A cocycle proof that reversible Fleming-Viot processes have uniform mutation

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Abstract

Why must the mutation operator associated with a reversible Fleming-Viot process be uniform? Our explanation is based on Handa's recent result that reversible distributions must be quasi-invariant under a certain flow, forcing the mutation operator to satisfy a cocycle identity.

Résumé

Pourquoi l'opérateur de mutation lié à un processus réversible de Fleming-Viot doit-il être uniforme? Notre explication est baseé sur le résultat récent de Handa que les distributions réversibles doivent être quasi-invariables sous un certain écoulement, forçant l'opérateur de mutation à satisfaire une identité de cocycle.

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1 Introduction

The Fleming-Viot process models the evolution of the genetic profile of a population. Each individual in the population has a genetic type belonging to the type space E, and X_t denotes the empirical distribution of types at time t. The process X_t lives on the space $M_1(E)$ of probability measures on E. The changes to the genetic makeup of this population come from two opposing sources; *genetic drift* which encourages conformity by favoring the offspring of individuals with dominant type and *mutation* which continually adds fresh variation. Other mechanisms such as *selection* and *recombination* can be added to the model for more realism. As with all Markov processes, it is of great interest to find an initial distribution Π which makes the process reversible in time, so that the different genetic forces are in perfect balance.

It has long been known ([2, Chapter 10, Exercise 14(a)], [1, Theorem 2.3], [8, page 276]) that if the mutation is uniform, then the Fleming-Viot process has a reversible distribution. In 1999, Li, Shiga, and Yao proved [5, Theorem 1.1] the "folklore" result (suggested in [3], and proved for finite E in [6]) that the converse is true. They considered a reversible Fleming-Viot process with mutation and selection, but not recombination. First reducing the problem to the inselective case, they used Dirichlet forms to argue that the mutation operator A is uniform[1].

In a very interesting recent paper [4], Handa shows that a probability measure on $\mathcal{M}_1(E)$ is reversible for the Fleming-Viot process (with mutation, selection, and recombination) if and only if it is quasi-invariant with respect to a certain flow on $\mathcal{M}_1(E)$. This leads to a cocycle identity (see (3) below) that Handa uses to explore reversible Fleming-Viot processes. For example, in [4, Lemma 3.5] he uses cocycles to reprove Li, Shiga, and Yao's result that a reversible distribution for the selective case is the reversible distribution for the inselective case multiplied by a simple density function. In [4, Proposition 3.1] Handa also uses cocycles to show that if the mutation operator generates an irreducible semigroup, then Π-almost every μ has full support on E. His main result [4, Theorem 2.2] extends Li, Shiga, and Yao's result to the case with recombination. Handa shows that[1] the Fleming-Viot process has a reversible distribution if and only if the recombination kernel is of a certain form, and the mutation operator A plus the diagonal part of the recombination kernel is uniform. His analysis mainly uses cocycles, but for the mutation he simply quotes [5] – though in a later remark [4, Section 4, Remark (i)] he outlines a direct proof using cocycles.

In this note we analyze an operator via the cocycle identity and show how the property of reversibility imposes uniformity.

[1]Provided the mutation operator generates an irreducible Markov semigroup on E

2 The analytical result

Let E be a compact metric space with Borel σ-algebra $\mathcal{B}(E)$, $\mathcal{M}_1(E)$ the space of probability measures on $\mathcal{B}(E)$, and $C(E)$ the space of continuous functions on E. For $f \in C(E)$ and $\mu \in \mathcal{M}_1(E)$ we set $\langle f, \mu \rangle := \int f(x)\, \mu(dx)$.

For every $f \in C(E)$ and $\mu \in \mathcal{M}_1(E)$, we define a "shifted" probability measure $S_f\mu$ by

$$S_f\mu(dx) := \frac{e^{f(x)}\mu(dx)}{\langle e^f, \mu \rangle}. \tag{1}$$

Note that $(S_f)_{f \in C(E)}$ forms a transformation group on $\mathcal{M}_1(E)$ since $S_{f+g} = S_f(S_g)$. This group is used in [9, 7] to study Fleming-Viot operators and it is core of Handa's work in [4]. One particularly nice property of this flow can be obtained by direct calculation:

$$\frac{d}{dt}\langle f, S_{tg}\mu \rangle = \mathrm{cov}_{S_{tg}\mu}(f, g).$$

Let $(B, D(B))$ be a densely defined linear operator on $C(E)$, and define for $f \in D(B)$

$$\Lambda(f, \mu) := \int_0^1 \langle Bf, S_{uf}\mu \rangle\, du. \tag{2}$$

We assume that the *cocycle identity* holds for all $f, g \in D(B)$ and $\mu \in \mathcal{M}_1(E)$:

$$\Lambda(f + g, \mu) = \Lambda(f, S_g\mu) + \Lambda(g, \mu). \tag{3}$$

Lemma 1. If $\mathrm{Var}_\mu(f) = 0$, then $\mathrm{Var}_\mu(Bf) = 0$.

