda/G_\lambda}{\psi^y(\lambda)}\psi^y(\lambda)'\right]\\
&= 0
\end{aligned} \tag{3.17}$$

since $\lim_{\lambda\to 0}\lambda G_\lambda(y,y) = 1$.

When $x \neq y$, it follows from (3.5) that

$$\begin{aligned}
&\lim_{t\to\infty} t^{n-1}(p(t,x,y)-1) = (-1)^{n-1}\lim_{\lambda\to 0}\lambda\partial^{n-1}D_\lambda(x,y)\\
&= (-1)^{n-1}\lim_{\lambda\to 0}\lambda\partial^{n-1}\left[D_\lambda(y,y) - \frac{s(y)(1-\mathbb{E}^x[e^{-\lambda H_y}])}{2\psi^y(\lambda)}\right]\\
&= 0 - (-1)^\ell\lim_{\lambda\to 0}\lambda\partial^{n-1}\left[\frac{s(y)(1-\mathbb{E}^x[e^{-\lambda H_y}])}{2\psi^y(\lambda)}\right].
\end{aligned}$$

As in the proof of Theorem 1.2 (cf. (3.11)),

$$\lim_{\lambda\to 0}\partial^{n-1}\left[\frac{s(y)(1-\mathbb{E}^x[e^{-\lambda H_y}])}{2\psi^y(\lambda)}\right]$$

exists and is finite provided $\int_0^\infty m(x)\mathbb{E}^x H_y^{n-1}dx < \infty$. Therefore for any $x, y \in [0,\infty)$, we have

$$\lim_{t\to\infty}(p(t,x,y)-1) = o(t^{-(n-1)}).$$

This completes the proof. □

4 Moments of hitting times

In this section, we will give the explicit formulae for high-order moments for the hitting times.

Theorem 4.1. *Assume that $a(x), b(x)$ are continuous in $x \in [0,\infty)$. For $n \ge 1$, then for $x, y \in [0,\infty)$,*

$$v_n(x,y) := \mathbb{E}^x[H_y^n] \quad \text{and} \quad v_0(x,y) \equiv 1$$

satisfy the following differential equations:

$$Lv_n(\cdot,y)(x) = -nv_{n-1}(x,y), \quad v_n(y,y) = 0. \tag{4.1}$$

Therefore, we have

$$v_1(x,y) = \begin{cases} \int_x^y s(\xi)d\xi \int_0^\xi m(\eta)d\eta, & \text{if} \quad x < y \\ \int_y^x s(\xi)d\xi \int_\xi^\infty m(\eta)d\eta, & \text{if} \quad x \ge y \end{cases} \tag{4.2}$$

and for $n \ge 2$

$$v_n(x,y) = \begin{cases} n\int_x^y s(\xi)d\xi \int_0^\xi m(\eta)v_{n-1}(\eta,y)d\eta, & \text{if} \quad x < y \\ n\int_y^x s(\xi)d\xi \int_\xi^\infty m(\eta)v_{n-1}(\eta,y)d\eta, & \text{if} \quad x \ge y. \end{cases} \tag{4.3}$$

Proof. The proof of (4.2) can be modified from [6, p.194]. we will prove (4.3).

Note that for $h \ge 0$, by Markov property,

$$\mathbb{E}[(H_y - h)^m | X(h) = z] = \mathbb{E}[H^m | X(0) = z] = v_m(z,y),$$

thus

$$v_n(x,y) = \mathbb{E}^x[(H_y - h + h)^n | X(h) = x] = \sum_{m=0}^n \binom{n}{m} h^{n-m}\mathbb{E}^x[v_m(X(h))]. \tag{4.4}$$

By a same argument as in [6, pp.193-194], we have

$$\mathbb{E}^x[v_m(X(h))] = \mathbb{E}^x[v_m(x+\Delta X, y)] = v_m(x,y) + (Lv_m(x,y))h + o(h),$$

so that (4.4) becomes

$$v_n(x,y) = v_n(x,y) + Lv_n(x,y) + nh[v_{n-1}(x,y) + (Lv_m(x,y))h] + O(h^2).$$

Then, subtracting $v_n(x,y)$ from the both sides, dividing by h and then letting $h \to 0$, we obtain (4.1).

□

Proof of Corollary 1.5 From (4.2), we have

$$\mathbb{E}^x H_0 = \int_0^x s(y)dy \int_y^\infty m(u)du,$$

which by (1.2) completes the proof. □

Proof of Corollary 1.6 (1) By Theorem in [7], we have the process is strongly ergodic if and only if

$$\int_0^\infty s(x)dx \int_x^\infty m(y)dy < \infty,$$

which implies (1.9) since $\int_0^\infty m(y)dy = 1$.

