Derived equivalences and almost $\mathcal{D}$-split sequences

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Abstract

In the paper, we introduce almost $\mathcal{D}$-split sequences and provide several new methods to construct derived equivalences. In particular, we obtain derived equivalences from Auslander-Reiten sequences (or n-almost split sequences), Auslander-Reiten triangles, stable equivalences, Auslander algebras and quotient algebras. Thus, our results establish a natural connection between derived equivalences and Auslander-Reiten sequences; and show that BB-tilting modules exist abundantly.

1 Introduction

Derived equivalence is one of the important equivalences in many mathematical branches. Especially, in the representation theory of groups and algebras, such an equivalence preserves many significant invariants of groups and algebras; for example, the number of irreducible representations, Cartan determinants, Hochschild cohomology groups, algebraic K-theory and G-theory(see [8], [12] and [10]). One of the fundamental results on derived categories may be the Morita theory for derived categories established by Rickard in his several papers [22, 23, 24], which says that two rings A and B are derived-equivalent if and only if there is a tilting complex T of A-modules such that B is isomorphic to the endomorphism ring of T. Thus, starting with a ring A, we may construct theoretically all rings which are derived-equivalent to A by finding all tilting complexes of A-modules. However, in practice, it is not easy to show that two given rings are derived-equivalent by finding a suitable tilting complex, as is indicated by the famous unsolved Broue's abelian defect group conjecture, which states that the module categories of a block algebra A of a finite group algebra and its Brauer correspondent B should have equivalent derived categories if their defect groups are abelian (see [8]). Therefore, the following questions related to constructing derived equivalences seem to be of interest:

(1) As is well-known, Auslander-Reiten sequence is of significant importance in the modern representation theory of algebras and finite groups. Is there any natural connection between Auslander-Reiten sequences and derived equivalences ? In other words, is it possible to construct derived equivalences from Auslander-Reiten sequences or n-almost split sequences or Auslander-Reiten triangles ?

Motivated by looking for an answer to this question, we are led to the following general question:

(2) Given a derived equivalence between two algebras A and B. Is it possible to construct new derived equivalences between algebras (possibly different from A and B) from the given derived equivalence ?

Another motivation of our questions comes from the construction of stable equivalences of Morita type in [19, 20, 21], where partial answers to both questions for stable equivalences

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of Morita type were provided. However, since stable equivalence of Morita type and derived equivalence are two independent concepts for general finite-dimensional algebras, we have to approach our questions in a completely different way.

In the present paper, we shall provide several constructions for derived equivalences. Surprisingly, we can affirmatively answer the question (1) and construct derived equivalences by the so-called almost $\mathcal{D}$-split sequences (see Definition 3.1 below). Such sequences include Auslander-Reiten sequences and occur very frequently in the representation theory of Artin algebras (see the examples in Section 3 below). Our result in this direction can be stated in the following general form:

Theorem 1.1 *Let $\mathcal{C}$ be an additive category and M be an object in $\mathcal{C}$. Suppose*

$$X \longrightarrow M' \longrightarrow Y$$

is an almost $\operatorname{add}(M)$-split sequence in $\mathcal{C}$. Then the endomorphism ring $\operatorname{End}_{\mathcal{C}}(M \oplus X)$ of $M \oplus X$ and the endomorphism ring $\operatorname{End}_{\mathcal{C}}(M \oplus Y)$ of $M \oplus Y$ are derived-equivalent. Moreover, the finitistic dimension of $\operatorname{End}_{\mathcal{C}}(M \oplus X)$ is finite if and only if so is the finitistic dimension of $\operatorname{End}_{\mathcal{C}}(M \oplus Y)$.

This result reveals a mysterious connection between Auslander-Reiten sequences and derived equivalences, namely we have the following corollary.

Corollary 1.2 *Let A be an Artin algebra.*

(1) *Suppose $0 \longrightarrow X_i \longrightarrow M_i \longrightarrow X_{i-1} \longrightarrow 0$ is an Auslander-Reiten sequence of finitely generated A-modules for $i = 1, 2, \cdots, n$. Let $M = \bigoplus_{i=1}^{n} M_i$. Then $\operatorname{End}_A(M \oplus X_n)$ and $\operatorname{End}_A(M \oplus X_0)$ are derived-equivalent. In particular, if $0 \longrightarrow X \longrightarrow M \longrightarrow Y \longrightarrow 0$ is an Auslander-Reiten sequence, then the endomorphism algebras $\operatorname{End}_A(X \oplus M)$ and $\operatorname{End}_A(M \oplus Y)$ are derived-equivalent, and have the same Cartan determinant.*

(2) *If A is self-injective and X is an A-module, then the endomorphism algebra $\operatorname{End}(A \oplus X)$ of $A \oplus X$ and the endomorphism algebra $\operatorname{End}_A(A \oplus \Omega(X))$ of $A \oplus \Omega(X)$ are derived-equivalent, where Ω is the syzygy operator.*

