Existence and Uniqueness of Stationary Solutions of Retarded Ornstein-Uhlenbeck Processes

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1. Retarded O-U Processes

Let $\mathcal{H} = H \times L_r^2 = H \times L^2([-r, 0]; H)$ and $L^2_{\mathcal{F}_0} = L^2_{\mathcal{F}_0}(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{H})$ denote the family of all $\mathcal{H}$-valued mappings $\phi(\omega) = (\phi_0(\omega), \phi_1(\omega))$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ such that both ϕ_0 and $\phi_1(\theta)$ are $\mathcal{F}_0$-measurable for any $\theta \in [-r, 0]$ and satisfy

$$\mathbb{E}\|\phi\|^2_{\mathcal{H}} = \mathbb{E}\|\phi_0\|^2_H + \mathbb{E}\|\phi_1\|^2_{L^2_r} < \infty.$$

We shall be concerned about the stochastic retarded evolution equation on H,

$$\begin{aligned} dy(t) = \Big[Ay(t) + \int_{-r}^{0} d\eta(\theta)y(\theta + t)\Big]dt + BdW(t), \\ t > 0, \\ y(0) = \phi_0,\ \ y_0 = \phi_1, \\ \phi = (\phi_0, \phi_1) \in L^2_{\mathcal{F}_0}(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{H}), \end{aligned} \tag{1.1}$$

where A generates some C_0-semigroup e^{tA}, $B \in \mathcal{L}(K, H)$, $W(t)$ is a K-valued Q-Wiener process on $(\Omega, \mathcal{F}, \mathbb{P})$ and η is the Stieltjes measure given by

$$\eta(\tau) = -\sum_{i=1}^{m} \chi_{(-\infty, -r_i]}(\tau) A_i - \int_{\tau}^{0} A_0(\theta) d\theta,$$

in which $0 < r_1 < r_2 < \cdots < r_m \leq r$, $A_i \in \boldsymbol{\mathcal{L}}(H)$, $A_0(\cdot) \in L^2([-r, 0]; \boldsymbol{\mathcal{L}}(H))$ and $y_0(\theta) := y(\theta)$, $\theta \in [-r, 0]$.

Definition 1.1. A stochastic processes $y = \{y(t); t \geq -r\}$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ is called a *mild solution* of (1.1) if

(i) $y(t,\phi)$ is adapted to $\mathcal{F}_t$, $t \geq 0$, and the trajectories of $y(\cdot,\phi)$ are almost surely Bochner integrable such that for arbitrary $t \geq 0$,

$$\mathbb{P}\Big\{\omega : \int_0^t \|y(s)\|_H^2 ds < \infty\Big\} = 1;$$

(ii) for any $t \geq 0$, we have

$$\begin{aligned} y(t) = e^{tA}\phi_0 + \int_0^t e^{(t-s)A} \int_{-r}^0 d\eta(\theta) y(\theta+s) ds \\ + \int_0^t e^{(t-s)A} B dW(s); \end{aligned} \tag{1.2}$$

(iii) $y(0) = \phi_0$, $y_0(\cdot) = \phi_1(\cdot)$, $\phi = (\phi_0, \phi_1) \in L^2_{\mathcal{F}_0}(\Omega, \mathcal{F}, \mathbb{P}; \mathcal{H})$.

For any $t \geq 0$, let

$$Q_t = \int_0^t G(s)BQB^*G^*(s)ds$$

where $G(t)$ is the Green operator of (1.1). It was shown [Liu (2006)] that for any $\phi = (\phi_0, \phi_1) \in L^2_{\mathcal{F}_0}$ if for any $t \geq 0$,

$$\text{Tr}\,[Q_t] = \int_0^t \text{Tr}\,[G(s)BQB^*G^*(s)]ds < \infty, \tag{1.3}$$

then there exists a unique mild solution $y(t, \phi)$ of (1.1).

2. Stationary Solutions

When ones try to investigate the stationary solutions of stochastic retarded systems, one of the major difficulties is that the solution, unlike the systems without memory, would not be Markovian any more.

Consequently, the important concepts of Markovian transition functions, semigroups, invariant measures etc. would not make sense for stochastic retarded systems.

But we can still define stationary solutions.

Definition 2.1. A solution $y = \{y(t); t \geq -r\}$ of (1.1) is called *strongly stationary*, or simply *stationary*, if the finite-dimensional distributions are invariant under time translations, i.e.,

$$\mathbb{P}\Big\{y(t+t_k) \in \Gamma_k,\ k = 1, \cdots, n\Big\}$$
$$= \mathbb{P}\Big\{y(t_k) \in \Gamma_k,\ k = 1, \cdots, n\Big\}$$

for all $t \geq 0$, $t_k \geq -r$ and Borel sets $\Gamma_k \in \mathcal{B}(H)$, $k = 1, \cdots, n$.

We say that (1.1) has a stationary solution y if there exists an initial $\phi = (\phi_0, \phi_1) \in L^2_{\mathcal{F}_0}$ such that $y(t, \phi)$, $t \geq 0$, is a stationary solution of (1.1) with $y(0) = \phi_0$, $y_0 = \phi_1$.

Definition 2.2. A stationary solution is said to be *uniquely determined* if any two stationary solutions of (1.1) have the same finite dimensional distributions.

Existence of Stationary Solutions

In the sequel, we always suppose that the condition (1.3) holds, i.e., for any $t \geq 0$,

$$\mathrm{Tr}\,[Q_t] = \int_0^t \mathrm{Tr}\,[G(s)BQB^*G^*(s)]ds < \infty,$$

which ensures the existence of mild solutions of (1.1).

