On the Convergence of the Spectral Empirical Process of Wigner Matrices

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Abstract:

A *complex Wigner matrix* of size n is a Hermitian matrix $W_n = (x_{i,j})_{1\le i,j\le n}$ where the upper-half entries $(x_{i,j})_{1\le i\le j\le n}$ are independent, zero-mean complex-valued random variables satisfying the following moment conditions: a) for all i, $\mathbb{E}|x_{ii}|^2 = \sigma^2 > 0$ and b) for all $i < j$, $\mathbb{E}|x_{ij}|^2 = 1$ and $\mathbb{E}x_{ij}^2 = 0$. Similarly , one can define a *real symmetric Wigner matrix* W_n of size n (then without the condition $\mathbb{E}x_{ij}^2 = 0$).

The empirical spectral distribution F_n of a Wigner matrix is the empirical distribution generated by the n eigenvalues of the normalized matrix $n^{-1/2}W_n$. This distribution is supported by the real line. Wigner (E.P. Wigner in the 1950's) first proved that as $n \to \infty$, $\mathbb{E}F_n$ converges to the semi-circle law $F(dx)$ with support [-2,2]. Later the convergence in probability or the almost sure convergence of F_n to F were also established.

The problem of the convergence rate has been considered more recently and several results are proposed by Z.D. Bai, O. Costin and J. Lebowitz, K. Johansson, Y. Sinai and A. Soshnikov. However, the exact convergence rate remains undiscoved for Wigner matrices. Results from numerical simulations lead to a conjecture for a rate on the order of $O(1/n)$. It thus seems natural to consider the asymptotics of the empirical process $G_n(x) = n(F_n(x)-F(x))$.

Let $\mathcal{U}$ be an open set of the complex plane including the interval $[-2, 2]$. Next define $\mathcal{A}$ to be the set of analytic functions $f \ : \mathcal{U} \to \mathbb{C}$. We then consider the empirical process $G_n := \{G_n(f)\}$ indexed by $\mathcal{A}$, *i.e.*, $G_n(f) := n\int_{-\infty}^{\infty} f(x)[F_n - F](dx)\ , \quad f \in \mathcal{A}$.

In this work we prove that, under suitable 4-th moment conditions on the entries of the Wigner matrix W_n, the spectral empirical process $G_n = (G_n(f))$ indexed by the set of analytic functions $\mathcal{A}$ converges weakly to a Gaussian process $G := \{G(f) :\ f \in \mathcal{A}\}$. The mean function $\mathbb{E}[G(f)]$ as well as the covariance function $c(f, g) := \mathbb{E}[\{G(f) - \mathbb{E}G(f)\}\{G(g) - \mathbb{E}G(g)\}]$ of the limiting Gaussian process are given.