Good tilting modules and recollements of derived module categories

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Abstract

Let T be an infinitely generated tilting module of projective dimension at most one over an arbitrary associative ring A, and let B be the endomorphism ring of T. In this paper, we prove that if T is good then there exists a ring C, a homological ring epimorphism $B \to C$ and a recollement among the (unbounded) derived module categories $\mathscr{D}(C)$ of C, $\mathscr{D}(B)$ of B, and $\mathscr{D}(A)$ of A. In particular, the kernel of the total left derived functor $T \otimes_B^{\mathbb{L}} -$ is triangle equivalent to the derived module category $\mathscr{D}(C)$. Conversely, if the functor $T \otimes_B^{\mathbb{L}} -$ admits a fully faithful left adjoint functor, then T is a good tilting module. We apply our result to tilting modules arising from ring epimorphisms, and can then describe the rings C as coproducts of two relevant rings. Further, in case of commutative rings, we can weaken the condition of being tilting modules, strengthen the rings C as tensor products of two commutative rings, and get similar recollements. Consequently, we can produce a large variety of examples (from commutative algebra and p-adic number theory, or Kronecker algebra) to show that two different stratifications of the derived module category of a ring by derived module categories of rings may have completely different derived composition factors (even up to ordering and up to derived equivalence), or different lengths. This shows that the Jordan-Hölder theorem fails even for stratifications by derived module categories, and also answers negatively an open problem by Angeleri-Hügel, König and Liu.

Contents

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2010 Mathematics Subject Classification: Primary 18E30, 16G10, 13B30; Secondary 16S10, 13E05.

Keywords: Commutative ring; Coproduct; Derived category; p-adic number; Recollement; Ring epimorphism; Tilting module; Universal localization

1 Introduction

The theory of finitely generated tilting modules has been successfully applied, in the representation theory of algebras and groups, to understanding different aspects of algebraic structure and homological features of (algebraic) groups, algebras and modules (for instance, see [14, 16, 17, 22], [25]-[28]). Recently, infinitely generated tilting modules over arbitrary associated rings have become of interest in and attracted increasingly attentions toward to understanding derived categories and equivalences of general rings ([1]-[7], [9]-[11], [20, 21], [36]-[39]). In this general situation, many classical results in the tilting theory appear in a very different new fashion. For example, Happel's Theorem (see also [17]) on derived equivalences induced by infinitely generated tilting modules comes up with a new formulation in which quotient categories are involved (see [9]). This more general context of tilting theory not only renews our view on features of finitely generated tilting modules, but also provides us completely different information about the whole tilting theory. Let us recall the definition of tilting modules over an arbitrary ring from [20].

Let A be a ring with identity, and let T be a left A-module which may be infinitely generated. The module T is called a tilting module (of projective dimension at most 1) provided that

$(T1)$ T has projective dimension at most one,

$(T2)$ $\mathrm{Ext}^i_A(T,T^{(\alpha)})=0$ for each $i\geq 1$ and each cardinal α, and

$(T3)$ there exists an exact sequence $0\to A\to T_0\to T_1\to 0$ of left A-modules, where T_0 and T_1 are isomorphic to direct summands of arbitrary direct sums of copies of T.

If, in addition, T is finitely presented, then we say that T is a classical tilting module. If the modules T_0 and T_1 in $(T3)$ are isomorphic to direct summands of finite direct sums of copies of T, then we say that T is a good tilting module, following [11]. Actually, each classical tilting module is good, furthermore, it is proved in [11] that, for an arbitrary tilting A-module T, there exists a good tilting A-module T' which is equivalent to T, that is, T and T' generate the same full subcategories in the category of all left A-modules.

One of the realizations of tilting modules is universal localizations. It is shown in [1] that every tilting module over a ring is associated in a canonical manner with a ring epimorphism which can be interpreted as a universal localization at a set of homomorphisms between finitely presented modules of projective dimension at most one.

As in the theory of classical tilting modules, a natural context for studying infinitely generated tilting modules is the relationship of derived categories and equivalences induced by infinitely generated tilting modules. In fact, if T is a good tilting module over a ring A, and if B is the endomorphism ring of T, then Bazzoni proves in [9] that the total right derived functor $\mathbb{R}\mathrm{Hom}_A(T,-)$ induces an equivalence between the (unbounded) derived category $\mathscr{D}(A)$ of A and the quotient category of the derived category $\mathscr{D}(B)$ of B modulo the full triangulated subcategory $\mathrm{Ker}(T\otimes^{\mathbb{L}}_B-)$ which is the kernel of the total left derived functor $T\otimes^{\mathbb{L}}_B-$. Thus, in general, the total right derived functor $\mathbb{R}\mathrm{Hom}_A(T,-)$ does not define a derived equivalence between A and B. This is a contrary phenomenon to the classical situation (see [17]). The condition for A and B to be derived-equivalent depends on the vanishing of $\mathrm{Ker}(T\otimes^{\mathbb{L}}_B-)$. It is shown in [9] that $\mathrm{Ker}(T\otimes^{\mathbb{L}}_B-)$ vanishes if and only if T is a classical tilting module. From this point of view, the triangulated category $\mathrm{Ker}(T\otimes^{\mathbb{L}}_B-)$ measures how far a good tilting module is from being classical, in other words, the difference between the two derived categories $\mathscr{D}(A)$ and $\mathscr{D}(B)$. It is certainly of interest to have a little bit knowledge about the categories $\mathrm{Ker}(T\otimes^{\mathbb{L}}_B-)$ for infinitely generated tilting modules T. This might help us to understand some new aspects of the tilting theory of infinitely generated tilting modules.

The main purpose of this paper is to give a characterization of the triangulated categories $\mathrm{Ker}(T\otimes^{\mathbb{L}}_B-)$ for infinitely generated tilting modules T, namely, we show that if the tilting module T is good then the triangulated category $\mathrm{Ker}(T\otimes^{\mathbb{L}}_B-)$ is equivalent to the derived category of a ring C, and therefore, there is a recollemment among the derived categories of rings A, B and C. Conversely, the existence of such a recollement implies that the given tilting module T is good. More precisely, our result can be stated as follows:

Theorem 1.1. *Let A be a ring, T a tilting A-module of projective dimension at most 1 and B the endomorphism ring of T.*

(1) *If T is good, then there is a ring C, a homological ring epimorphism $\lambda : B \to C$ and a recollement among the unbounded derived categories of the rings A, B and C:*

$$\begin{tikzcd} \mathscr{D}(C) \arrow[r] & \mathscr{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "j^!"] & \mathscr{D}(A) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

such that the triangle functor $j^!$ is isomorphic to the total left derived functor ${}_AT \otimes^{\mathbb{L}}_B -$. In this case, the kernel of the functor $T \otimes^{\mathbb{L}}_B -$ is equivalent to the unbounded derived category $\mathscr{D}(C)$ of C as triangulated categories.

(2) *If the triangle functor $T \otimes^{\mathbb{L}}_B - : \mathscr{D}(B) \to \mathscr{D}(A)$ admits a fully faithful left adjoint $j_! : \mathscr{D}(A) \to \mathscr{D}(B)$, then the given tilting module T is good.*

Let us remark that a noteworthy difference of Theorem 1.1(1) from the result [4, Proposition 1.7] is that our recollement is over derived module categories of precisely determined rings, while the recollement in [4, Proposition 1.7] involves a triangulated category. Theorem 1.1(1) realizes this abstract triangulated category by a derived module category via describing the kernel of the functor $T \otimes^{\mathbb{L}}_B -$. Our result also distinguishes itself from the one in [41] where C is a differential graded ring instead of a usual ring, and where the consideration is restricted to ground ring being a field.

If we apply Theorem 1.1 to tilting modules arising from ring epimorphisms, then we can see that, in most cases, the recollements given in Theorem 1.1 are different from the usual ones induced from the structure of triangular matrix rings. The following corollary is a consequence of the proof of Theorem 1.1.

Corollary 1.2. (1) *Let $R \to S$ be an injective ring epimorphism such that $\mathrm{Tor}^R_1(S,S) = 0$ and that ${}_RS$ has projective dimension at most one. Then there is a recollement of derived module categories:*

$$\begin{tikzcd} \mathscr{D}(S \sqcup_R S') \arrow[r] & \mathscr{D}(\mathrm{End}_R(S \oplus S/R)) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd},$$

where S' is the endomorphism ring of the R-module S/R, and $S \sqcup_R S'$ is the coproduct of S and S' over R.

(2) *Suppose that $\lambda : R \to S$ is an injective homological ring epimorphism between commutative rings R and S. Then there is a recollement of derived module categories:*

$$\begin{tikzcd} \mathscr{D}(S \otimes_R S') \arrow[r] & \mathscr{D}(\mathrm{End}_R(S \oplus S/R)) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd},$$

where $S' := \mathrm{End}_R(S/R)$ is a commutative ring, and $S \otimes_R S'$ is the tensor product of S and S' over R.

(3) *For every prime number $p \geq 2$, the derived category of the ring $\begin{pmatrix} \mathbb{Q} & \mathbb{Q}_p \\ 0 & \mathbb{Z}_p \end{pmatrix}$ admits two stratifications, one of which clearly has composition factors $\mathbb{Q}$ and $\mathbb{Z}_p$, and the other has composition factors $\mathbb{Q}_{(p)}$ and $\mathbb{Q}_p$, where $\mathbb{Q}_{(p)}$, $\mathbb{Q}$, $\mathbb{Z}_p$ and $\mathbb{Q}_p$ denote the rings of p-integers, rational numbers, p-adic integers and p-adic numbers, respectively.*

As pointed out in [5], the Jordan-Hölder theorem fails for stratifications of derived module categories by triangulated categories. Our Corollary 1.2(3) (see also the example in Section 8 below) shows that the Jordan-Hölder theorem fails even for stratifications of derived module categories by derived module categories, and therefore the problem posed in [5] gets a negative answer.

The paper is organized as follows: In Section 2, we recall some definitions, notations and useful results which are needed for our proofs. In Section 3, we shall first establish a connection between universal localizations and recollements of triangulated categories, and then prove Proposition 3.6 which is crucial for the proof of the main result. In Section 4, we discuss some homological properties of good tilting modules, and establish another crucial result, Proposition 4.6, for the proof of the main result Theorem 1.1. After these preparations, we apply the results obtained in Section 3 to prove Theorem 1.1(1). In Section 5, we prove the second part of Theorem 1.1. This may be regarded as a converse statement of the first part. In Section 6, we apply Theorem 1.1 to good tilting modules arising from ring epimorphisms, and prove Corollary 1.2(1). In these cases the universal localization rings in Theorem 1.1 can be given by coproducts of rings. Our discussion in this section is actually carried out under the general assumption of injective homological ring epimorphisms. In Section 7, we first consider the existence of the recollements in Theorem 1.1 for commutative rings without assumption that the involved modules are tilting modules, and then make special consideration of localizations of commutative one-Gorenstein rings. In particular, we prove Corollary 1.2(2) and Corollary 1.2(3). It turns out that many derived module categories of rings possess stratifications by derived module categories of rings, such that, even up to ordering and up to derived equivalence, not all of their composition factors are the same; for instance, the derived category of the endomorphism ring of the abelian group $\mathbb{Q}\oplus\mathbb{Q}/\mathbb{Z}$ (or its variation $\mathbb{Q}\oplus\mathbb{Q}/\mathbb{Q}_{(p)}$). Note that, in the examples presented in this section, the two stratifications all have the same lengths. In Section 8, we give an example of a non-commutative algebra over which the derived category of the endomorphism ring of a tilting module has two stratifications of different finite lengths. This, together with the examples in Section 7, gives a complete answer to an open problem in [5] negatively.

The research work of the corresponding author C.C. Xi is partially supported by the Fundamental Research Funds for the Central Universities (2009SD-17), while the author H.X. Chen is partially supported by the Doctor Funds of the Beijing Normal University. Also, C.C. Xi thanks Lidia Angeleri-Hügel for some discussions on localizations of commutative rings.

2 Preliminaries

In this section, we shall recall some definitions, notations and basic results which are related to our proofs. In particular, we recall the notions of recollements and TTF triples as well as their relationship.

2.1 Some conventions

In this subsection, we recall some standard notations which will be used throughout this paper.

All rings considered in this paper are assumed to be associative and with identity, and all ring homomorphisms preserve identity.

Let A be a ring. We denote by A-Mod the category of all unitary left A-modules. For an A-module M, we denote by $\mathrm{add}(M)$ (respectively, $\mathrm{Add}(M)$) the full subcategory of A-Mod consisting of all direct summands of finite (respectively, arbitrary) direct sums of copies of M. In many circumstances, we shall write A-proj and A-Proj for $\mathrm{add}(_AA)$ and $\mathrm{Add}(_AA)$, respectively. If I is an index set, we denote by $M^{(I)}$ the direct sum of I copies of M. If there is a surjective homomorphism from $M^{(I)}$ to an A-module X, we say that X is generated by M, or M generates X. By $\mathrm{Gen}(M)$ we denote the full subcategory of A-Mod generated by M.