Theorem 2.1. *For the equation (1.1), the following (i), (ii), (iii) and (iv) are equivalent:*

(i) there exists a stationary mild solution $y(t)$ of (1.1);

(ii)

$$\sup_{t\geq 0} Tr[Q_t] = \sup_{t\geq 0} \int_0^t Tr[G(s)BQB^*G^*(s)]ds < \infty;$$

(iii) *there exists a trace-class operator P in the space $\mathcal{L}^+(H)$ of all symmetric nonnegative operators on H, satisfying the Lyapunov equation*

$$2\langle PA^*\varphi(0),\varphi(0)\rangle_H + 2\Big\langle P\int_{-r}^{0} d\eta^*(\theta)\varphi(\theta),\varphi(0)\Big\rangle_H + \langle BQB^*\varphi(0),\varphi(0)\rangle_H = 0$$

for any $\varphi \in C([-r,0];H)$ with $\varphi(0) \in \mathcal{D}(A^*)$;

(iv) *the mild solution $y(t,0)$ with null initial is bounded in mean square.*

Uniqueness of Stationary Solutions

For each $\lambda \in \mathbb{C}^1$, we define the densely defined closed linear operator $\Delta(\lambda, A, \eta)$ by

$$\Delta(\lambda, A, \eta) = \lambda I - A - \int_{-r}^{0} e^{\lambda\theta} d\eta(\theta), \quad \lambda \in \mathbb{C}^1.$$

The *retarded resolvent set* $\rho(A, \eta)$ is defined as the set of all values λ in $\mathbb{C}^1$ for which the operator $\Delta(\lambda, A, \eta)$ has a bounded inverse on H.

The *retarded spectrum* is defined to be

$$\sigma(A, \eta) = \mathbb{C}^1 \setminus \rho(A, \eta).$$

In particular, $\lambda \in \sigma_P(A, \eta)$, the retarded point spectrum of (A, η), if and only if there exists a nonzero $x \in \mathcal{D}(A)$ such that

$$\Delta(\lambda, A, \eta)x = 0.$$

The value $\lambda \in \sigma_P(A, \eta)$ is often called the *characteristic value* of $\Delta(\lambda, A, \eta)$.

Let $\mathcal{L}_2(K,H)$ denote the space of all Hilbert-Schmidt operators from K to H.

Theorem 2.2. *If A generates a compact C_0-semigroup e^{tA}, $t \geq 0$,*

$$\alpha_0 = \max\{Re\,\lambda : \lambda \in \sigma_P(A,\eta)\} < 0$$

and either

(1) $Tr\,Q < \infty$, $B \in \mathcal{L}(K,H)$, or

(2) $Q = I$, $B \in \mathcal{L}_2(K,H)$,

then

(a) there exists a unique stationary solution $y(t)$, $t \geq -r$, of (1.1) and it is a zero mean Gaussian process with the covariance operator $K(\cdot)$ given by

$$K(t) = \int_0^\infty G(t+s)BQB^*G^*(s)ds,\ t \geq 0;$$

(b) let $\phi = (\phi_0, \phi_1) \in L^2_{\mathcal{F}_0}$ *be an initial process and* $y(t, \phi)$ *the corresponding solution, which is nonstationary in general, of (1.1), the distribution of*

$$\{y(t + t_1), \cdots, y(t + t_n)\}$$

where $n \in \mathbb{N}$, $0 \leq t_1 < t_2 < \cdots < t_n$, *are fixed, tends for* $t \longrightarrow \infty$ *to a zero mean normal distribution with covariance operator* $K = (K(t_i, t_j))_{1 \leq i, j \leq n}$ *given by*

$$K(t_i, t_j) = \int_0^\infty G(|t_i - t_j| + s) BQB^* G^*(s) ds;$$

(c) for any $x \in \mathcal{D}(A)$, $K(t)x \in \mathcal{D}(A)$ *and* $K(t)x$ *is strongly differentiable such that*

$$\frac{dK(t)}{dt} x = AK(t)x + \int_{-r}^0 d\eta(\theta) K(\theta + t)x,$$

for any $t \geq 0$.

Examples

Consider the stochastic retarded partial differential equation

$$\frac{\partial y(t,\xi)}{\partial t} = a\frac{\partial^2 y(t,\xi)}{\partial \xi^2} + \sum_{i=1}^{m} a_i y(t-r_i,\xi) + b(\xi)\dot{B}_t,$$

$$t > 0,\ \ \xi \in (0,1),$$

$$y(t,0) = y(t,1) = 0,\ t \geq 0;$$

$$y(t,\xi) = \varphi(t,\xi),\ \ 0 \leq \xi \leq 1,\ \ -r \leq t \leq 0,$$

where $a > 0$ and a_i are real numbers, $b(\cdot),\ \varphi(t,\cdot) \in L^2(0,1)$ and B_t is a one dimensional real Brownian motion.

For the system it is easy to see that

$$\sigma_P(A,\eta) = \Big\{\lambda \in \mathbb{C}^1 :\ \lambda + an^2\pi^2 - \sum_{i=1}^{m} a_i e^{-\lambda r_i} = 0 \quad \text{for some} \ \ n = 1, \cdots \Big\}.$$

The condition $\alpha_0 < 0$ in Theorem 2.2 is now reduced to a verifiable one that

$$\begin{cases} \text{all roots of the equations} \\ \quad \lambda = -an^2\pi^2 + \sum_{i=1}^{m} a_i e^{-\lambda r_i},\ n = 1, \cdots \\ \text{has negative real parts.} \end{cases}$$

It can be shown that this holds for all $r_i \geq 0$ if and only if

$$\sum_{i=1}^{m} |a_i| \leq a\pi^2 \ \text{ and } \ \sum_{i=1}^{m} a_i < a\pi^2.$$