Symmetric Lévy Type Operators

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Outline

Lévy Type Operators

For any $f \in C_0^\infty(\mathbb{R}^d)$,

$$Lf(x) = \sum_{i,j=1}^{d} a_{ij}(x)\partial_{ij} f(x) + \sum_{i=1}^{d} b_i(x)\partial_i f(x) + \int \Big(f(x+z) - f(x) - \nabla f(x)\cdot z \mathbb{1}_{\{|z|\le 1\}} \Big)\nu(x,dz),$$

where $\nu(x,dz)$ is a Lévy kernel, i.e., $\nu(x,dz)$ is σ-finite measure on $\mathbb{R}^d \setminus \{0\}$ such that $\int (1 \wedge |z|^2)\nu(x,dz) < +\infty$ for each $x \in \mathbb{R}^d$.

Symmetry

- $(L, C_0^\infty(\mathbb{R}^d))$ is symmetric with respect to μ, i.e., for any $f, g \in C_0^\infty(\mathbb{R}^d) \cap \mathbb{L}^2(\mu)$,

$$(f, Lg)_{\mathbb{L}^2(\mu)} = (Lf, g)_{\mathbb{L}^2(\mu)}.$$

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$$(f, Lg)_{\mathbb{L}^2(\mu)} = (Lf, g)_{\mathbb{L}^2(\mu)}.$$
- Let
$$L := \sum_{i,j=1}^d a_{i,j}\partial_{i,j} + \sum_{i=1}^d b_i\partial_i,$$
and $\mu(dx) = \rho(x)dx$. Then L is symmetric with respect to μ if and only if
$$b_i(x) = \sum_{j=1}^d \Big(a_{i,j}(x)\rho(x)^{-1}\partial_j\rho(x) + \partial_j a_{i,j}(x)\Big), \qquad 1 \le i \le d.$$

Known Results [K. Sato, 1999]

- Lévy process is symmetric with respect to dx if and only if the drift term $b = 0$ and the Lévy measure ν is symmetric.
- Lévy process is not symmetric with respect to any probability measure.

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- Lévy process is not symmetric with respect to any probability measure.
- Infinitesimal Generator of Lévy Process

$$Lf(x) = \sum_{i,j=1}^{d} a_{ij}\partial_{ij}f(x) + \sum_{i=1}^{d} b_i\partial_i f(x) + \int \big(f(x+z) - f(x) - \nabla f(x)\cdot z\mathbb{1}_{\{|z|\le 1\}}\big)\nu(dz),$$

Symmetric Lévy Type Operator

- Lévy Type Jump Operator

$$Lf(x) = \int \big(f(x+z) - f(x) - \mathbb{1}_{\{|z|\le 1\}} \nabla f(x) \cdot z\big)\nu(x, dz) + b(x) \cdot \nabla f(x).$$

Symmetric Lévy Type Operator

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Assumptions

(1) $\mu(dx) = \rho(x)dx$.

(2) $\int (1\wedge |z|^2)\nu(x,dz) \in \mathbb{L}^2_{\mathrm{loc}}(\mu)$, and for r large enough

$$f_r(x) := \mathbb{1}_{B_{2r}(0)}(x)\int_{\{|z|>r\}} \nu(x,dz) \in \mathbb{L}^2(\mu).$$

$L : C_0^\infty(\mathbb{R}^d) \to \mathbb{L}^2(\mu)$

Main Result

Theorem

L is symmetric with respect to the measure μ, if

$$\begin{aligned} \nu(x, dz) =& j(x, x+z)\rho(x+z)dz, \\ b_i(x) =& \frac{1}{2}\int_{\{|z|\leq 1\}} z_i\big[\nu(x, dz) - \nu(x, -dz)\big], \end{aligned}$$

where $j(x, y)$ is a non-negative symmetric function.

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Assumptions

(3) For each $x \in \mathbb{R}^d$, $\nu(x, dz)$ satisfies

$$\int_{\{|z|\le 1\}} |z|\big[\nu(x, dz) - \nu(x, -dz)\big] \in \mathbb{L}^1_{\text{loc}}(\mu).$$

Example

Set

$$Lf(x) = -(-\Delta)^{\alpha/2} f(x) - xf'(x)$$

on $\mathbb{R}$. Then

(1) L is positive recurrent; moreover, its invariant measure $\mu(dx) = \rho(x)dx$ is a symmetric stable distribution and

$$\hat{\rho}(x) = \exp\Big(-\frac{c_\alpha}{\alpha\sqrt{2\pi}}|x|^\alpha\Big).$$

S. Albeverio et al.(2001), Wang (2008)

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(2) For $\alpha \neq 2$, L is not symmetric with respect to $\mu(dx) = \rho(x)dx$.