If $f: M\to N$ is a homomorphism of A-modules, then the image of $x\in M$ under f is denoted by $(x)f$ instead of $f(x)$. Also, for any A-module X, the induced morphisms $\mathrm{Hom}_A(X,f): \mathrm{Hom}_A(X,M)\to \mathrm{Hom}_A(X,N)$ and $\mathrm{Hom}_A(f,X): \mathrm{Hom}_A(N,X)\to \mathrm{Hom}_A(M,X)$ is denoted by f^* and f_*, respectively.

Let $\mathcal{C}$ be an additive category.

Given two morphisms $f: X\to Y$ and $g: Y\to Z$ in $\mathcal{C}$, we denote the composition of f and g by fg which is a morphism from X to Z, while we denote the composition of a functor $F:\mathcal{C}\to\mathcal{D}$ between categories $\mathcal{C}$ and $\mathcal{D}$ with a functor $G:\mathcal{D}\to\mathcal{E}$ between categories $\mathcal{D}$ and $\mathcal{E}$ by GF which is a functor from $\mathcal{C}$ to $\mathcal{E}$. The image of the functor F is denoted by $\mathrm{Im}(F)$ which is a full subcategory of $\mathcal{D}$.

Throughout the paper, a full subcategory $\mathcal{D}$ of $\mathcal{C}$ is always assumed to be closed under isomorphisms, that is, if X and Y are objects in $\mathcal{C}$, then $Y\in\mathcal{D}$ whenever $Y\simeq X$ with $X\in\mathcal{D}$.

Let $\mathcal{Y}$ be a full subcategory of $\mathcal{C}$. By $\mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(-,\mathcal{Y}))$ we denote the left orthogonal subcategory with respect to $\mathcal{Y}$, that is, the full subcategory of $\mathcal{C}$ consisting of the objects X such that $\mathrm{Hom}_{\mathcal{C}}(X,Y)=0$ for all objects Y in $\mathcal{Y}$. Similarly, $\mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(\mathcal{Y},-))$ stands for the right orthogonal subcategory of $\mathcal{C}$ with respect to $\mathcal{Y}$.

By a complex $X^\bullet$ over $\mathcal{C}$ we mean a sequence of morphisms d_X^i between objects X^i in $\mathcal{C}$: $\cdots \to X^i \xrightarrow{d_X^i} X^{i+1} \xrightarrow{d_X^{i+1}} X^{i+2} \to \cdots$, such that $d_X^i d_X^{i+1}=0$ for all $i\in\mathbb{Z}$. In this case, we write $X^\bullet=(X^i,d_X^i)_{i\in\mathbb{Z}}$, and call d_X^i a differential of $X^\bullet$. Sometimes, for simplicity, we write $(X^i)_{i\in\mathbb{Z}}$ for $X^\bullet$ without mentioning the morphisms d_X^i. For a fixed integer n, we denote by $X^\bullet[n]$ the complex obtained from $X^\bullet$ by shifting n degrees, that is, $(X^\bullet[n])^0=X^n$, and by $H^n(X^\bullet)$ the cohomology of $X^\bullet$ in degree n.

Let $\mathscr{C}(\mathcal{C})$ be the category of all complexes over $\mathcal{C}$ with chain maps, and $\mathscr{K}(\mathcal{C})$ the homotopy category of $\mathscr{C}(\mathcal{C})$. We denote by $\mathscr{C}^b(\mathcal{C})$ and $\mathscr{K}^b(\mathcal{C})$ the full subcategories of $\mathscr{C}(\mathcal{C})$ and $\mathscr{K}(\mathcal{C})$ consisting of bounded complexes over $\mathcal{C}$, respectively. When $\mathcal{C}$ is abelian, the derived category of $\mathcal{C}$ is denoted by $\mathscr{D}(\mathcal{C})$, which is the localization of $\mathscr{K}(\mathcal{C})$ at all quasi-isomorphisms. The full subcategory of $\mathscr{D}(\mathcal{C})$ consisting of bounded complexes over $\mathcal{C}$ is denoted by $\mathscr{D}^b(\mathcal{C})$. As usual, for a ring A, we simply write $\mathscr{C}(A)$ for $\mathscr{C}(A\text{-Mod})$, $\mathscr{K}(A)$ for $\mathscr{K}(A\text{-Mod})$, $\mathscr{C}^b(A)$ for $\mathscr{C}^b(A\text{-Mod})$, and $\mathscr{K}^b(A)$ for $\mathscr{K}^b(A\text{-Mod})$. Similarly, we write $\mathscr{D}(A)$ and $\mathscr{D}^b(A)$ for $\mathscr{D}(A\text{-Mod})$ and $\mathscr{D}^b(A\text{-Mod})$, respectively. Furthermore, we always identify A-Mod with the full subcategory of $\mathscr{D}(A)$ consisting of all stalk complexes concentrated on degree zero.

Now we recall some basic facts about derived functors defined on derived module categories. We refer to [15] for details and proofs.

Let R and S be rings, and let H be an additive functor from R-Mod to S-Mod.

(1) For each complex $X^\bullet$ in $\mathscr{D}(R)$, there is a complex $I^\bullet\in\mathscr{C}(R\text{-Inj})$ such that $X^\bullet$ is quasi-isomorphic to $I^\bullet$, where R-Inj is the full subcategory of R-Mod consisting of all injective R-modules. Dually, for each complex $Y^\bullet$ in $\mathscr{D}(R)$, there is a complex $P^\bullet\in\mathscr{C}(R\text{-Proj})$ such that $P^\bullet$ is quasi-isomorphic to $Y^\bullet$.

(2) There is a total right derived functor $\mathbb{R}H$ and a total left derived functor $\mathbb{L}H$ defined on $\mathscr{D}(R)$. If $X^\bullet, Y^\bullet\in\mathscr{D}(R)$, then $\mathbb{R}H(X^\bullet)=H(I^\bullet)$ and $\mathbb{L}H(Y^\bullet)=H(P^\bullet)$, where $I^\bullet$ and $P^\bullet$ are chosen as in (1). Here we think of H as an induced functor between homotopy categories, and if $X^\bullet=(X^i,d_X^i)_{i\in\mathbb{Z}}$ then $H(X^\bullet):=\big(H(X^i),H(d_X^i)\big)_{i\in\mathbb{Z}}$.

In case T is an R-S-bimodule, the total right derived functor of $\mathrm{Hom}_R(T,-)$ is denoted by $\mathbb{R}\mathrm{Hom}_R(T,-)$, and the total left derived functor of $T\otimes_S -$ is denoted by $T\otimes_S^{\mathbb{L}} -$.

(3) Any adjoint pair of additive functors (G,H) between R-Mod and S-Mod induces an adjoint pair $(\mathbb{L}G,\mathbb{R}H)$ between the unbounded derived categories of R and S.

2.2 Homological ring epimorphisms

Let R and S be rings. Recall that a homomorphism $\lambda: R\to S$ of rings is called a ring epimorphism if, for any two homomorphisms $f_1,f_2: S\to T$ of rings, the equality $\lambda f_1=\lambda f_2$ implies that $f_1=f_2$. It is known that λ is a ring epimorphism if and only if the multiplication map $S\otimes_R S\to S$ is an isomorphism as S-S-bimodules if and only if $x\otimes 1=1\otimes x$ in $S\otimes_R S$ for any $x\in S$. It follows that, for a ring epimorphism, we have $X\otimes_S Y\simeq X\otimes_R Y$ for any S-modules X_S and ${}_SY$. An example of ring epimorphisms is the inclusion $\mathbb{Z}\hookrightarrow\mathbb{Q}$. Note that $\mathbb{Q}$ is an injective and a flat $\mathbb{Z}$-module.

Given a ring epimorphism $\lambda: R\to S$ between two rings R and S, we can regard S-Mod as a full subcategory of R-Mod via λ. This means that $\mathrm{Hom}_S(X,Y)\simeq\mathrm{Hom}_R(X,Y)$ for all S-modules X and Y.

Two ring epimorphisms $\lambda: R\to S$ and $\lambda': R\to S'$ are said to be equivalent if there is a ring isomorphism $\psi: S\to S'$ such that $\lambda'=\lambda\psi$. This defines an equivalence relation on the class of ring epimorphisms $R\to S$ with R fixed. The equivalence classes with respect to this equivalence relation are called the epiclasses of R. This notion is associated with bireflective subcategories of module categories.

Recall that a full subcategory $\mathcal{D}$ of R-Mod is said to be reflective if every R-module X admits a $\mathcal{D}$-reflection, that is, there exists an R-module $D' \in \mathcal{D}$ and a homomorphism $f : D' \to X$ of R-modules such that $\mathrm{Hom}_R(D, f) : \mathrm{Hom}_R(D, D') \to \mathrm{Hom}_R(D, X)$ is an isomorphism as abelian groups for any module $D \in \mathcal{D}$. Dually, one defines the notion of coreflective subcategories of R-Mod. The full subcategory $\mathcal{D}$ of R-Mod is called bireflective if it is both reflective and coreflective.

Ring epimorphisms are related to bireflective subcategories in the following way.

Lemma 2.1. [1, Theorm 1.4] *For a full subcategory $\mathcal{D}$ of R-Mod, the following statements are equivalent.*

(1) *There is a ring epimorphism $\lambda : R \to S$ such that the category $\mathcal{D}$ is the image of the restriction functor $\lambda_* : S$-Mod $\to R$-Mod.*

(2) *$\mathcal{D}$ is a bireflective subcategory of R-Mod.*

(3) *$\mathcal{D}$ is closed under direct sums, products, kernels and cokernels.*

Thus, there is a bijection between the epiclasses of R and the bireflective subcategories of R-Mod. Furthermore, the map $\lambda : R \to S$ in (1), viewed as a homomorphism of R-modules, is a $\mathcal{D}$-reflection of R.

Following Geigle and Lenzing [24], we say that a ring epimorphism $\lambda : R \to S$ is homological if $\mathrm{Tor}_i^R(S, S) = 0$ for all $i > 0$. This is equivalent to saying that the restriction functor $\lambda_* : \mathscr{D}(S) \to \mathscr{D}(R)$ induced by λ is fully faithful. In [24, Theorem 4.4], the following lemma is proved.

Lemma 2.2. *For a homomorphism $\lambda : R \to S$ of rings, the following assertions are equivalent:*

(1) *λ is homological,*

(2) *For all right S-modules X and all left S-modules Y, the natural map $\mathrm{Tor}_i^R(X,Y) \to \mathrm{Tor}_i^S(X,Y)$ is an isomorphism for all $i \geq 0$.*

(3) *For all S-modules X and Y, the natural map $\mathrm{Ext}_S^i(X,Y) \to \mathrm{Ext}_R^i(X,Y)$ is an isomorphism for all $i \geq 0$.*

Note that the condition (3) in Lemma 2.2 can be replaced by the corresponding version of right modules. For more details, one may look at [24] and [35, Section 5.3].

On ring epimorphisms, we have the following property which will be used in Section 7.

Lemma 2.3. *Let $g : \Lambda \to \Gamma$ and $h : \Gamma \to \Delta$ be ring homomorphisms such that $gh : \Lambda \to \Delta$ is a ring epimorphism. Then h is a ring epimorphism. Suppose further that h is injective. If Γ_Λ and ${}_\Lambda\Delta$ (respectively, ${}_\Lambda\Gamma$ and Δ_Λ) are flat modules, then both g and h are homological ring epimorphisms.*

Proof. By the definition of ring epimorphisms, we can readily show that h is a ring epimorphism. Note that we always have the following commutative diagram:

$$\begin{tikzcd} \Gamma \otimes_\Lambda \Gamma \arrow[r, "h\otimes_\Lambda h"] \arrow[d, "\mu_1"'] & \Delta \otimes_\Lambda \Delta \arrow[d, "\mu_2"'] \\ \Gamma \arrow[r, "h"] & \Delta, \end{tikzcd}$$

where μ_1 and μ_2 are the canonical multiplication maps. Suppose that h is injective. If Γ_Λ and ${}_\Lambda\Delta$ are flat modules, then the map $h \otimes_\Lambda h$ is injective. Since $gh : \Lambda \to \Delta$ is a ring epimorphism, the map μ_2 is an isomorphism. It follows that μ_1 is injective, and therefore it is an isomorphism. This means that $g : \Lambda \to \Gamma$ is a ring epimorphism. Note that Γ_Λ is a flat module. Thus g is a homological ring epimorphism. To prove that h also is a homological ring epimorphism, we claim that the module ${}_\Gamma\Delta$ is flat. In fact, this follows from Lemma 2.2 because g is a homological ring epimorphism and because Δ is flat as a Λ-module. Similarly, we can prove that if ${}_\Lambda\Gamma$ and Δ_Λ are flat, then both g and h are homological ring epimorphisms. $\square$

2.3 Recollements and TTF triples

In this subsection, we first recall the definitions of recollements and TTF triples, and then state a correspondence between them.

