Stochastic Analysis on p-adics

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March, 2002

Outline of this talk

- Development and Application
- p-adics
- Introduction of my recent work
  - The study on Lévy processes on p-adics
  - The study on measure-symmetric stochastic processes on p-adics
  - Stochastic process induced by a map
  - The study on superprocesses on p-adics
- Open Problems

Development and Applications

- Development
  - Stochastic processes on $\mathbb{Q}_p$ (A-K 1991, 1994, 1998, A-Z 2000–2002, Evans 1988–2000, Talamanca et al 1993,4, Vladimirov (1988), Kochubei 1991–2000, Yasuda 1996, some my students and so on)
  - Dirichlet Form: *A-K 1991, 1994; A-K-Z 1999; Kaneko 1999*
  - Stochastic Integral: *Kochubei 1997, Kaneko 2000, Kaneko-Z 2001*
  - Trace Formula for p-adics: *A-K-Yasuda 1999, Yasuda 2001*
  - Other processes: Bipartite Markov Chain: *Aldous-Evans 1998*; Gaussian processes, p-adic white noise and stochastic integral, K-Brownian motion: *Evans 1988a,b, 1993, 1995, 1998*; Extensive Brownian motions *Bikulov-Volovich 1997*.

- Applications
  - Application to certain system of quasi-differential equations. (*A-Z 2000, 1999c*).
  - Application of "Diffusion" on local fields or trees to Dynamic System, Data structure, and Quantum Physics (*Gregory and Krishnamurthy 1984, Rizzi 1995, Freund and Witten 1987, Ruelle and Thiran 1989, Khrennikov 1990, 1994, A-Khrennikov 1998 and so on*)
  - Application of Hierarchical Structure to Econometrics: (*Pagliacci 1997, A-L-Z*)

p–adics

What is p–adics fields, $\mathbb{Q}_p$?

Let $\mathbb{Q}$ be the field of the rational numbers. For a given prime $p > 1$, any $a \in \mathbb{Q}$ can be expressed uniquely by

$$a = p^M \frac{q}{r},$$

for $M \in \mathbb{Z}$ and $q,\ r \in \mathbb{Z}$, $(q,r,p) = 1,\ (q,r) = (p,q) = (p,r) = 1$. We set

$$\|a\|_p = \begin{cases} p^{-M}, & a \neq 0; \\ 0, & a = 0. \end{cases}$$

FACT: $\|\cdot\|_p$ is a **norm**.

It satisfies:

$$\|x+y\|_p \leq \max\{\|x\|_p, \|y\|_p\}.$$

This is a **non–Archimedean** norm.

Ostrowski' Theorem says that non-trivial metric on $\mathbb{Q}$ is equivalent to a $\|\cdot\|_p$ for some prime number $p > 1$ or absolute value ($p = \infty$).

Completion: $\mathbb{Q}_p$ is the completion of Q under norm $\|\cdot\|_p$.

Hensel expansion: Each $a \in \mathbb{Q}_p$ has a unique representation as follows:

$$a = \sum_{i=0}^{\infty} a_{m+i} p^{m+i},$$

where m is an integer and $1 \leq a_m \leq p-1,\ 0 \leq a_i \leq p-1,\ i = m+1, m+2, \ldots,$.

$$-1 = \sum_{k=0}^{\infty} (p-1)p^k.$$

See Koblitz (1984) for Arithmetic in $\mathbb{Q}_p$.

$$K(a, p^M) := \{x \in \mathbb{Q}_p;\ \|x - a\| \leq p^M\}$$

for $a \in \mathbb{Q}_p$ and $M \in \mathbb{Z}$.

$(\mathbb{Q}_p, \mathcal{Q}_p)$ is a measurable space.

$\mathbb{Q}_p$ is **different** from Archimedean metric space (c.f., $\mathbb{R}^d$).

Measures on $\mathbb{Q}_p$

Define,

$$\mathcal{L}(K(a, p^M)) = p^M.$$

$\mathcal{L}$ is a **Haar Measure**, i.e., a spatially invariant measure finite in any compact set, satisfying

$$\mathcal{L}(K(a, 1)) = 1.$$

Denote by dx the normalized Haar measure.

Fourier Analysis on $\mathbb{Q}_p$

Gel'fand et al. (1969).

Our recent results

We mainly studied the following four aspects.

