

Liouville Theorems for Symmetric Diffusion Operators on Complete Riemannian Manifolds

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Abstract:

In a series of seminal papers in 1975-1978, S.T. Yau developed the gradient estimate method in geometric analysis and proved that there exists no positive or bounded harmonic function on complete non-compact Riemannian manifolds with non-negative Ricci curvature, and that the heat semigroup generated by the Laplace-Beltrami operator has the C_0 -diffusion property and is conservative on complete Riemannian manifolds on which the Ricci curvature is bounded from below. Based on the Li-Yau differential Harnack inequality and some other earlier results due to S.Y. Cheng, L. Karp, R. Schoen, S. T. Yau and himself, P. Li in 1984 gave the optimal condition on the Ricci curvature for the L^1 -Liouville theorem of harmonic functions and the L^1 -uniqueness of the heat semigroup on complete non-compact Riemannian manifolds.

Let $L = \Delta - \nabla\phi \cdot \nabla$ be a symmetric diffusion operator with an invariant measure $\mu(dx) = e^{-\phi(x)}dx$ on a complete non-compact Riemannian manifold M . Using an interplay between geometric analysis and stochastic analysis, we give the optimal conditions on “the Ricci curvature associated with L ” so that various Liouville theorems hold for L -harmonic functions, and the heat semigroup $P_t = e^{tL}$ has the C_0 -diffusion property and is unique in $L^1(M, \mu)$. As applications, we give the optimal conditions for the uniqueness of the positive L -invariant measure and the L^1 -uniqueness of the intrinsic Schrödinger operators on complete non-compact Riemannian manifolds. We also prove a criterion for the finiteness of the total mass of the L -invariant measure and extend the Calabi-Yau theorem to the L -invariant measure on complete Riemannian manifolds with non-negative m -dimensional Bakry-Emery Ricci curvature. Finally, we give a new proof of the Bakry-Qian Laplacian comparison theorem which plays an important role in this work.

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