From now on, $\mathcal{D}$ denotes a triangulated category with small coproducts (that is, coproducts indexed over sets exist in $\mathcal{D}$), and with $[1]$ the shift functor of $\mathcal{D}$.

The notion of recollements was first defined by Beilinson, Bernstein and Deligne in [12] to study "exact sequences" of derived categories of coherent sheaves over geometric objects.

Definition 2.4. Let $\mathcal{D}'$ and $\mathcal{D}''$ be triangulated categories. We say that $\mathcal{D}$ is a recollement of $\mathcal{D}'$ and $\mathcal{D}''$ if there are six triangle functors as in the following diagram

$$\begin{tikzcd} \mathcal{D}'' \arrow[r, "i_*=i_!"] & \mathcal{D} \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^!=j^*"] & \mathcal{D}' \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

such that

(1) $(i^*, i_*), (i_!, i^!), (j_!, j^!)$ and (j^*, j_*) are adjoint pairs;

(2) i_*, j_* and $j_!$ are fully faithful functors;

(3) $i^! j_* = 0$ (and thus also $j^! i_! = 0$ and $i^* j_! = 0$); and

(4) for each object $C \in \mathcal{D}$, there are two triangles in $\mathcal{D}$:

$$i_! i^!(C) \longrightarrow C \longrightarrow j_* j^*(C) \longrightarrow i_! i^!(C)[1],$$

$$j_! j^!(C) \longrightarrow C \longrightarrow i_* i^*(C) \longrightarrow j_! j^!(C)[1].$$

Recollements are closely related to TTF triples which are defined in terms of torsion pairs. So, let us first recall the notion of torsion pairs in triangulated categories.

Definition 2.5. [14] A torsion pair in $\mathcal{D}$ is a pair $(\mathcal{X}, \mathcal{Y})$ of full subcategories $\mathcal{X}$ and $\mathcal{Y}$ of $\mathcal{D}$ satisfying the following conditions:

(1) $\mathrm{Hom}_{\mathcal{D}}(\mathcal{X}, \mathcal{Y}) = 0$;

(2) $\mathcal{X}[1] \subseteq \mathcal{X}$ and $\mathcal{Y}[-1] \subseteq \mathcal{Y}$; and

(3) for each object $C \in \mathcal{D}$, there is a triangle

$$X_C \longrightarrow C \longrightarrow Y^C \longrightarrow X_C[1]$$

in $\mathcal{D}$ such that $X_C \in \mathcal{X}$ and $Y^C \in \mathcal{Y}$. In this case, $\mathcal{X}$ is called a torsion class and $\mathcal{Y}$ is called a torsion-free class. If, in addition, $\mathcal{X}$ is a triangulated subcategory of $\mathcal{D}$ (or equivalently, $\mathcal{Y}$ is a triangulated subcategory of $\mathcal{D}$), then the torsion pair $(\mathcal{X}, \mathcal{Y})$ is said to be hereditary (see [14, Chapter I, Proposition 2.6]).

Note that, if $(\mathcal{X}, \mathcal{Y})$ is a torsion pair in $\mathcal{D}$, then $\mathcal{X} = \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(-, \mathcal{Y}))$ which is closed under small coproducts, and $\mathcal{Y} = \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(\mathcal{X}, -))$ which is closed under small products.

Definition 2.6. [14] A torsion torsionfree triple, or TTF triple for short, in $\mathcal{D}$ is a triple $(\mathcal{X}, \mathcal{Y}, \mathcal{Z})$ of full subcategories $\mathcal{X}, \mathcal{Y}$ and $\mathcal{Z}$ of $\mathcal{D}$ such that both $(\mathcal{X}, \mathcal{Y})$ and $(\mathcal{Y}, \mathcal{Z})$ are torsion pairs. In this case, $\mathcal{X}$ is said to be a smashing subcategory of $\mathcal{D}$.

It follows from [14, Chapter I.2.] that, associated with a TTF triple $(\mathcal{X},\mathcal{Y},\mathcal{Z})$ in $\mathcal{D}$, there are seven triangle functors demonstrated in the following diagram

$$\begin{tikzcd}
\mathcal{X} \arrow[r, bend left, "\mathbf{i}"] & \mathcal{D} \arrow[l, bend left, "\mathbf{R}"] \arrow[r, bend left, "\mathbf{L}"] & \mathcal{Y} \arrow[l, bend left, "\mathbf{j}"] & \mathcal{D} \arrow[l, bend left, "\mathbf{V}"] \arrow[r, bend left, "\mathbf{U}"] & \mathcal{Z} \arrow[l, bend left, "\mathbf{k}"]
\end{tikzcd}$$

such that

(1) $\mathbf{i},\mathbf{j}$ and $\mathbf{k}$ are canonical inclusions; and

(2) $(\mathbf{i},\mathbf{R}),(\mathbf{L},\mathbf{j}),(\mathbf{j},\mathbf{V})$ and $(\mathbf{U},\mathbf{k})$ are adjoint pairs; and

(3) the composition functor $\mathbf{Ui}:\mathcal{X}\to\mathcal{Z}$ of th functors $\mathbf{i}$ and $\mathbf{U}$ is a triangle equivalence with the quasi-inverse functor $\mathbf{Rk}$ which is the composition of the functors $\mathbf{k}$ and $\mathbf{R}$.

Note that if $(\mathcal{X},\mathcal{Y},\mathcal{Z})$ is a TTF triple in $\mathcal{D}$, then it is easy to check that $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ are automatically triangulated subcategories of $\mathcal{D}$.

Observe also that the existence of the functors $\mathbf{R}$ and $\mathbf{L}$ in the above diagram follows from the fact that $(\mathcal{X},\mathcal{Y})$ is a torsion pair in $\mathcal{D}$ (see [14, Chapter I, Proposition 2.3] for details). Furthermore, $\mathcal{Y}$ is closed under small coproducts and products.

Now, we state a correspondence between recollements and TTF triples given in [30, Section 9.2] and [35, Section 4.2]. For more details, we refer the reader to these papers.

Lemma 2.7. (1) *If $\mathcal{D}$ is a recollement of $\mathcal{D}'$ and $\mathcal{D}''$ in Definition 2.4, then $\big(j_!(\mathcal{D}'),i_*(\mathcal{D}''),j_*(\mathcal{D}')\big)$ is a* TTF *triple in $\mathcal{D}$.*

(2) *If $(\mathcal{X},\mathcal{Y},\mathcal{Z})$ is a* TTF *triple in $\mathcal{D}$, then $\mathcal{D}$ is a recollement of $\mathcal{X}$ and $\mathcal{Y}$ as follows:*

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, "\mathbf{j}"] & \mathcal{D} \arrow[l, bend right, "\mathbf{L}"'] \arrow[l, bend left, "\mathbf{V}"] \arrow[r, "\mathbf{R}"] & \mathcal{X} \arrow[l, bend right, "\mathbf{i}"'] \arrow[l, bend left, "\mathbf{kUi}"]
\end{tikzcd}.$$

2.4 Generators and compact objects

In this subsection, we shall recall some definitions and facts on generators in triangulated categories.

Given a class of objects $\mathcal{U}$ in $\mathcal{D}$, we denote by $\mathrm{Tria}(\mathcal{U})$ the smallest full triangulated subcategory of $\mathcal{D}$ which contains $\mathcal{U}$ and is closed under small coproducts. If $\mathcal{U}$ consists of only one single object U, then we simply write Tria(U) for Tria({U}).

Definition 2.8. A class $\mathcal{U}$ of objects in $\mathcal{D}$ is called a class of generators of $\mathcal{D}$ if an object D in $\mathcal{D}$ is zero whenever $\mathrm{Hom}_{\mathcal{D}}(U[n],D)=0$ for every object U of $\mathcal{U}$ and every n in $\mathbb{Z}$.

An object P in $\mathcal{D}$ is called compact if the functor $\mathrm{Hom}_{\mathcal{D}}(P,-)$ preserves small coproducts, that is, $\mathrm{Hom}_{\mathcal{D}}(P,\oplus_{i\in I}X_i)\simeq\oplus_{i\in I}\mathrm{Hom}_{\mathcal{D}}(P,X_i)$, where I is a set; and exceptional if $\mathrm{Hom}_{\mathcal{D}}(P,P[i])=0$ for all $i\neq 0$. The object P is called a tilting object if P is compact, exceptional and a generator of $\mathcal{D}$. Note that, for a compact generator P, we have $\mathrm{Tria}(P)=\mathcal{D}$ (see [35], for instance).

The category $\mathcal{D}$ is said to be compactly generated if $\mathcal{D}$ admits a set $\mathcal{V}$ of compact generators. In this case, $\mathcal{D}=\mathrm{Tria}(\mathcal{V})$, and we say that $\mathcal{D}$ is compactly generated by $\mathcal{V}$.

It is well-known that, for a ring A, the unbounded derived category $\mathscr{D}(A)$ is a compactly generated triangulated category, and one of its compact generators is ${}_AA$. Moreover, a complex $P^\bullet\in\mathscr{D}(A)$ is compact if and only if it is quasi-isomorphic to a bounded complex of finitely generated projective A-modules. The importance of compact objects can be seen from the following lemma, due to Keller in [28, Corollary 8.4, Theorem 8.5].

Lemma 2.9. *Let A be a ring. If $P^\bullet$ is a compact, exceptional object in $\mathscr{D}(A)$, then* $\mathrm{Tria}(P^\bullet)$ *is equivalent to* $\mathscr{D}(\mathrm{End}_{\mathscr{D}(R)}(P^\bullet))$ *as triangulated categories.*

The following result is proved in [13, Proposition 5.14], which shows that, under certain natural assumptions, torsion pairs in compactly generated triangulated categories can be lifted to TTF triples.

Lemma 2.10. *Let $\mathcal{C}$ be a compactly generated triangulated category which admits all small coproducts and products. Suppose that $(\mathcal{Y},\mathcal{Z})$ is a hereditary torsion pair in $\mathcal{C}$. Then we have the following.*

(1) *If $\mathcal{Y}$ is closed under all small products, then there exists a* TTF *triple $(\mathcal{X},\mathcal{Y},\mathcal{Z})$ in $\mathcal{C}$. In this case, $\mathcal{Y}$ is compactly generated.*

(2) *If $\mathcal{Z}$ is closed under all small coproducts, then there exists a* TTF *triple $(\mathcal{Y},\mathcal{Z},\mathcal{W})$ in $\mathcal{C}$. In this case, $\mathcal{Z}$ is compactly generated.*

The relationship between compact objects and TTF triples is explained in the next result, which states that any set of compact objects in a triangulated category with small coproducts gives rise to a TTF triple. For more details, we refer the reader to [14, Chapter III, Theorem 2.3; Chapter IV, Proposition 1.1].

Lemma 2.11. *Let $\mathcal{C}$ be a triangulated category which admits all small coproducts. Suppose that $\mathcal{P}$ is a set of compact objects in $\mathcal{C}$. Set $\mathcal{X} := \mathrm{Tria}(\mathcal{P}), \mathcal{Y} := \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(\mathcal{X},-))$ and $\mathcal{Z} := \mathrm{Ker}(\mathrm{Hom}_{\mathcal{C}}(\mathcal{Y},-))$. Then $(\mathcal{X},\mathcal{Y},\mathcal{Z})$ is a* TTF *triple in $\mathcal{C}$. Moreover, $\mathcal{Y}$ coincides with the full subcategory of $\mathcal{C}$ consisting of the objects Y such that $\mathrm{Hom}_{\mathcal{C}}(P[n],Y) = 0$ for every $P \in \mathcal{P}$ and $n \in \mathbb{Z}$.*

3 Universal localizations and recollements

In this section, we shall further generalize and develop some known results and connections between universal localizations and recollements of triangulated categories in literature. In this consideration, homological ring epimorphisms and perpendicular categories will play a role.

Now, we fix a ring R, and suppose that Σ is a set of homomorphisms between finitely generated projective R-modules. For each $f : P^{-1} \to P^0$ in Σ, we denote by $P_f^\bullet$ the following complex of R-modules:

$$\cdots \longrightarrow 0 \longrightarrow P^{-1} \xrightarrow{f} P^0 \longrightarrow 0 \longrightarrow \cdots,$$

where P^{-1} and P^0 are of degrees -1 and 0, respectively.