The Study of Lévy Processes on p-adics

- Lévy Processes: Its Lévy–Khinchine representation is given by

  $$\hat{\Phi}(t, \xi) = \exp\{t \int_{\mathbb{Q}_p} [\chi(x\xi) - 1] \nu(dx)\},$$

  $(t, \xi) \in \mathbb{R}_+ \times \mathbb{Q}_p$ for some measure ν on $\mathbb{Q}_p$, satisfying

  $$\nu(\mathbb{Q}_p \backslash \{\|x\|_p \leq p^n\}) < \infty, \ n \in \mathbb{Z}.$$

  $\hat{\Phi}(t, \xi)$, the Fourier transform of transition function of X (see Evans 1989, A-Z 2001)

  Call ν **Lévy measure**.

Any Lévy process on $\mathbb{Q}_p$ is pure jump process: no "drift" or "Brownian" components — nor is there any need to introduce some analogue of the "compensation" term presented in the Itô representation of a Euclidean Lévy process.

- Several construction methods:
  1. Albeverio-Karwowski (1991, 1994)
  2. Vladimirov (1988)

$$D^\alpha f(x) = \frac{1-p^\alpha}{1-p^{-\alpha-1}} \int_{\mathbb{Q}_p} \frac{f(x-y)-f(x)}{\|y\|_p^{\alpha+1}} dy, \ \alpha \neq -1$$

     This operator is a generator when $\alpha > 0$.

  3. Figà-Talamanca (1993, 1994, 1998)

Theorem (A-Z 2000) Any of the above three classes of random walks is a Lévy process.

- Introducing some new classes of Lévy processes and decomposing Lévy processes on p-adics

Theorem (A-Z 2001) For a Lévy measure ν on $\mathbb{Q}_p$, suppose the Radon–Nikodym derivative of $\nu - \nu_s$ is **locally constant**. Then X_t^ν can be uniquely decomposed in distribution by

$$X_t^\nu = A_t + \sum_{\lambda \in \Lambda} X_t^\lambda + Y_t,$$

Moreover, all these processes are independent.

Theorem (A-Z 2001) Denote by X_t^μ the Lévy process associated with μ. Let I be a countable set. If $\{\mu_i,\ i \in I\}$ is a set of measures on $\mathbb{Q}_p$

and $\mu := \Sigma_{i \in I} \mu_i$ is a Lévy measure, then

$$X_t^{\mu} \stackrel{d}{=} \sum_{i \in I} X_t^{\mu_i}$$

- **Equivalence Theorem** (A-Z 2000): For any Lévy measure ν,

$$\nu := \nu_0 + \sum_{i \in I} \nu_i,$$

where $\nu_0 := \nu|_{K(0,p^n)}$, and $\nu_i := \nu|_{K(a_i,p^n)}$, $i \in I$, $a_i \in \mathbb{Q}_p$,

$$X_t^{\nu} \stackrel{d}{=} \sum_{i \in I} a_i \mathcal{P}_{c_i}(t) + K_n(t)$$

where $c_i := \nu_i(K(a_i, p^n))$ and K_n is a process confined in $K(0, p^n)$.

This establishes an one-to-one mapping (up to modulo p^{-n}) between

Lévy processes X^{ν} on $\mathbb{Q}_p$

and

MDPP $\mathcal{P}_{\mathbf{c}} := (\mathcal{P}_{c_i})_{i \in I}$

on corresponding $(\mathbb{N}_0)^I$.

This theorem is very useful, e.g., in consideration of some problems of hitting balls: for any $n \in \mathbb{Z}$,

– decomposing a Lévy process into p^n-ball-supported Lévy processes,
– condensing each Lévy process associated with measure supported by ball into a multiplicative Poisson process,
– translating the hitting problems,
– solving the problem by introducing an linear indefinite equations.

In fact, when the Lévy measure has a bounded support, we obtain some interesting results, e.g., recurrence, range denseness, oscillation and etc. (see A-Z 2000).

- Transition density of Lévy processes on $\mathbb{Q}_p$ (A-Z 2001).

- Limit theorem (A-Z 2000)

- Path properties (A-K-Z 1999, A-Z 2000a, 2000b) (e.g., Hausdorff dimension and packing dimension)

- Infinitesimal generator, spectrum theorem and Dirichlet form (A-K-Z 1999).

The study of μ-symmetric stochastic processes on $\mathbb{Q}_p$

- μ-symmetric stochastic processes:

  If $p(t, x, dy),\ t \geq 0,\ x, y \in \mathbb{Q}_p$ is a transition function, then

  $$\int_A \mu(dx) \int_B p(t, x, dy) = \int_B \mu(dx) \int_A p(t, x, dy)$$

  for any $A,\ B \in \mathcal{Q}_p$.

  The symmetric property is relative to $L^2(\mathbb{Q}_p, \mu)$ (see Fukushima et al. (1980, 1994), Ma-Roeckner (1992))

- A-Z (1999d) has constructed the stochastic processes symmetric to any given σ-finite measures on $\mathbb{Q}_p$.