Thus, by Corollary 1.2 or more generally, by Proposition 3.14 in Section 3 below, one can produce a lot of derived equivalences from Auslander-Reiten sequences or n-almost split sequences. We stress that the algebra $\operatorname{End}_A(X \oplus M)$ and the algebra $\operatorname{End}_A(M \oplus Y)$ in Corollary 1.2 may be very different from each other (see Example 1 in Section 8 and the example at the end of Section 4), though the mesh diagram of the Auslander-Reiten sequence is somehow symmetric. Another result related to Corollary 1.2 is Proposition 5.1 in Section 5 below, which produces derived equivalences from Auslander-Reiten triangles in a triangulated category. In particular, we have

Corollary 1.3 *Let A be a self-injective Artin algebra. Suppose $0 \longrightarrow X \longrightarrow M \longrightarrow Y \longrightarrow 0$ is an Auslander-Reiten sequence such that $\Omega^{-1}(X) \notin \operatorname{add}(M \oplus Y)$. Then $\underline{\operatorname{End}}_A(M \oplus X)$ and $\underline{\operatorname{End}}_A(M \oplus Y)$ are derived-equivalent, where $\underline{\operatorname{End}}_A(M)$ denotes the stable endomorphism algebra of an A-module ${}_AM$.*

The above results stimulate investigations in different directions. Note that the syzygy functor Ω is an equivalence from the stable category of A-modules to itself if A is self-injective. Corollary 1.2(2) suggests to consider the derived equivalences between the endomorphism algebras of modules linked by a stable equivalence. Our result in this direction is the following theorem, in which $\mathscr{D}^b(A)$ stands for the derived category of bounded complexes of finitely generated A-modules over an Artin algebra A.

Theorem 1.4 *Let A and B be two self-injective Artin algebras. Suppose there is a derived equivalence $F : \mathscr{D}^b(A) \longrightarrow \mathscr{D}^b(B)$ and $T^\bullet$ is the tilting complex associated to F. Let $\overline{F}$ be the stable equivalence induced by F. If X is an A-module such that $\operatorname{Hom}_{\mathscr{D}^b(A)}(T^\bullet, X[n]) = 0$ for all $n \neq 0, 1$, then $\operatorname{End}_A(A \oplus X)$ is derived-equivalent to $\operatorname{End}_B(B \oplus \overline{F}(X))$.*

As a consequence of Theorem 1.4, we have

Corollary 1.5 *Let A and B be self-injective Artin algebras of finite representation type. If A and B are derived-equivalent, then so are their Auslander algebras.*

Finally, inspired by Corollary 1.3, we consider derived equivalences between quotient algebras.

Theorem 1.6 *Let A be an Artin algebra, and let $T^\bullet$ be a tilting complex over A with $B = \mathrm{End}_{\mathscr{D}^b(A)}(T^\bullet)$. Let I be an ideal in A with $\overline{A}$ the quotient algebra of A modulo I. Let J be the ideal of B consisting of all those b which factor through the canonical embedding of $IT^\bullet$ into $T^\bullet$, and $\overline{B} = B/J$. Then $\overline{T}^\bullet$ is a tilting complex and induces a derived equivalence between $\overline{A}$ and $\overline{B}$ if and only if $\mathrm{Hom}_{\mathscr{K}^b(A)}(T^\bullet, IT^\bullet[i]) = 0$ for all $i \neq 0$ and $\mathrm{Hom}_{\mathscr{K}^b(A)}(\overline{T}^\bullet, \overline{T}^\bullet[-1]) = 0$, where $\mathscr{K}^b(A)$ denotes the homotopy category of bounded complexes of finitely generated A-modules.*

As a special case of Theorem 1.6, we have the following result.

Corollary 1.7 *Let $F : \mathscr{D}^b(A) \longrightarrow \mathscr{D}^b(B)$ be a derived equivalence between two self-injective basic Artin algebras. Suppose P is a direct summand of ${}_AA$, and Q is a direct summand of ${}_BB$ such that $F(\mathrm{soc}\,(P))$ is isomorphic to $\mathrm{soc}\,(Q)$, where $\mathrm{soc}\,(P)$ denotes the socle of the module P. Then the quotient algebras $A/\mathrm{soc}\,(P)$ and $B/\mathrm{soc}\,(Q)$ are derived-equivalent.*

The paper is organized as follows: In Section 2, we recall briefly some basic notion and a fundamental result of Rickard on derived categories. Our main results, Theorem 1.1, Theorem 1.4 and Theorem 1.6, are proved in Section 3, Section 6 and Section 7, respectively. In Section 3, we also provide several generalizations of Corollary 1.2; among others is a formulation of Corollary 1.2(1) for n-almost split sequences. In section 4, we point out that if an almost $\mathcal{D}$-split sequence is given by an Auslander-Reiten sequence then Theorem 1.1 can be viewed as a "generalized" version of BB-tilting module. Thus an n-almost split sequence or concatenating n Auslander-Reiten sequences provides us a natural way to get an n-BB-tilting module (for definition, see Section 4). In Section 5, we discuss how to get derived equivalences from Auslander-Reiten triangles in a triangulated category. In particular, Corollary 1.3 is proved in this section. In the last section, we present examples to illustrate our main results in this paper.

In the paper [17], we shall discuss relationship between derived equivalence and stable equivalence of Morita type for general finite-dimensional algebras. In particular, a well-known result of Richard for self-injective algebras is generalized for finite-dimensional algebras.