Set

$$\Sigma^\bullet := \{P_f^\bullet \mid f \in \Sigma\},$$

$$\Sigma^\perp := \{X \in R\text{-Mod} \mid \mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, X[i]) = 0 \text{ for all } P^\bullet \in \Sigma^\bullet \text{ and all } i \in \mathbb{Z}\},$$

$$\mathscr{D}(R)_{\Sigma^\perp} := \{Y^\bullet \in \mathscr{D}(R) \mid H^n(Y^\bullet) \in \Sigma^\perp \text{ for all } n \in \mathbb{Z}\},$$

where $H^n(Y^\bullet)$ is the n-th cohomology of the complex $Y^\bullet$. Note that some special cases of $\Sigma^\perp$ have been discussed in literature (see, for example, [1, 4, 21, 24]). For example, the set Σ consists of injective homomorphisms or only one single homomorphism. In some papers, such a category $\Sigma^\perp$ is called the perpendicular category of Σ.

Universal localizations were pioneered by Ore and Cohn, in order to study embedding of noncommutative rings in skew fields.

Before recalling the definition of universal localizations, we mention the following result, due initially to Cohen (see also [37]), which explains how universal localizations arise.

Lemma 3.1. [19] *Let R and Σ be as above. Then there is a ring R_Σ and a homomorphism $\lambda : R \to R_\Sigma$ of rings with the following properties:*

(1) *λ is Σ-inverting, that is, if $\alpha : P \to Q$ belongs to Σ, then $R_\Sigma \otimes_R \alpha : R_\Sigma \otimes_R P \to R_\Sigma \otimes_R Q$ is an isomorphism of R_Σ-modules, and*

(2) *λ is universal Σ-inverting, that is, if S is a ring such that there exists a Σ-inverting homomorphism $\varphi : R \to S$, then there exists a unique homomorphism $\psi : R_\Sigma \to S$ of rings such that $\varphi = \lambda\psi$.*

The homomorphism $\lambda : R \to R_\Sigma$ in Lemma 3.1 is a ring epimorphism with $\mathrm{Tor}_1^R(R_\Sigma, R_\Sigma) = 0$. It is called the universal localization of R at Σ.

The left-right symmetry holds for universal localizations (see [37, Chapter 4, p.51-52]).

Lemma 3.2. *Let R_Σ be the universal localization of R at Σ in Lemma 3.1, and let $\Gamma := \{\mathrm{Hom}_R(f,R) \mid f \in \Sigma\}$ which is a set of homomorphisms between finitely generated projective right R-modules. Then R_Σ is isomorphic to the universal localization of R at Γ.*

The proof of this lemma is actually a consequence of the following two observations: (a) For any finitely generated projective R-module P, we have a natural isomorphism: $\mathrm{Hom}_R(P,R) \otimes_R - \simeq \mathrm{Hom}_R(P,-)$, and (b) the functor $\mathrm{Hom}_R(-, {}_RR) : \mathrm{add}({}_RR) \to \mathrm{add}(R_R)$ defines an equivalence of categories.

It is easy to see that if R has weak dimension at most 1, then the localization $\lambda : R \to R_\Sigma$ of R at any set Σ is homological, and moreover, the weak dimension of R_Σ is also at most 1 by Lemma 2.2.

If Σ is a finite set, then we may assume that Σ contains only one homomorphism since the universal localization at Σ is the same as the universal localization at the direct sum of the homomorphisms in Σ.

The following result is a more general formulation of the case discussed in [1] and [4]. Nevertheless, many arguments of the proof there work in this general situation. We outline here a modified proof.

Proposition 3.3. (1) *$\Sigma^\perp$ is closed under isomorphic images, extensions, kernels, cokernels, direct sums and products.*

(2) *$\Sigma^\perp$ coincides with the image of the restriction functor $\lambda_* : R_\Sigma\text{-Mod} \to R\text{-Mod}$ induced by the ring homomorphism λ defined in Lemma 3.1. In this sense, we can identify $\Sigma^\perp$ with R_Σ-Mod via λ.*

(3) $\mathscr{D}(R)_{\Sigma^\perp} = \mathrm{Ker}(\mathrm{Hom}_{\mathscr{D}(R)}(\mathrm{Tria}(\Sigma^\bullet), -))$.

In order to prove Proposition 3.3, we need the following known homological result.

Lemma 3.4. *Suppose that $W^\bullet = (W^i)_{i\in\mathbb{Z}}$ is a complex in $\mathscr{C}(A\text{-Proj})$ such that $W^i = 0$ for all $i \in \mathbb{Z}\backslash\{-1,0\}$. Then, for each $X^\bullet \in \mathscr{D}(R)$ and $n \in \mathbb{Z}$, there is an exact sequence of abelian groups:*

$$0 \longrightarrow \mathrm{Hom}_{\mathscr{D}(R)}(W^\bullet, H^{n-1}(X^\bullet)[1]) \longrightarrow \mathrm{Hom}_{\mathscr{D}(R)}(W^\bullet, X^\bullet[n]) \longrightarrow \mathrm{Hom}_{\mathscr{D}(R)}(W^\bullet, H^n(X^\bullet)) \longrightarrow 0.$$

Proof. It is sufficient to show the statement for $n = 0$. In this case, it follows from the triangle $W^{-1} \to W^0 \to W^\bullet \to W^{-1}[1]$ that the following diagram is commutative and exact:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{K}(R)}(W^0[1],X^\bullet) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}_{\mathscr{K}(R)}(W^{-1}[1],X^\bullet) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}_{\mathscr{K}(R)}(W^\bullet,X^\bullet) \arrow[r] & \mathrm{Hom}_{\mathscr{K}(R)}(W^0,X^\bullet) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Hom}_{\mathscr{K}(R)}(W^{-1},X^\bullet) \arrow[d, "\simeq"'] \\
\mathrm{Hom}_{\mathscr{K}(R)}(W^0,H^{-1}(X^\bullet)) \arrow[r] & \mathrm{Hom}_{\mathscr{K}(R)}(W^{-1},H^{-1}(X^\bullet)) & & \mathrm{Hom}_{\mathscr{K}(R)}(W^0,H^0(X^\bullet)) \arrow[r] & \mathrm{Hom}_{\mathscr{K}(R)}(W^{-1},H^0(X^\bullet)).
\end{tikzcd}$$

Here we use the fact that $\mathrm{Hom}_{\mathscr{D}(R)}(P, X^\bullet[n]) = \mathrm{Hom}_{\mathscr{K}(R)}(P, X^\bullet[n]) \simeq \mathrm{Hom}_R(P, H^n(X^\bullet))$ for every projective module P and $n \in \mathbb{Z}$. Thus Lemma 3.4 follows. $\square$

Proof of Proposition 3.3. (1) Clearly, $\Sigma^\perp$ is closed under isomorphic images and extensions. In the following, we shall prove that $\Sigma^\perp$ is closed under kernels and cokernels. Recall that $\Sigma^\perp$ is defined to be the full subcategory of R-Mod consisting of those R-modules X that $\mathrm{Hom}_{\mathscr{D}(R)}(U^\bullet, X) = \mathrm{Hom}_{\mathscr{D}(R)}(U^\bullet, X[1]) = 0$

for all $U^\bullet \in \Sigma^\bullet$. Suppose that $f : Y \to Z$ is a homomorphism between two modules Y and Z in $\Sigma^\perp$. Set $K := \mathrm{Ker}(f), I := \mathrm{Im}(f)$ and $C := \mathrm{Coker}(f)$. Then we have two exact sequences of R-modules:

$$0 \to K \to Y \to I \to 0 \quad \text{and} \quad 0 \to I \to Z \to C \to 0.$$

Since every short exact sequence in R-Mod can canonically be extended to a triangle in $\mathscr{D}(R)$, we get two triangles in $\mathscr{D}(R)$:

$$K \to Y \to I \to K[1] \quad \text{and} \quad I \to Z \to C \to I[1].$$

For convenience, we will write ${}_{\mathscr{D}(R)}(X^\bullet, Y^\bullet)$ for the Hom-set $\mathrm{Hom}_{\mathscr{D}(R)}(X^\bullet, Y^\bullet)$, with $X^\bullet, Y^\bullet \in \mathscr{D}(R)$. Let $P^\bullet \in \Sigma^\bullet$. Then, by applying ${}_{\mathscr{D}(R)}(P^\bullet, -)$ to these triangles, we obtain two long exact sequences of abelian groups

$$0 \to {}_{\mathscr{D}(R)}(P^\bullet, K) \to {}_{\mathscr{D}(R)}(P^\bullet, Y) \to {}_{\mathscr{D}(R)}(P^\bullet, I) \to {}_{\mathscr{D}(R)}(P^\bullet, K[1]) \to {}_{\mathscr{D}(R)}(P^\bullet, Y[1]) \to {}_{\mathscr{D}(R)}(P^\bullet, I[1] \to 0;$$

$$0 \to {}_{\mathscr{D}(R)}(P^\bullet, I) \to {}_{\mathscr{D}(R)}(P^\bullet, Z) \to {}_{\mathscr{D}(R)}(P^\bullet, C) \to {}_{\mathscr{D}(R)}(P^\bullet, I[1]) \to {}_{\mathscr{D}(R)}(P^\bullet, Z[1]) \to {}_{\mathscr{D}(R)}(P^\bullet, C[1] \to 0.$$

Since Y and Z lie in $\Sigma^\perp$, we know ${}_{\mathscr{D}(R)}(P^\bullet, Y) = {}_{\mathscr{D}(R)}(P^\bullet, Z) = {}_{\mathscr{D}(R)}(P^\bullet, Y[1]) = {}_{\mathscr{D}(R)}(P^\bullet, Z[1]) = 0$. It follows that ${}_{\mathscr{D}(R)}(P^\bullet, K) = {}_{\mathscr{D}(R)}(P^\bullet, I) = 0$, and so ${}_{\mathscr{D}(R)}(P^\bullet, K[1]) = 0$. This implies $K \in \Sigma^\perp$. Similarly, we can conclude that I and C belong to $\Sigma^\perp$. Hence $\Sigma^\perp$ is closed under kernels, images and cokernels. By the definition of $\Sigma^\perp$ and the fact that Hom-funcors commute with products, we infer that $\Sigma^\perp$ is closed under products. Since $\Sigma^\bullet$ is a set of bounded complexes over finitely generated projective R-modules, these complexes are compact, and therefore $\Sigma^\perp$ is closed under direct sums.

(2) Observe that, for each element $f : P^{-1} \to P^0$ in Σ, there is a canonical triangle in $\mathscr{D}(R)$:

$$(*) \quad P^{-1} \xrightarrow{f} P^0 \longrightarrow P_f^\bullet \longrightarrow P^{-1}[1].$$

If, in addition, f is injective, then we have a short exact sequence of R-modules:

$$(**) \quad 0 \longrightarrow P^{-1} \xrightarrow{f} P^0 \longrightarrow \mathrm{Coker}(f) \longrightarrow 0.$$

In this case, we get $P_f^\bullet \simeq \mathrm{Coker}(f)$ in $\mathscr{D}(R)$. Note that the same statement as (2) is obtained in [1, Lemma 1.6, Proposition 1.7] under the extra assumption that each element in Σ is injective, where the sequence $(**)$ is used. In fact, this assumption is not necessary since we can replace $(**)$ by $(*)$ and modify the proof there to show the general case. For more details, we refer the reader to [1].

(3) This follows directly from Lemma 2.11 and Lemma 3.4. $\square$

Combining Lemma 2.1 with Proposition 3.3, we have the following result, which says that, in some sense, Morita equivalences preserve universal localizations.