(2) Set

$$M = \sup_x \int_0^x s(y)dy \int_x^\infty m(y)dy.$$

It is prove in [2] that gap(L) > 0 whenever $M < \infty$, thus we need only to prove that (1.9) holds under $M < \infty$. Indeed, by Fubini theorem, we have

$$\begin{aligned}
&\int_0^\infty s(x)dx \left[\int_x^\infty m(y)dy\right]^2 = \int_0^\infty m(x)dx \int_0^x s(y)dy \int_y^\infty m(u)du \\
&= \int_0^\infty m(x)dx \int_0^x s(y)dy \int_x^\infty m(u)du + \int_0^\infty m(x)dx \int_0^x s(y)dy \int_y^x m(u)du \\
&= \int_0^\infty m(x)dx \int_0^x s(y)dy \int_x^\infty m(u)du + \int_0^\infty m(u)du \int_0^u s(y)dy \int_u^\infty m(x)dx \\
&\leq 2M \int_0^\infty m(x)dx = 2M.
\end{aligned}$$

□

5 Examples

Example 5.1. *Let $a(x) = (1+x)^\gamma, b(x) = 0$. If $\gamma > 2 - 1/(n+2)$ for some $n \geq 0$, then $d^{(n)}(x,y)$ exists. Specially, $d(x,y)$ exists when $\gamma > 3/2$.*

Proof. Note that $m(x) = (1+x)^{-\gamma}$ and $s(x) = 1$, then for $\gamma > 1$

$$\mathbb{E}^x H_0 = \int_0^x s(y)dy \int_y^\infty m(z)dz = \frac{(1+x)^{2-\gamma} - 1}{(\gamma-1)(2-\gamma)}.$$

thus $\int m(x)\mathbb{E}^x H_0 dx < \infty$ provided $\gamma > 3/2$. If $\gamma > 2 - 1/(n+2)$ for some $n \geq 0$, it follows from Theorem 4.1 that

$$\mathbb{E}^x H_0^{n+1} \sim (1+x)^{(n+1)(2-\gamma)}$$

and then $\int_0^\infty m(x)\mathbb{E}^x H_0^{n+1} dx < \infty$.

□

Example 5.2. *Let $a(x) = 1, b(x) = -\alpha/(1+x)$. If $\alpha > 2n+3$ for some $n \geq 0$, then $d^{(n)}(x,y)$ exists. Specially, $d(x,y)$ exists when $\alpha > 3$.*

Proof. We have that $m(x) = (1+x)^{-\alpha}$ and $s(x) = (1+x)^\alpha$. If $\alpha > 2n+3$ for some $n \geq 0$, then it follows from Theorem 4.1 that

$$\mathbb{E}^x H_0^{n+1} \sim (1+x)^{2(n+1)}$$

and then $\int_0^\infty m(x)\mathbb{E}^x H_0^{n+1} dx < \infty$.

□

Example 5.3. *Let $a(x) = 1, b(x) = -1/\sqrt{x}$. Then $d^{(n)}(x,y)$ exists for all $n \geq 0$ but the spectral gap of the generator is null.*

Proof. Note that $m(x) = e^{-2\sqrt{x}}$ and $s(x) = e^{2\sqrt{x}}$. A direct computation shows that

$$\mathbb{E}^x H_0 = \int_0^x e^{2\sqrt{y}} dy \int_y^\infty e^{-2\sqrt{z}} dz \sim \int_0^x e^{2\sqrt{y}} \sqrt{y} e^{-2\sqrt{y}} dy \sim x^{3/2}$$

and $\mathbb{E}^x H_0^n \sim x^{3n/2}$. Thus $\int_0^\infty m(x) \mathbb{E}^x H_0^n dx = \int_0^\infty x^{3n/2} e^{-2\sqrt{x}} dx < \infty$ for any $n \geq 0$.

On the other hand, as in the proof of Corollary 1.6, we know that gap(L) > 0 if and only if

$$\sup_{x \in (0,\infty)} \int_0^x s(y) dy \int_x^\infty m(y) dy < \infty.$$

But in our case, we have that as $x \to \infty$,

$$\int_0^x s(y) dy \int_x^\infty m(y) dy \sim \sqrt{x} e^{-2\sqrt{x}} \int_0^x e^{2\sqrt{y}} dy \to \infty$$

by applying L'Höspital rule. Therefore gap(L) $= 0$.

□

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