Estimates on the Density of Brownian Motion with Singular Drift

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Abstract:
Let $d \geq 3$ and let $\mu = (\mu^1, \dots, \mu^d)$ with each μ^i being a signed measure on R^d satisfying

$$\lim_{r\to 0} \sup_{x\in R^d} \int_{B(x,r)} \frac{|\mu^i|(dy)}{|x-y|^{d-1}} = 0.$$

The existence and uniqueness of a continuous Markov process X on R^d, called a Brownian motion with singular drift μ, was established by Bass and Chen recently. In this paper we study the potential theory of X. We show that X has a density q^μ and that there exist positive constant c_i, $i = 1, \cdots, 9$ such that

$$c_1 e^{-c_2 t} t^{-\frac{d}{2}} e^{-\frac{c_3|x-y|^2}{2t}} \leq q^\mu(t,x,y) \leq c_4 e^{c_5 t} t^{-\frac{d}{2}} e^{-\frac{c_6|x-y|^2}{2t}}$$

and

$$|\nabla_x q^\mu(t,x,y)| \leq c_7 e^{c_8 t} t^{-\frac{d+1}{2}} e^{-\frac{c_9|x-y|^2}{2t}}$$

for all $(t,x,y) \in (0,\infty) \times R^d \times R^d$. We further show that, for every bounded $C^{1,1}$-domain D, the density $q^{\mu,D}$ of X^D, the process obtained from X by killing upon exiting from D, has the following estimates: For any $T > 0$, there exist positive constants C_i, $i = 1, \cdots, 5$ such that

$$C_1(1 \wedge \frac{\rho(x)}{\sqrt{t}})(1 \wedge \frac{\rho(y)}{\sqrt{t}}) t^{-\frac{d}{2}} e^{-\frac{C_2|x-y|^2}{t}} \leq q^{\mu,D}(t,x,y) \leq C_3(1 \wedge \frac{\rho(x)}{\sqrt{t}})(1 \wedge \frac{\rho(y)}{\sqrt{t}}) t^{-\frac{d}{2}} e^{-\frac{C_4|x-y|^2}{t}}$$

and

$$|\nabla_x q^{\mu,D}(t,x,y)| \leq C_5(1 \wedge \frac{\rho(y)}{\sqrt{t}}) t^{-\frac{d+1}{2}} e^{-\frac{C_4|x-y|^2}{t}}$$

where $\rho(x)$ is the distance between x and ∂D. Using the above estimates, we prove a parabolic Harnack inequality for X and boundary Harnack inequality for nonnegative harmonic functions of X.

References

[1.] Bass, R.F. and Chen, Z.-Q. (2003). Brownian motion with singular drift, *Ann. Probab.*, 31, 791-817.