Corollary 3.5. *Let $\lambda : R \to R_\Sigma$ be the universal localization of the ring R at the set Σ. Suppose that P is a finitely generated projective generator for R-Mod. Set $\Delta := \{\mathrm{Hom}_R(P, f) \mid f \in \Sigma\}$. Then the ring homomorphism $\mu : \mathrm{End}_R(P) \to \mathrm{End}_{R_\Sigma}(R_\Sigma \otimes_R P)$, defined by $g \mapsto R_\Sigma \otimes_R g$ for any $g \in \mathrm{End}_R(P)$, is the universal localization of the ring $\mathrm{End}_R(P)$ at the set Δ.*

Proof. Let $S := \mathrm{End}_R(P)$. Since ${}_RP$ is a finitely generated projective generator for R-Mod, the Hom-functor $\mathrm{Hom}_R(P, -) : R\text{-Mod} \to S\text{-Mod}$ is an equivalence, which extends to a triangle equivalence between $\mathscr{D}(R)$ and $\mathscr{D}(S)$. By the definitions of $\Sigma^\perp$ and $\Delta^\perp$, the restriction of $\mathrm{Hom}_R(P, -)$ induces an equivalence from $\Sigma^\perp$ to $\Delta^\perp$. Note that $R_\Sigma \otimes_R P$ is a finitely generated projective generator for R_Σ-Mod. Since the functor

$\lambda_* : R_\Sigma\text{-Mod} \to R\text{-Mod}$ is fully faithful and since the image of λ_* coincides with $\Sigma^\perp$ by Proposition 3.3(2), it follows from the following commutative diagram of functors:

$$\begin{tikzcd}
R_\Sigma\text{-Mod} \arrow[r, "\mathrm{Hom}_{R_\Sigma}(R_\Sigma\otimes_R P,-)", "\simeq"'] \arrow[d, "\lambda_*"'] & \mathrm{End}_{R_\Sigma}(R_\Sigma \otimes_R P)\text{-Mod} \arrow[d, "\mu_*"] \\
R\text{-Mod} \arrow[r, "\mathrm{Hom}_R(P,-)", "\simeq"'] & S\text{-Mod}
\end{tikzcd}$$

that μ_* is fully faithful, and that the image of μ_* coincides with $\Delta^\perp$. This implies also that μ is a ring epimorphism. Note that, under our conventions, full subcategories are always closed under isomorphic images.

On the other hand, if $\varphi : S \to S_\Delta$ is the universal localization of S at Δ, then, by Proposition 3.3(2), the image of φ_* coincides with $\Delta^\perp$. Thus the two ring epimorphisms μ and φ are equivalent by Lemma 2.1. This means that the two rings S_Δ and $\mathrm{End}_{R_\Sigma}(R_\Sigma \otimes_R P)$ are isomorphic. Thus μ is the universal localization of S at Δ. $\square$

Motivated by [31, Theorem 10.8], see also [4, Thereom 4.8 (3)], we shall establish the following connection between universal localizations and recollements of triangulated categories. The last condition (5) of Proposition 3.6 below seems to appear for the first time in the work, and will be used in Section 4 to prove Theorem 1.1(1).

Proposition 3.6. (a) *Let* $\mathbf{j}$ *be the canonical embedding of* $\mathscr{D}(R)_{\Sigma^\perp}$ *into* $\mathscr{D}(R)$*. Then there is a recollement*

$$\begin{tikzcd}
\mathscr{D}(R)_{\Sigma^\perp} \arrow[r, "\mathbf{j}"] & \mathscr{D}(R) \arrow[l, bend right, "\mathbf{L}"'] \arrow[l, bend left] \arrow[r] & \mathrm{Tria}(\Sigma^\bullet) \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}$$

such that $\mathbf{L}$ *is the left adjoint of* $\mathbf{j}$ *and* $T^\bullet := \mathbf{L}(R)$ *is a compact generator of* $\mathscr{D}(R)_{\Sigma^\perp}$.

(b) *The following statements are equivalent:*

(1) $\lambda : R \to R_\Sigma$ *is a homological epimorphism of rings;*

(2) $\lambda_* : \mathscr{D}(R_\Sigma) \xrightarrow{\sim} \mathscr{D}(R)_{\Sigma^\perp}$;

(3) *the complex* $T^\bullet$ *in (a) is a tilting object in* $\mathscr{D}(R)_{\Sigma^\perp}$;

(4) *the complex* $T^\bullet$ *in (a) is isomorphic to* R_Σ *in* $\mathscr{D}(R)$;

(5) *the complex* $T^\bullet$ *in (a) is isomorphic in* $\mathscr{D}(R)$ *to a complex* $X^\bullet := (X^i)_{i\in\mathbb{Z}}$ *such that* $X^i \in \Sigma^\perp$ *for all* $i \in \mathbb{Z}$.

Proof. The existence of the above recollement is an immediate consequence of Lemmata 2.7(2), 2.11 and Proposition 3.3. The property in (a) follows from the proof in [14, Chapter IV, Proposition 1.1]. As to the property (b), we notice that the equivalences among the first four statements in (b) can be deduced from [4, Proposition 1.7, Lemma 4.6]. Clearly, the statement (4) implies the statement (5). We shall show that (5) implies (4).

Let $\lambda : R \to R_\Sigma$ be the universal localization of R at Σ. In what follows, we always identify $\Sigma^\perp$ with R_Σ-Mod via λ. This is due to Proposition 3.3(2).

Suppose that $T^\bullet \simeq X^\bullet := (X^i)_{i\in\mathbb{Z}}$ in $\mathscr{D}(R)$ such that $X^i \in R_\Sigma$-Mod for all $i \in \mathbb{Z}$. Since λ is a ring epimorphism, we get $\mathrm{Hom}_{R_\Sigma}(X,Y) \simeq \mathrm{Hom}_R(X,Y)$ for all $X,Y \in R_\Sigma$-Mod. Thus $X^\bullet$ can be considered as a complex over R_Σ-Mod, that is, $X^\bullet \in \mathscr{C}(R_\Sigma)$. Let λ_1 be the map $\mathrm{Hom}_{\mathscr{D}(R)}(\lambda, X^\bullet) : \mathrm{Hom}_{\mathscr{D}(R)}(R_\Sigma, X^\bullet) \to \mathrm{Hom}_{\mathscr{D}(R)}(R, X^\bullet)$. We claim that λ_1 is surjective. In fact, there is a commutative diagram:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathscr{K}(R)}(R_\Sigma, X^\bullet) \arrow[r, "q_1"] \arrow[d, "\lambda_2"] & \mathrm{Hom}_{\mathscr{D}(R)}(R_\Sigma, X^\bullet) \arrow[d, "\lambda_1"] \\
\mathrm{Hom}_{\mathscr{K}(R)}(R, X^\bullet) \arrow[r, "q_2"] & \mathrm{Hom}_{\mathscr{D}(R)}(R, X^\bullet),
\end{tikzcd}$$

where $\lambda_2 = \mathrm{Hom}_{\mathscr{K}(R)}(\lambda, X^\bullet)$, and where q_1 and q_2 are induced by the localization functor $q : \mathscr{K}(R) \to \mathscr{D}(R)$. Clearly, q_2 is a bijection. To prove that λ_1 is surjective, it suffices to show that λ_2 is bijective. Indeed, λ_2 is a composition of the following series of isomorphisms:

$$\mathrm{Hom}_{\mathscr{K}(R)}(R_\Sigma, X^\bullet) \simeq H^0(\mathrm{Hom}_R(R_\Sigma, X^\bullet)) = H^0(\mathrm{Hom}_{R_\Sigma}(R_\Sigma, X^\bullet)) \simeq \mathrm{Hom}_{\mathscr{K}(R)}(R, X^\bullet),$$

where the equality follows from the fact that λ is a ring epimorphism. More precisely, for $\bar{f}^\bullet := \overline{(f^i)} \in \mathrm{Hom}_{\mathscr{K}(R)}(R_\Sigma, X^\bullet)$ with $(f^i)_{i\in\mathbb{Z}}$ a chain map, the series of the above maps are defined by:

$$\overline{(f^i)} \mapsto \overline{f^0} = \overline{f^0} \mapsto \overline{\lambda_*(f^\bullet)},$$

where $\lambda_*(f^\bullet)$ is a chain map from R to $X^\bullet$ with λf^0 in degree 0 and zero in all other degrees. Thus λ_2 is bijective, which implies that λ_1 is surjective. Now, let λ' be the map $\mathrm{Hom}_{\mathscr{D}(R)}(\lambda, T^\bullet) : \mathrm{Hom}_{\mathscr{D}(R)}(R_\Sigma, T^\bullet) \to \mathrm{Hom}_{\mathscr{D}(R)}(R, T^\bullet)$. Since $T^\bullet \simeq X^\bullet$ in $\mathscr{D}(R)$, we know that λ' is also surjective. Suppose that $\varphi : R \to T^\bullet := \mathbf{L}(R)$ is the unit adjunction morphism with respect to the adjoint pair $(\mathbf{L}, \mathbf{j})$. Then there exists $g : R_\Sigma \to T^\bullet$ in $\mathscr{D}(R)$ such that $\varphi = \lambda g$. Since R_Σ belongs to $\mathscr{D}(R)_{\Sigma^\perp}$, there exists $f : T^\bullet \to R_\Sigma$ in $\mathscr{D}(R)$ such that $\lambda = \varphi f$. This gives rise to the following commutative diagram in $\mathscr{D}(R)$:

$$\begin{tikzcd} R \arrow[r, equal] \arrow[d, "\varphi"'] & R \arrow[r, equal] \arrow[d, "\lambda"'] & R \arrow[d, "\varphi"] \\ T^\bullet \arrow[r, dashed, "f"] & R_\Sigma \arrow[r, dashed, "g"] & T^\bullet. \end{tikzcd}$$

Consequently, $\varphi = \varphi f g$ and $\lambda = \lambda g f$. On the one hand, since φ is the unit adjunction morphism, we have $fg = 1_{T^\bullet}$. On the other hand, it follows from [1, Theorem 1.4] that λ is an R_Σ-Mod-reflection of R, that is, the morphism of abelian groups $\mathrm{Hom}_R(\lambda, Z) : \mathrm{Hom}_R(R_\Sigma, Z) \to \mathrm{Hom}_R(R, Z)$ is bijective for any $Z \in R_\Sigma$-Mod. This yields $gf = 1_{R_\Sigma}$. Thus f is an isomorphism. In other words, $T^\bullet \simeq R_\Sigma$ in $\mathscr{D}(R)$. Therefore, (5) implies (4). $\square$

Remark. Note that every tilting module is associated to a class of finitely presented modules of projective dimension at most one (see [1, 10]) and thus to a universal localization since each finitely presented module of projective dimension at most one is determined by an injective homomorphism between finitely generated projective modules. In Proposition 3.6, we do not require that each homomorphism in Σ is injective. From this point of view, the formulation of Proposition 3.6(b) seems to be more general than that in [4, Thereom 4.8(3)].

Corollary 3.7. *Let $R \subseteq S$ be an extension of rings, that is, R is a subring of the ring S with the same identity, and let B be the endomorphism ring of the R-module $S \oplus S/R$. Then there is a recollement of triangulated categories:*

$$\begin{tikzcd} \mathscr{D}(B)_{\Sigma^\perp} \arrow[r] & \mathscr{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd},$$

where $\Sigma := \{\pi^*\}$, and the homomorphism $\pi^* : \mathrm{Hom}_R(S \oplus S/R, S) \to \mathrm{Hom}_R(S \oplus S/R, S/R)$ of B-modules is defined by $f \mapsto f\pi$ for any $f \in \mathrm{Hom}_R(S \oplus S/R, S)$, which is induced by the canonical map $\pi : S \to S/R$.

Proof. It follows from Proposition 3.6(a) that we have the following recollement:

$$\begin{tikzcd} \mathscr{D}(B)_{\Sigma^\perp} \arrow[r] & \mathscr{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathrm{Tria}(\Sigma^\bullet) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}.$$

To show that $\mathrm{Tria}(\Sigma^\bullet)$ is equivalent to $\mathscr{D}(R)$ as triangulated categories, it suffices to prove that the complex $\Sigma^\bullet \in \mathscr{K}^b(B\text{-proj})$ is exceptional with $\mathrm{End}_{\mathscr{D}(B)}(\Sigma^\bullet) \simeq R$.

In fact, let ${}_RT := S \oplus S/R$ and $B := \mathrm{End}_R(T)$. Then $\mathrm{add}({}_RT)$ and B-proj are equivalent, and therefore $\mathscr{K}^b(\mathrm{add}({}_RT))$ and $\mathscr{K}^b(B\text{-proj})$ are equivalent as triangulated categories via the functor $\mathrm{Hom}_R(T,-)$. Thus, to show that the complex $\Sigma^\bullet \in \mathscr{K}^b(B\text{-proj})$ is exceptional with $\mathrm{End}_{\mathscr{D}(B)}(\Sigma^\bullet) \simeq R$, it is sufficient to show that the complex

$$\Pi^\bullet : \quad 0 \longrightarrow S \xrightarrow{\pi} S/R \longrightarrow 0$$

in $\mathscr{K}^b(\mathrm{add}(T))$ is exceptional with $\mathrm{End}_{\mathscr{K}^b(\mathrm{add}(T))}(\Pi^\bullet) \simeq R$ since $\mathrm{Hom}_R(T,\Pi^\bullet) = \Sigma^\bullet$.