Example

Set

$$L_0 f(x) = \frac{1}{(2\pi)^{d/2}} \int \Big(f(x+z) - f(x) - \mathbb{1}_{\{|z|\le 1\}} \nabla f(x) \cdot z \Big) \times \frac{c(x, x+z)}{|z|^{d+\alpha}} e^{-\frac{|x+z|^2}{2}} dz,$$

$$b(x) = \frac{1}{2(2\pi)^{d/2}} \int_{\{|z|\le 1\}} z \left(\frac{c(x, x+z)}{|z|^{d+\alpha}} e^{-\frac{|x+z|^2}{2}} - \frac{c(x, x-z)}{|z|^{d+\alpha}} e^{-\frac{|x-z|^2}{2}} \right) dz.$$

Then $L := L_0 + b(x) \cdot \nabla$ is symmetric with respect to

$$\mu(dx) = (2\pi)^{-d/2} e^{-|x|^2/2} dx.$$

Dirichlet Forms

- Dirichlet Forms corresponding to Symmetric Lévy Type Operators

$$D(f) = \frac{1}{2}\int\int (f(x) - f(y))^2 j(x, y)\mu(dx)\mu(dy)$$

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Example

$$D(f) = \int\int (f(x) - f(y))^2 \frac{c(x, y)}{|x - y|^{d+\alpha}}\mu(dx)\mu(dy),$$

where $\mu(dx) = (2\pi)^{-d/2}e^{-|x|^2/2}dx$.

Exponential Ergodicity and Dirichlet Eigenvalue

First Poincaré Inequality

$$\int f^2 d\mu \le \lambda_0^{-1} D(f) \qquad f|_{(-\infty,0]} = 0.$$

Second Poincaré Inequality [M.F. Chen (2005)]

$$\int f^2 d\mu \le \lambda_1^{-1} D(f) \qquad \mu(f) = 0,$$

which implies $\|P_t(x,) - \mu\|_{\mathrm{Var}} \le C(x)e^{-\lambda_1 t}$.

Simple Sufficient Condition

If $f|_{(-\infty,0]} = 0$, then

$$D(f) = \frac{1}{2}\int_0^\infty \int_0^\infty \big(f(x) - f(y)\big)^2 j(x,y)\mu(dx)\mu(dy) + \int_0^\infty f^2(x)\mu(dx)\int_{-\infty}^0 j(x,y)\mu(dy).$$

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Simple Sufficient Condition

$$\inf_{x>0}\int_{-\infty}^0 j(x,y)\rho(y)dy > 0$$

Sufficient Condition

Hardy Inequality

$$\sup_{x>0}\int_0^x j(x,y)^{-1}\rho(y)dy<\infty$$

Sufficient Condition

Variational Formula

there exists $\beta \in (0,1)$ such that

$$\lim_{x\to\infty}\left[\beta x^{-2}A(x) - x^{-1}\left(\overline{b}(x) + D(x)\right)\right] > 0,$$

where

$$A(x) = \frac{1}{2}\int_{-1}^{1} z^2\nu(x,dz),$$

$$\overline{b}(x) = \frac{1}{2}\int_{\{|z|\le x\}} z\left[\nu(x,dz) - \nu(x,-dz)\right],$$

$$D(x) = \int_x^{\infty} z\nu(x,dz).$$

Example

Let

$$j(x,y) = \frac{c(x,y)}{|x-y|^{1+\alpha}} \mathbb{1}_{\{x \neq y\}},$$

where c is non-negative symmetric and $\alpha \in (0,2)$.

(1) $\rho(x) = \pi^{-1/2} e^{-x^2}$ and
$c(x,y) = [(1+|x|)(1+|y|)]^{\beta} (\beta \geq 1+\alpha)$.

(2) $\rho(x) = c_{\gamma}(1+|x|)^{-\gamma}$ $(\gamma \geq 2)$ and
$c(x,y) = [(1+|x|)(1+|y|)]^{\beta} \mathbb{1}_{\{|x-y| \leq 1\}}$ with
$\beta \geq 1+\gamma/2$.

Questions

- What criterion for $\lambda_0 > 0$?
- How to present the exact expression for λ_0?

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Thank You!