# Semicircle Law for Hadamard Products

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**Abstract**:

Since large random matrices were first used in the field of nuclear physics to model the interaction between the huge body of interacting atomic nuclei in a system, spectral theory concerning their eigenvalue behavior has been attracting the interest of researchers from both theoretical and practical field ever since. The first rigorous theoretical result in this field is usually considered to be the result proving the convergence to the semicircle law for the class of random matrices called Wigner matrices. Later on, the spectral theory has been further developed for the sample covariance matrices, for which the limiting spectral distribution is the Marcenko-Pastur law. Spectral theory concerning a class of sample covariance type of random matrix which is of the form of the product of a sample covariance matrix and another Hermitian matrix has then been analyzed very thoroughtly. Recently, sparse random matrices, which has a large portion of 0 entries, have received special attention in quantum mechanics, atomic physics, neural networks and many other areas. This is due to their application background in these fields. Indeed, when a real system is not of full connectivity, a random matrix provides a more natural and relevant description of the system. Partially connceted system occurs in nuclei physics, in which case, since the particles move in a very high velocity in a small range, many exciting states in very short time cannot be observed. Moreover, in neural network theory, the number of neurons in one person's brain is probably of several orders of magnitude larger than that of the dendrites connected with one individual neuron. Recent works on large sparse matrices and their applications can be found in various areas such as Linear Algebra, Neural Networks, Algorithms and Computing, Finance Modelling, Electrical Engineering, Biointeractions and Theoretical Physics.

A sparse matrix can be considered to be a special case of matrix in the form of Hadamard product of matrices: $A_p = B_m \circ D_m$. Then $A_p$ is sparse if the elements $d_{ij}$'s of $D_m$ take values 0 and 1 with $\sum_{i=1}^m P(d_{ij}=1) = p = o(m)$. The index $p$ usually stands for the level of sparseness, i.e. after performing the Hadamard product, the resulting matrix will have $p$ nonzero elements per row on the average. In this case, it is commonly assumed that the matrix $D_m$ is symmetric and its entries $\{d_{ij} : i \le j\}$ are independent Bernoulli trials with $P(d_{ij}=1) = p_{ij}$ and independent of the entries of the matrix $B_m$. However, the Hadamard product can obviously include more general situations than these sparse matrices by applying more general sparseness condition on the matrix $D_m$. In this paper, as $p/n \to 0$ as $n \to \infty$, we will prove the weak and strong convergence to the semicircle law of the empirical spectral distribution of the Hadamard product of a normalized sample covariance matrix and a sparsing matrix, which is of the form $A_p = \frac{1}{\sqrt{np}}(X_{m,n}X_{m,n}^* - \sigma^2 n I_m) \circ D_m$, where the matrices $X_{m,n}$ and $D_m$ are independent and the entries of $X_{m,n}$ $(m \times n)$ are independent, the matrix $D_m$ $(m \times m)$ is Hermitian with independent entries above and on the diagonal, $p$ is the sum of the second moments of the row (and column) entries of $D_m$, and "$\circ$" denotes the Hadamard product of matrices.