It is easy to see $\mathrm{Hom}_{\mathscr{K}^b(\mathrm{add}(T))}(\Pi^\bullet,\Pi^\bullet[-1]) = 0$. To show $\mathrm{Hom}_{\mathscr{K}^b(\mathrm{add}(T))}(\Pi^\bullet,\Pi^\bullet[1]) = 0$, we pick up a homomorphism $f : S \to S/R$ of R-modules, suppose $(1)f = s+R \in S/R$ and define $g : {}_RS \to {}_RS$ by $x \mapsto xs$ for $x \in S$. Clearly, g is a homomorphism of R-modules and $(f-g)|_R = 0$. Thus there exists a homomorphism $h : S/R \to S/R$ such that $f - g = \pi h$. This implies that f is zero in $\mathscr{K}^b(\mathrm{add}(T))$, that is, $\mathrm{Hom}_{\mathscr{K}^b(\mathrm{add}(T))}(\Pi^\bullet,\Pi^\bullet[1]) = 0$. Hence we have shown that $\Pi^\bullet$ is exceptional.

Now, we define a ring homomorphism α from $\mathrm{End}_{\mathscr{K}^b(\mathrm{add}(T))}(\Pi^\bullet)$ to R as follows: Given $f = (f^0, f^1) \in \mathrm{End}_{\mathscr{K}^b(\mathrm{add}(T))}(\Pi^\bullet)$, let $(f)\alpha$ be the unique map determined by the following exact commutative diagram of R-modules:

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, "\lambda"] \arrow[d, "(f)\alpha"'] & S \arrow[r, "\pi"] \arrow[d, "f^0"'] & S/R \arrow[r] \arrow[d, "f^1"'] & 0 \\
0 \arrow[r] & R \arrow[r, "\lambda"] & S \arrow[r, "\pi"] & S/R \arrow[r] & 0.
\end{tikzcd}$$

Note that if f is null-homotopic then $(f)\alpha$ is zero. This means that α is well-defined. Clearly, α is a ring homomorphism. We claim that α is an isomorphism of rings. It is easy to check that α is injective. We shall show that α is surjective. Let $r \in R$. We define $f^0 : S \to S$ to be the right multiplication of r. Then there is a homomorphism $f^1 : S/R \to S/R$ of R-modules such that $f^0\pi = \pi f^1$. This means that α is surjective. Hence α is an isomorphism of rings. So, $\Sigma^\bullet$ is exceptional with $\mathrm{End}_{\mathscr{D}(B)}(\Sigma^\bullet) \simeq R$. By Lemma 2.9, we may identify $\mathrm{Tria}(\Sigma^\bullet)$ with $\mathscr{D}(R)$. This proves Corollary 3.7. $\square$

As another corollary of Proposition 3.6, we have the following result.

Corollary 3.8. *If the weak dimension of R is at most* 1*, then there is a recollement*

$$\begin{tikzcd}
\mathscr{D}(R_\Sigma) \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathrm{Tria}(\Sigma^\bullet) \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd} ,$$

where Σ is a set of homomorphisms between finitely generated projective R-modules.

Proof. Under the assumption, the universal localization map λ_Σ is trivially a homological ring epimorphism. So, this corollary follows from Proposition 3.6(b). $\square$

As a consequence of Corollary 3.8, we have the following result which is a generalization of [5, Theorem 2.5, Corollary 3.3].

Corollary 3.9. *Suppose that R is a left semi-hereditary ring (that is, every finitely generated submodule of a projective left R-module is projective). If $T^\bullet$ is a compact exceptional object in $\mathscr{D}(R)$, then there is a ring S, a homological ring epimorphism $\lambda : R \to S$ and a recollement*

$$\begin{tikzcd}
\mathscr{D}(S) \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(\mathrm{End}_{\mathscr{D}(R)}(T^\bullet)) \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd} .$$

Proof. Since $T^\bullet$ is a compact object in $\mathscr{D}(R)$, there exists a complex $P^\bullet \in \mathscr{K}^b(R\text{-proj})$ such that $T^\bullet \simeq P^\bullet$ in $\mathscr{D}(R)$. Suppose that $P^\bullet$ is of the following form

$$\cdots \longrightarrow 0 \longrightarrow P^s \longrightarrow \cdots \longrightarrow P^i \xrightarrow{d^i} P^{i+1} \longrightarrow \cdots \longrightarrow P^t \longrightarrow 0 \longrightarrow \cdots,$$

where $P^i \in \mathrm{add}({}_RR)$ for $s \leq i \leq t$. Since R is left semi-hereditary, we have $\mathrm{Im}(d^i) \in \mathrm{add}({}_RR)$ for all i. This implies that $P^\bullet$ is isomorphic to a direct sum of finitely many two-term complexes in $\mathscr{K}^b(R\text{-proj})$, say $P^\bullet \simeq \oplus_{i=1}^n P_i^\bullet$, where $n \in \mathbb{N}$ and $P_j^\bullet$ is of the form: $0 \to P_j^{s_j-1} \xrightarrow{d_j} P_j^{s_j} \to 0$ with $s_j \in \mathbb{Z}$ for each $1 \leq j \leq n$. Now we choose $\Sigma := \{d_j \mid 1 \leq j \leq n\}$. Then, by definition, we have $\Sigma^\bullet = \{P_j^\bullet[s_j] \mid 1 \leq j \leq n\}$ (see notations at the beginning of Section 3). Consequently, $\mathrm{Tria}(T^\bullet) = \mathrm{Tria}(P^\bullet) = \mathrm{Tria}(\Sigma^\bullet)$. Since $T^\bullet$ is compact and $\mathrm{Hom}_{\mathscr{D}(R)}(T^\bullet, T^\bullet[i]) = 0$ for all $i \neq 0$, it follows from Lemma 2.9 that $\mathrm{Tria}(T^\bullet)$ is triangle equivalent to the derived category $\mathscr{D}(\mathrm{End}_{\mathscr{D}(R)}(T^\bullet))$ of $\mathrm{End}_{\mathscr{D}(R)}(T^\bullet)$. Note that R has weak dimension at most 1 because it is left semi-hereditary. Thus the condition of Corollary 3.8 is fulfilled, and therefore Corollary 3.9 follows from Corollary 3.8 if we define $S = R_\Sigma$. $\square$

In general, it is hard to compute R_Σ. However, if Σ consists of only one element with an orthogonality assumption, one can construct R_Σ explicitly in terms of endomorphism rings of modules. To this purpose, we first establish the following result which generalizes [21, Proposition 1.3] where only stalk complexes (or modules) were considered.

Proposition 3.10. *Suppose that $P^\bullet := (P^i)_{i\in\mathbb{Z}}$ is a complex in $\mathscr{C}(R\text{-Proj})$ such that $P^n = 0$ for all $n \in \mathbb{Z}\backslash\{-1,0\}$. Define ${P^\bullet}^\perp := \{X \in R\text{-Mod} \mid \mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, X[i]) = 0$ for all $i \in \mathbb{Z}\}$. If $\mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, P^{\bullet(\delta)}[1]) = 0$ for each cardinal δ, then the inclusion $j : {P^\bullet}^\perp \to R\text{-Mod}$ admits a left adjoint $l : R\text{-Mod} \to {P^\bullet}^\perp$.*

Proof. The proof will be divided into three steps. We define $\mathscr{X}$ to be the full subcategory of R-Mod consisting of the objects X such that $\mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, X[1]) = 0$. Then, it follows from $\mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, X[1]) \simeq \mathrm{Hom}_{\mathscr{K}(R)}(P^\bullet, X[1])$ for $X \in R$-Mod that $\mathscr{X}$ is closed under quotients.

Let M and N be R-modules.

Step (1). For the given M, we shall construct an R-module, denoted by $l(M)$, which belongs to ${P^\bullet}^\perp$ and is endowed with an R-homomorphism $\eta_M : M \to l(M)$.

Since $P^\bullet \in \mathscr{C}^b(R\text{-Proj})$, we have $\mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, M[1]) = \mathrm{Hom}_{\mathscr{K}(R)}(P^\bullet, M[1])$. Let α be a generating set of $\mathrm{Hom}_{\mathscr{K}(R)}(P^\bullet, M[1])$ as an $\mathrm{End}_{\mathscr{D}(R)}(P^\bullet)$-module. Thus each element of α is a chain map from $P^\bullet$ to $M[1]$. We define $\omega_M : P^{\bullet(\alpha)} \to M[1]$ to be the coproduct of the elements of α. Then, it is clear that

$$\omega'_M := \mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, \omega_M) : \mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, P^{\bullet(\alpha)}) \longrightarrow \mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, M[1])$$

is a surjective homomorphism of $\mathrm{End}_{\mathscr{D}(R)}(P^\bullet)$-modules. Let $\mathrm{cone}(\omega_{\mathrm{M}})$ be the mapping cone of ω_M, and $\overline{M}^\bullet := \mathrm{cone}(\omega_{\mathrm{M}})[-1]$. Then we obtain the following canonical triangle in $\mathscr{D}(R)$:

$$(*) \qquad M \xrightarrow{\varphi_M} \overline{M}^\bullet \xrightarrow{\psi_M} P^{\bullet(\alpha)} \xrightarrow{\omega_M} M[1],$$

where φ_M and ψ_M can be constructed explicitly. For more details, we refer the reader to any standard textbook of homological algebra (for instance, [40]). Note that the complex $\overline{M}^\bullet$ is of the form

$$\overline{M}^\bullet: \quad \cdots \longrightarrow 0 \longrightarrow M^{-1} \xrightarrow{d_M} M^0 \longrightarrow 0 \longrightarrow \cdots,$$

where d_M is a homomorphism between R-modules M^{-1} and M^0, which are of degrees -1 and 0, respectively. Let $C^\bullet$ denote the complex: $0 \to \mathrm{Im}(d_M) \to M^0 \to 0$. Then we have an exact sequence of complexes: $0 \to H^{-1}(\overline{M}^\bullet)[1] \to \overline{M}^\bullet \to C^\bullet \to 0$, this gives us an triangle

$$H^{-1}(\overline{M}^\bullet)[1] \to \overline{M}^\bullet \to C^\bullet \to H^{-1}(\overline{M}^\bullet)[2].$$

Since $C^\bullet$ is quasi-isomorphic to the stalk complex $H^0(\overline{M}^\bullet)$, we get the following triangle in $\mathscr{D}(R)$:

$$(**) \quad H^{-1}(\overline{M}^\bullet)[1] \longrightarrow \overline{M}^\bullet \xrightarrow{\gamma_M} H^0(\overline{M}^\bullet) \longrightarrow H^{-1}(\overline{M}^\bullet)[2],$$

where the chain map γ_M is induced by the homomorphism d_M such that $H^0(\gamma_M) = 1_{H^0(\overline{M}^\bullet)}$. Applying $\mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, -)$ to $(*)$, we get a long exact sequence of $\mathrm{End}_{\mathscr{D}(R)}(P^\bullet)$-modules:

$$\mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, P^{\bullet(\alpha)}) \xrightarrow{\omega'_M} \mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, M[1]) \longrightarrow \mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, \overline{M}^\bullet[1]) \longrightarrow \mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, P^{\bullet(\alpha)}[1]).$$

Since ω'_M is surjective and $\mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, P^{\bullet(\alpha)}[1]) = 0$, we have $\mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, \overline{M}^\bullet[1]) = 0$. Now it follows from Lemma 3.4 that

$$\mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, H^0(\overline{M}^\bullet)[1]) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, \overline{M}^\bullet[1]) = 0.$$

This implies $H^0(\overline{M}^\bullet) \in \mathscr{X}$.

Now, we first fix a chain map ω_M for the given M, and then define $l(M) := H^0(\overline{M}^\bullet)/t(M)$, where $t(M)$ denotes the trace of $H^0(P^\bullet)$ in $H^0(\overline{M}^\bullet)$, that is, the sum of the images of all homomorphisms from $H^0(P^\bullet)$ to $H^0(\overline{M}^\bullet)$. Thus $l(M) \in \mathscr{X}$ since $\mathscr{X}$ is closed under quotients. By Lemma 3.4, we have $\mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, H^0(P^\bullet)[1]) \simeq \mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, P^\bullet[1]) = 0$. This means $H^0(P^\bullet) \in \mathscr{X}$. Since H^0 commutes with coproducts, we infer from Lemma 3.4 that coproducts of copies of $H^0(P^\bullet)$ lie in $\mathscr{X}$. This shows $t(M) \in \mathscr{X}$ because it is an image of a coproduct of $H^0(P^\bullet)$.

In the following, we shall prove $l(M) \in P^{\bullet\perp}$. Clearly, we have $\mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, l(M)[i]) = 0$ for $i \neq 0$, and $\mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, l(M)) \simeq \mathrm{Hom}_R(H^0(P^\bullet), l(M))$. So, in order to show $l(M) \in P^{\bullet\perp}$, it is sufficient to prove $\mathrm{Hom}_R(H^0(P^\bullet), l(M)) = 0$.