  Karwowski–Vilela'Mendes (1994) for $\rho(x)dx$.

- μ-symmetric stochastic process on $\mathbb{Q}_p$ is usually not a Lévy process.

  Theorem: (A-Z 1999d) μ-symmetric stochastic process on $\mathbb{Q}_p$ is a Lévy process iff $\mu = cdx$ for some constant $c \geq 0$.

- For K-V stochastic processes, we also obtained a criterion of recurrence (see A-K-Z 1999).

- Hitting probabilities (A-K-Z 1999, A-Z 2000).

Stochastic process induced by a map

For any map ϕ from $\mathbb{Q}_p$ to $\mathbb{Q}_p$, define

$$\phi^{(n)} = \underbrace{\phi \circ \cdots \circ \phi}_{n \text{ times}}.$$

Then, we can construct a semi-group:

Theorem (Kaneko-Zhao 2001) If a map $\phi : \mathbb{Q}_p \to \mathbb{Q}_p$ satisfies

$$\|x - y\|_p \leq p^r \Rightarrow \|\phi(x) - \phi(y)\|_p \leq p^r$$

$\forall x, y \in \mathbb{Q}_p$ for some integer r, then the transition probability measure $p_t(x, dy)$ defined by

$$\begin{aligned} &p_t(x, dy) \\ &\quad = e^{-ctp^r}\{\delta_{\{x\}}(dy) + p^{-r}\sum_{n=1}^{\infty}\frac{(ctp^r)^n}{n!} \\ &\qquad 1_{B(\phi^{(n)}(x),p^r)}(y)\mu(dy)\} \end{aligned}$$

enjoys

$$\int_{\mathbb{Q}_p} p_t(x,dy) \int_{\mathbb{Q}_p} p_s(y,dz) f(z)$$
$$= \int_{\mathbb{Q}_p} p_{t+s}(x,dz) f(z) \qquad (\forall t, s \geq 0)$$

for any real valued locally constant function f with compact support.

Superprocesses on p-adics

- Superprocesses: Measure-Valued Branching Processes, called as (c, β)-superprocess
  - underlying processes — A-K random processes associated with parameter $c \in (0,1)$
  - branching characteristic — $\Psi(\lambda) = \lambda^{1+\beta}$, $0 < \beta \leq 1$.

Archimedean Space: very popular in the world, e.g. Dawson (1975–1998), Dynkin (1990–1998), Watanabe (1968, 1997), and etc; Li (1990–2002) Wang (1990-2002), Wu and so on.

Non-Archimedean Space: No study

- A-Z (2000) introduced the measure-valued branching processes.

- A-Z (2000) studied several questions, for example, Self-Similarity and Local Extinction.

  Some technique difficulties.

  We obtained some results different from Euclidean case.

  – Quasi-self-similarity

    Theorem (Albeverio-Zhao (2000)) Suppose $c = p^{-2(1+\beta)}$, then for $t > 0$, $a \in \mathbb{Q}_p$, $M \in \mathbb{Z}$ and $\mu \in M_q(\mathbb{Q}_p)$,

    $$(X_{ct}(K(a, p^M)),\ P^\mu) \stackrel{d}{=} (X_t(K(p^{-1}a, p^{M+1})), P^{\mu'}),$$

    where μ':

    $$\int_{\mathbb{Q}_p} \phi(x)\mu'(dx) = p^2 \int_{\mathbb{Q}_p} \phi(p^{-1}x)\mu(dx)$$

    for $\phi \in C_c(\mathbb{Q}_p)_+$.

– Local extinction: a superprocess $\{X_t, t \geq 0\}$ is **locally extinct** if

$$\lim_{t\to\infty} X_t(K) = 0$$

in probability for any compact set K.

Theorem (Albeverio-Zhao (2000)) If the initial measure is a **Haar measure**, then the (c, β)-superprocess is locally extinct if and only if $\beta \leq (-\log_p c) \wedge 1$.

That is, the Hausdorff dimension of the image of corresponding stable random walk is the critical value of local extinction (see A-Z 2000a).

Generally speaking, if the initial measure is not Haar measure we also give a sufficient condition of initial measure for the associated superprocesses to be locally extinct.

Open Problems

- Superprocesses and nonlinear Quasi-DFEs.
- Large deviation of transition functions.
- F-K functional.
- Potential Theory and global properties of Lévy processes (when the support of Lévy measure is unbounded).
- The study of Evans' stochastic processes associated with "p-adic time" and p-adic state.
- Multi-dimensional Poisson processes.
- Applications.