Proof. Choose $f \in D(B)$ and $\mu \in \mathcal{M}_1(E)$ so that f is a constant μ-almost everywhere. Then $S_{uf}\mu = \mu$ for all $0 \le u \le 1$ so $\Lambda(f, S_h\mu) = \langle Bf, S_h\mu \rangle$ for any $h \in C(E)$. Applying this at $h \equiv 0$ gives $\Lambda(f, \mu) = \langle Bf, \mu \rangle$. The cocycle identity implies

$$\Lambda(f, S_g\mu) + \Lambda(g, \mu) = \Lambda(g, S_f\mu) + \Lambda(f, \mu),$$

or

$$\Lambda(f, S_g\mu) = \Lambda(f, \mu).$$

Therefore $\langle Bf, S_g\mu \rangle = \langle Bf, \mu \rangle$ is independent of g, so setting $g = t(Bf)$ and differentiating gives $0 = \frac{d}{dt}\big|_{t=0} \langle Bf, S_{t(Bf)}\mu \rangle = \mathrm{cov}_\mu(Bf, Bf)$. □

Proposition 1. If $(B, D(B))$ is a closed operator satisfying the cocycle identity (3), then B must be of the form

$$Bf(x) = \alpha f(x) + \langle f, \nu \rangle, \quad f \in C(E), \tag{4}$$

for some $\alpha \in \mathbb{R}$ and some finite signed measure ν.

Proof. Let $x \neq y \in E$ and $f, g \in D(B)$, and define the function

$$F = [g(x) - g(y)]f + [f(y) - f(x)]g.$$

Since $F(x) = F(y)$, we can apply Lemma 1 at $\mu = (\delta_x + \delta_y)/2$ and conclude that $BF(x) = BF(y)$. This can be rearranged to read

$$[g(x) - g(y)][Bf(x) - Bf(y)] = [f(x) - f(y)][Bg(x) - Bg(y)].$$

Since $D(B)$ is dense in $C(E)$ we may choose $g \in D(B)$ with $g(x) \neq g(y)$ and define $\alpha_{xy} = [Bg(x) - Bg(y)]/[g(x) - g(y)]$, so that

$$Bf(x) - Bf(y) = \alpha_{xy}[f(x) - f(y)]$$

for all $f \in D(B)$. Now take three distinct points $x, y, z \in E$ and $f \in D(B)$ with $f(z) \neq f(x)$. Then

$$\begin{aligned} \alpha_{zx} = \frac{Bf(z) - Bf(x)}{f(z) - f(x)} &= \frac{Bf(z) - Bf(y)}{f(z) - f(x)} + \frac{Bf(y) - Bf(x)}{f(z) - f(x)} \\ &= \alpha_{zy} \frac{f(z) - f(y)}{f(z) - f(x)} + \alpha_{yx} \left(1 - \frac{f(z) - f(y)}{f(z) - f(x)} \right). \end{aligned}$$

Once again, since $D(B)$ is dense we may choose $g \in D(B)$ with $g(z) \neq g(x)$, and so that $f(z) - f(y)/f(z) - f(x) \neq g(z) - g(y)/g(z) - g(x)$. Applying the equation for α_{zx} to f and g then taking the difference, we obtain

$$0 = \left(\frac{f(z) - f(y)}{f(z) - f(x)} - \frac{g(z) - g(y)}{g(z) - g(x)} \right) [\alpha_{zy} - \alpha_{yx}],$$

and conclude that $\alpha_{zy} = \alpha_{yx}$. Thus, all these α's are the same, and we can denote the common value as α. We have

$$Bf(x) - \alpha f(x) = Bf(y) - \alpha f(y)$$

for all $x, y \in E$ and $f \in D(B)$. This means that the operator $B - \alpha I$ takes $D(B)$ into the space of constant functions.

Since $(B, D(B))$ is closed, it follows that $(B - \alpha I, D(B - \alpha I))$ is also closed, where $D(B - \alpha I) := D(B)$. Then its null space $N(B - \alpha I)$ is closed and the quotient space $D(B)/N(B - \alpha I)$ is a vector space with norm

$$\|[f]\| = \inf\{\|f - g\| : g \in N(B - \alpha I)\}.$$

The canonical projection $\pi : D(B) \to D(B)/N(B - \alpha I)$ defined by $\pi(f) = [f] = f + N(B - \alpha I)$ is continuous.

Fix any $x \in E$ and define $\phi : D(B)/N(B - \alpha I) \to \mathbb{R}$ by $\phi([f]) = (B - \alpha I)f(x)$. This is a one-to-one linear map so that $D(B)/N(B - \alpha I)$ must be either zero or one dimensional, and hence ϕ must be continuous. Thus $f \to (B - \alpha I)f(x) = \phi(\pi(f))$ is a continuous linear functional on $D(B)$ and can be extended continuously to $C(E)$ where it is represented by a finite signed measure ν. For $f \in D(B)$ we have

$$(B - \alpha I)f(x) = \langle f, \nu \rangle,$$

and since the right hand side is a continuous operator, and since B is closed we conclude that $D(B) = C(E)$ and

$$Bf(x) = \alpha f(x) + \langle f, \nu \rangle, \quad f \in C(E). \qquad \square$$

3 Application to the Fleming-Viot process

Handa [4, Step 2, Proof of Theorem 2.2] proved that if Π is reversible for the Fleming-Viot process, then Π is quasi-invariant with respect to the shifts S_f, and that the cocycle identity (3) holds for the operator

$$Bf(x) := Af(x) + \rho \left(\int_E f(z)\eta(x, x; dz) - f(x) \right),$$

where A is the mutation operator and $\eta(x, y; dz)$ the recombination kernel.

Unfortunately, this gives the cocycle identity only for Π-almost every μ. However, if A generates an irreducible Markov semigroup on E, Handa's result [4, Proposition 3.1] says that Π-almost every μ has full support on E. From here it is an easy exercise to show that Π has full support on $\mathcal{M}_1(E)$, so by continuity the cocycle identity extends to all of $\mathcal{M}_1(E)$.

In this way, Proposition 1 can be fitted in to a purely cocycle based proof of Handa's main result [4, Theorem 2.2].

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