One the one hand, applying $\mathrm{Hom}_R(H^0(P^\bullet), -)$ to the the canonical exacts sequence

$$0 \longrightarrow t(M) \longrightarrow H^0(\overline{M}^\bullet) \longrightarrow l(M) \longrightarrow 0$$

of R-modules, we can see that $\mathrm{Hom}_R(H^0(P^\bullet), l(M))$ can be embedded into $\mathrm{Hom}_{\mathscr{D}(R)}(H^0(P^\bullet), t(M)[1])$ because $\mathrm{Hom}_R(H^0(P^\bullet), t(M)) \longrightarrow \mathrm{Hom}_R(H^0(P^\bullet), H^0(\overline{M}^\bullet))$ is always bijective by definition. On the other hand, applying $\mathrm{Hom}_{\mathscr{D}(R)}(-, t(M)[1])$ to the following canonical triangle induced from $P^\bullet$

$$H^{-1}(P^\bullet)[1] \longrightarrow P^\bullet \longrightarrow H^0(P^\bullet) \longrightarrow H^{-1}(P^\bullet)[2]$$

in $\mathscr{D}(R)$ and using the fact that $\mathrm{Hom}_{\mathscr{D}(R)}(H^{-1}(P^\bullet)[1], t(M)) = 0$ and that $t(M) \in \mathscr{X}$, we can deduce that $\mathrm{Hom}_{\mathscr{D}(R)}(H^0(P^\bullet), t(M)[1]) = 0$. Consequently, $\mathrm{Hom}_R(H^0(P^\bullet), l(M)) = 0$, as desired. Thus $l(M) \in P^{\bullet\perp}$.

We define $\eta_M := \varphi_M\gamma_M\pi_M : M \to l(M)$, which is clearly a homomorphism of R-modules.

Similarly, for the module N, we fix, once and for all, a chain map $\omega_N : P^{\bullet(\beta)} \to N[1]$, and then define $l(N)$ and $\eta_N : N \to l(N)$, where β is a cardinal. Clearly, we have the following triangle in $\mathscr{D}(R)$:

$$N \xrightarrow{\varphi_N} \overline{N}^\bullet \xrightarrow{\psi_N} P^{\bullet(\beta)} \xrightarrow{\omega_N} N[1]$$

with $\overline{N}^\bullet := \mathrm{cone}(\omega_N)[-1]$.

Step (2). For any homomorphism $g : M \to N$ in R-Mod, we claim that there is a unique homomorphism $l(g) : l(M) \to l(N)$ such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\eta_M"] \arrow[d, "g"'] & l(M) \arrow[d, "l(g)"] \\ N \arrow[r, "\eta_N"] & l(N). \end{tikzcd}$$

In fact, since $\mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, \overline{N}^\bullet[1]) = 0$ by Step (1), we know from homological algebra that $\mathrm{Hom}_{\mathscr{D}(R)}(P^{\bullet(\alpha)}[-1], \overline{N}^\bullet) \simeq \Pi_\alpha \mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet[-1], N) = 0$. In particular, the homomorphism $\omega_M[-1]g\varphi_N : P^{\bullet(\alpha)}[-1] \to \overline{N}^\bullet$ must be zero. Consequently, there is a homomorphism $g' : \overline{M}^\bullet \to \overline{N}^\bullet$ such that $g\varphi_N = \varphi_M g'$. So, we have the following commutative diagram

$$\begin{tikzcd}
P^{\bullet(\alpha)}[-1] \arrow[r, "-\omega_M[-1]"] & M \arrow[r, "\varphi_M"] \arrow[d, "g"] & \overline{M}^\bullet \arrow[r, "\psi_M"] \arrow[d, dashed, "g'"] & P^{\bullet(\alpha)} \\
P^{(\beta)}[-1] \arrow[r, "-\omega_N[-1]"] & N \arrow[r, "\varphi_N"] & \overline{N}^\bullet \arrow[r, "\psi_N"] & P^{(\beta)}.
\end{tikzcd}$$

Notice that such g' is not unique in general. Suppose that there exists another morphism $h : \overline{M}^\bullet \to \overline{N}^\bullet$ such that $g\varphi_N = \varphi_M h$. Then $g' - h = \psi_M s$ for some morphism $s : P^{\bullet(\alpha)} \to \overline{N}^\bullet$. Applying the 0-th cohomological functor $H^0(-) : \mathscr{D}(R) \to R$-Mod to this equality, we get

$$H^0(g') - H^0(h) = H^0(\psi_M)H^0(s) : H^0(\overline{M}^\bullet) \longrightarrow H^0(\overline{N}^\bullet),$$

with $H^0(s) : H^0(P^{\bullet(\alpha)}) \to H^0(\overline{N}^\bullet)$. Since the functor $H^0(-)$ commutes with coproducts, we get $H^0(P^{\bullet(\alpha)}) \simeq H^0(P^\bullet)^{(\alpha)}$. This means that the image of $H^0(g') - H^0(h)$ is contained in $t(N)$ by the definition of $t(N)$. Thus

$$\overline{H^0(g')} = \overline{H^0(h)} : l(M) \longrightarrow l(N),$$

which shows that $\overline{H^0(g')}$ depends on g and not on the choice of g'. Thus, given $g : M \to N$, we can define $l(g) := \overline{H^0(g')} : l(M) \to l(N)$ which is a homomorphism in $P^{\bullet\perp}$. To prove the equality $\eta_M l(g) = g\eta_N$, we observe that $H^0(g')$ is the unique homomorphism from $H^0(\overline{M}^\bullet)$ to $H^0(\overline{N}^\bullet)$ such that the following diagram is commutative:

$$\begin{tikzcd}
H^{-1}(\overline{M}^\bullet)[1] \arrow[r] & \overline{M}^\bullet \arrow[r, "\gamma_M"] \arrow[d, "g'"'] & H^0(\overline{M}^\bullet) \arrow[r] \arrow[d, "H^0(g')"] & H^{-1}(\overline{M}^\bullet)[2] \\
H^{-1}(\overline{N}^\bullet)[1] \arrow[r] & \overline{N}^\bullet \arrow[r, "\gamma_N"] & H^0(\overline{N}^\bullet) \arrow[r] & H^{-1}(\overline{N}^\bullet)[2]
\end{tikzcd}$$

because $\mathrm{Hom}_{\mathscr{D}(R)}(H^{-1}(\overline{M}^\bullet)[2], H^0(\overline{N}^\bullet)) = 0$. Then

$$\eta_M l(g) = \varphi_M \gamma_M \pi_M l(g) = \varphi_M \gamma_M H^0(g')\pi_N = \varphi_M g' \gamma_N \pi_N = g\varphi_N \gamma_N \pi_N = g\eta_N.$$

Hence, we have proved the existence of a homomorphism $l(g)$ from $l(M)$ to $l(N)$ with the desired property.

Now we show the uniqueness of $l(g)$. Let $t_1, t_2 : l(M) \to l(N)$ be two homomorphisms such that $\eta_M t_i = g\eta_N$ for $i = 1, 2$. Set $t := t_1 - t_2$. Then $\eta_M t = \varphi_M(\gamma_M \pi_M t) = 0$. It follows from the triangle $(*)$ that there exists a morphism $u : P^{\bullet(\alpha)} \to l(N)$ in $\mathscr{D}(R)$ such that $\gamma_M \pi_M t = \psi_M u$. But $\mathrm{Hom}_{\mathscr{D}(R)}(P^\bullet, l(N)) = 0$ since $l(N) \in P^{\bullet\perp}$. This shows $u = 0$ and $\gamma_M \pi_M t = 0$. By the triangle $(**)$, we know $\pi_M t = 0$ since $\mathrm{Hom}_{\mathscr{D}(R)}(H^{-1}(\overline{M}^\bullet)[2], l(N)) = 0$. Note that π_M is surjective. Hence $t = 0$, that is, $t_1 = t_2$. This finishes the proof of the uniqueness.

Consequently, if g is an isomorphism, then $l(g)$ is an isomorphism. This shows also that, regardless of different choices of ω_M, the module $l(M)$ is unique up to isomorphism.

Step (3). We define a functor $l : R\text{-Mod} \to P^{\bullet\perp}$ by sending M in R-Mod to $l(M)$, and homomorphism $g : M \to N$ to $l(g) : l(M) \to l(N)$. By Step (2), l is well-defined. Clearly, we have $l(Y) \simeq Y$ for any $Y \in P^{\bullet\perp}$ by definition. Now, it follows from Step (2) that $\mathrm{Hom}_R(l(U), V) \simeq \mathrm{Hom}_R(U, j(V))$ for any $U \in R$-Mod and $V \in P^{\bullet\perp}$. This isomorphism is natural in U and V. Thus (l, j) is an adjoint pair of functors. $\square$

As a consequence of Proposition 3.10, we obtain the following promised result which provides an effective description of R_Σ as endomorphism rings.

Corollary 3.11. *Let $f: P^{-1} \to P^0$ be a homomorphism between finitely generated projective R-modules and $\Sigma := \{f\}$. If $\mathrm{Hom}_{\mathscr{D}(R)}(P_f^\bullet, P_f^\bullet[1]) = 0$, then the inclusion $j: \Sigma^\perp \to R\text{-Mod}$ admits a left adjoint $l: R\text{-Mod} \to \Sigma^\perp$, which can be constructed explicitly. In particular, R_Σ is isomorphic to the endomorphism ring $\mathrm{End}_R(l(R))$.*

Proof. Since $P_f^\bullet \in \mathscr{C}^b(R\text{-proj})$, we know that the functor $\mathrm{Hom}_{\mathscr{D}(R)}(P_f^\bullet, -)$ commutes with direct sums. Consequently, if $\mathrm{Hom}_{\mathscr{D}(R)}(P_f^\bullet, P_f^\bullet[1]) = 0$, then $\mathrm{Hom}_{\mathscr{D}(R)}(P_f^\bullet, P_f^{\bullet(\delta)}[1]) = 0$ for any cardinal δ. Thus the existence of the functor l in Corollary 3.11 follows immediately from Proposition 3.10.

In the following, we shall prove that R_Σ is isomorphic to the endomorphism ring $\mathrm{End}_R(l(R))$. Indeed, by Proposition 3.3(1), we know that $\Sigma^\perp$ is closed under extensions, kernels, cokernels, arbitrary direct sums and products. Further, since the inclusion j admits a left adjoint functor l, the full subcategory $\Sigma^\perp$ of R-Mod satisfies all assumptions of [24, Proposition 3.8]. Define $S := \mathrm{End}_R(l(R))$. Then it follows directly from [24, Proposition 3.8] that $l(R)$ is a projective generator for $\Sigma^\perp$, and there exists a ring epimorphism $\rho: R \to S$ such that the following diagram

$$\begin{tikzcd}
\Sigma^\perp \arrow[r, shift left, "j"] \arrow[d, shift left, "G"] & R\text{-Mod} \arrow[l, shift left, "l"] \arrow[dl, shift left, "\rho^*"] \\
S\text{-Mod} \arrow[u, shift left, "F"] \arrow[ur, shift left, "\rho_*"] &
\end{tikzcd}$$

commutes, where $F := {}_R l(R) \otimes_S -$ and $G := \mathrm{Hom}_R(l(R), -)$ are mutually inverse functors, and where $\rho_* := {}_R S \otimes_S -$ is the canonical embedding, and $\rho^* := {}_S S \otimes_R -$ is a left adjoint of ρ_*. Since the both ring epimorphisms $\lambda: R \to R_\Sigma$ and $\rho: R \to S$ give rise to the same bireflective subcategory $\Sigma^\perp$ of R-Mod, we conclude from Lemma 2.1 that R_Σ is isomorphic to S. $\square$

Finally, we remark that, in general, the two-term complex $P^\bullet$ in Proposition 3.10 cannot be replaced by a complex in $\mathscr{C}(R\text{-Proj})$ with more than two terms. A counterexample is the following:

Let A be the algebra given by the following quiver with relations:

$$\begin{tikzcd}
3 \arrow[r, bend right, "\gamma"'] & 2 \arrow[l, bend right, "\beta"'] \arrow[r, "\alpha"] & 1,
\end{tikzcd} \qquad \beta\gamma = \gamma\alpha = 0.$$

We denote by S_i, I_i and P_i the simple, injective and projective modules corresponding to the vertex i, respectively. Let $P^\bullet$ be the minimal projective resolution of S_3. Then $P^\bullet$ is a three-term complex. We can easily check that $P^{\bullet\perp}$ has only two indecomposable modules, they are the indecomposable modules I_1 and I_2. Note that A is representation-finite and every indecomposable module is finitely generated. Also, we have $\mathrm{Ext}_A^i(S_3, S_3) = 0$ for $i = 1, 2$. Actually, this is true for all $i > 0$. Since $P^\bullet$ is compact, one has $\mathrm{Hom}_{\mathscr{D}(A)}(P^\bullet, P^{\bullet(\alpha)}[1]) = 0$ for all cardinals α. If the inclusion functor j from $P^{\bullet\perp}$ into A-Mod would have a left adjoint, then j would preserve monomorphisms. One can verify that there is a non-zero homomorphism from I_1 to I_2 which is a monomorphism in $P^{\bullet\perp}$, but not a monomorphism in A-Mod. This is a contradiction and shows that the inclusion functor from $P^{\bullet\perp}$ into A-Mod cannot possess a left adjoint.

Note that the simple module corresponding to the vertex 1 is of injective dimension 3, and defines a 2-APR-tilting module $S_3 \oplus P_2 \oplus P_3$ (see [27] for unexplained definitions).

4 Recollements of derived categories and infinitely generated tilting modules

In this section, we shall use our results in Section 3 to show the first statement of the main result, Theorem 1.1. More precisely, we first recall the definition of infinitely generated tilting modules, and then discuss some of their homological properties. Especially, we shall establish a crucial result, Proposition 4.6, which will play a role in our proof of the main result.

Let A be a ring with identity.

Definition 4.1. [20] An A-module T is called a tilting module (of projective dimension at most one) if the following conditions are satisfied:

$(T1)$ the projective dimension of T is at most 1, that is, there exists a projective resolution of T: $0 \to P_1 \to P_0 \to T \to 0$, where P_i is projective for $i = 0, 1$.

$(T2)$ $\mathrm{Ext}_A^i(T, T^{(\alpha)}) = 0$ for each $i \geq 1$ and every cardinal α; where $T^{(\alpha)}$ stands for the direct sum of α copies of T; and

$(T3)$ there exists an exact sequence

$$0 \longrightarrow {}_AA \longrightarrow T_0 \longrightarrow T_1 \longrightarrow 0$$

of A-modules such that $T_i \in \mathrm{Add}(T)$ for $i = 0, 1$.

If P_1 and P_0 in the condition $(T1)$ are finitely generated, then the tilting module T is called a classical tilting module (see [16] and [26]).

Two tilting A-modules T and T' are said to be equivalent if $\mathrm{Add}(T) = \mathrm{Add}(T')$, or equivalently, $\mathrm{Gen}(T) = \mathrm{Gen}(T')$, where $\mathrm{Gen}(T)$ denotes the full subcategory of A-Mod generated by T. Recall that an A-module M is generated by T if there is an index set I and a surjective homomorphism $f : T^{(I)} \to M$.

An A-module T is said to be good if it satisfies $(T1)$, $(T2)$ and

$(T3)'$ there is an exact sequence

$$0 \longrightarrow {}_AA \longrightarrow T_0 \xrightarrow{\varphi} T_1 \longrightarrow 0$$

in A-Mod such that $T_i \in \mathrm{add}(T)$ for $i = 0, 1$.

Note that each classical tilting module is good. Moreover, for any given tilting module ${}_AT$ with (T1) and (T2), the module $T' := T_0 \oplus T_1$ is a good tilting module which is equivalent to the given one.

From now on, we assume in this section that T is a **good** tilting A-module. Let $B := \mathrm{End}_A(T)$. We define

$$T^{\perp} := \{X \in A\text{-Mod} \mid \mathrm{Ext}_A^i(T, X) = 0 \text{ for all } i \geq 1\},\ \mathscr{E} := \{Y \in B\text{-Mod} \mid \mathrm{Tor}_i^B(T, Y) = 0 \text{ for all } i \geq 0\};$$

$$G := {}_AT \otimes_B^{\mathbb{L}} - : \mathscr{D}(B) \longrightarrow \mathscr{D}(A), \quad H := \mathbb{R}\mathrm{Hom}_A(T, -) : \mathscr{D}(A) \longrightarrow \mathscr{D}(B);$$

$$\mathcal{Y} := \mathrm{Ker}(G), \qquad \mathcal{Z} := \mathrm{Im}(H),$$

$$Q^{\bullet} := \quad \cdots \longrightarrow 0 \longrightarrow \mathrm{Hom}_A(T, T_0) \xrightarrow{\varphi^*} \mathrm{Hom}_A(T, T_1) \longrightarrow 0 \longrightarrow \cdots \in \mathscr{C}^b(B\text{-proj}),$$

where $\varphi^* := \mathrm{Hom}_A(T, \varphi)$, and where the finitely generated projective B-modules $\mathrm{Hom}_A(T, T_0)$ and $\mathrm{Hom}_A(T, T_1)$, as terms of the complex $Q^{\bullet}$, are of degrees 0 and 1, respectively. Clearly, $H(A) = Q^{\bullet}$ in $\mathscr{D}(A)$.

In the next lemma we mention a few basic properties of tilting modules. For proofs we refer to [11, Proposition 1.4, Lemma 1.5] and [9].

Lemma 4.2. *Let T be a tilting A-module. Then:*

(1) *T_B has a projective resolution $0 \longrightarrow Q_1 \xrightarrow{\psi} Q_0 \longrightarrow T_B \longrightarrow 0$ such that $Q_i \in \mathrm{add}(B_B)$ for $0 \leq i \leq 1$.*

(2) *$\mathrm{End}_{B^{\mathrm{op}}}(T) \simeq A^{\mathrm{op}}$ and $\mathrm{Ext}_{B^{\mathrm{op}}}^i(T, T) = 0$ for all $i \geq 1$.*

(3) *For each $Y \in \mathrm{Add}({}_BB)$, we have $\mathrm{Ext}_A^i(T, {}_AT \otimes_B Y) = 0$ for all $i \geq 1$.*

(4) *For each $X \in T^{\perp}$, we have* $\mathrm{Tor}_i^B({}_AT_B, \mathrm{Hom}_A(T, X)) \simeq \begin{cases} X, & i = 0, \\ 0, & i > 0. \end{cases}$

(5) *$T^{\perp}$ is closed under direct sums.*

The following result is shown in [9, Theorem 5.1], which says that the unbounded derived category of B-Mod is bigger than that of A-Mod in general.

Lemma 4.3. *The functor H is fully faithful, and the functor G induces a triangle equivalence between $\mathscr{D}(B)/\mathrm{Ker}(G)$ and $\mathscr{D}(A)$. Here we denote by $\mathscr{D}(B)/\mathrm{Ker}(G)$ the Verdier quotient of $\mathscr{D}(B)$ by the subcategory* $\mathrm{Ker}(G)$.

The following lemma supplies a method to obtain modules in $\mathscr{E}$, and is also useful for our later calculations.

Lemma 4.4. *Suppose that I is a cardinal and $X_i \in T^{\perp}$ for each $i \in I$. Consider the canonical exact sequence*

$$0 \longrightarrow \bigoplus_{i\in I} \mathrm{Hom}_A(T, X_i) \xrightarrow{\delta_I} \mathrm{Hom}_A(T, \bigoplus_{j\in I} X_j) \longrightarrow \mathrm{Coker}(\delta_I) \longrightarrow 0$$

*in B-*Mod*, where δ_I is defined by $(f_i)_{i\in I} \mapsto \sum_{i\in I} f_i\lambda_i$ with $f_i \in \mathrm{Hom}_A(T, X_i)$ and $\lambda_i : X_i \to \bigoplus_{j\in I} X_j$ the canonical inclusion for each $i \in I$. Then $\mathrm{Coker}(\delta_I) \in \mathscr{E}$. Particularly, for each projective B-module P, the unit adjunction morphism $\eta'_P : P \to \mathrm{Hom}_A(T, T \otimes_B P)$ is injective with $\mathrm{Coker}(\eta'_P) \in \mathscr{E}$.*

Proof. Note that δ_I is well-defined. By the definition of δ_I, we can see easily that δ_I is injective. So, there is a canonical exact sequence

$$(*) \quad 0 \longrightarrow \bigoplus_{i\in I} \mathrm{Hom}_A(T, X_i) \xrightarrow{\delta_I} \mathrm{Hom}_A(T, \bigoplus_{j\in I} X_j) \longrightarrow \mathrm{Coker}(\delta_I) \longrightarrow 0.$$

Since $T^{\perp}$ is closed under direct sums by Lemma 4.2(5), we have $\bigoplus_{j\in I} X_j \in T^{\perp}$. It then follows from Lemma 4.2(4) that

$$\mathrm{Tor}^B_m(T, \mathrm{Hom}_A(T, \bigoplus_{j\in I} X_j)) \simeq \begin{cases} \bigoplus_{j\in I} X_j, & m = 0, \\ 0, & m > 0. \end{cases}$$

Similarly, for any $i \in I$, we have

$$\mathrm{Tor}^B_n(T, \mathrm{Hom}_A(T, X_i)) \simeq \begin{cases} X_i, & n = 0, \\ 0, & n > 0. \end{cases}$$

Since the right module T_B has projective dimension at most 1, we see that $\mathrm{Tor}^B_t(T, \mathrm{Coker}(\delta_I)) = 0$ for any $t > 1$. By applying the functor ${}_AT \otimes_B - : B\text{-Mod} \to A\text{-Mod}$ to the sequence $(*)$, we can easily form the following exact commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Tor}^B_1(T, \mathrm{Coker}(\delta_I)) \arrow[r] & T \otimes_B (\bigoplus_{i\in I} \mathrm{Hom}_A(T, X_i)) \arrow[r] \arrow[d, "\simeq"] & T \otimes_B \mathrm{Hom}_A(T, \bigoplus_{j\in I} X_j) \arrow[r] \arrow[d, "\simeq"] & T \otimes_B \mathrm{Coker}(\delta_I) \arrow[r] & 0 \\ & & \bigoplus_{i\in I} X_i \arrow[r, equal] & \bigoplus_{j\in I} X_j. & & \end{tikzcd}$$

This implies $T \otimes_B \mathrm{Coker}(\delta_I) = 0 = \mathrm{Tor}^B_1(T, \mathrm{Coker}(\delta_I))$. Hence $\mathrm{Coker}(\delta_I) \in \mathscr{E}$.

To prove the last statement of Lemma 4.4, we note that the unit adjunction

$$\eta' : 1_{B\text{-Mod}} \longrightarrow \mathrm{Hom}_A(T, T \otimes_B -)$$

is a natural transformation of functors from B-Mod to itself, and that $\mathscr{E}$ is closed under direct summands. Thus, it is sufficient to show that the statement holds for free B-modules. Let α be any cardinal. Then we may form the following exact commutative diagram:

$$\begin{tikzcd}
 & B^{(\alpha)} \arrow[r, "\eta'_{B^{(\alpha)}}"] \arrow[d, equal] & \mathrm{Hom}_A(T, T\otimes_B B^{(\alpha)}) \arrow[d, "\simeq"] & & \\
0 \arrow[r] & \mathrm{Hom}_A(T,T)^{(\alpha)} \arrow[r, "\delta_\alpha"] & \mathrm{Hom}_A(T,T^{(\alpha)}) \arrow[r] & \mathrm{Coker}(\delta_\alpha) \arrow[r] & 0.
\end{tikzcd}$$

Since δ_α is injective, we conclude that $\eta'_{B^{(\alpha)}}$ also is injective, and therefore $\mathrm{Coker}(\eta'_{B^{(\alpha)}}) \simeq \mathrm{Coker}(\delta_\alpha) \in \mathscr{E}$. This finishes the whole proof. $\square$

In the next lemma we give a description of the category $\mathscr{E}$.

Lemma 4.5. *The following statements hold.*

(1) $\mathscr{E} = \{X \in B\text{-Mod} \mid \mathrm{Hom}_{\mathscr{D}(B)}(Q^\bullet, X[i]) = 0$ *for all* $i \in \mathbb{Z}\}$. *In particular, $\mathscr{E}$ is closed under direct sums and products.*

(2) *$\mathscr{E}$ is closed under isomorphic images, extensions, kernels and cokernels. In particular, $\mathscr{E}$ is an abelian subcategory of B-Mod.*

Proof. (1) Let X be a B-module and i an integer. Then

$$\mathrm{Hom}_{\mathscr{D}(B)}(Q^\bullet, X[i]) \simeq \mathrm{Hom}_{\mathscr{K}(B)}(Q^\bullet, X[i]) \simeq H^i(\mathrm{Hom}_B(Q^\bullet, X)) \simeq H^i(\mathrm{Hom}_B(Q^\bullet, B)\otimes_B X),$$

where the last isomorphism follows from the fact that the restriction of the natural transformation $\mathrm{Hom}_B(-,B)\otimes_B X \to \mathrm{Hom}_B(-,X)$ to $\mathscr{C}(B\text{-proj})$ is a natural isomorphism. By the definition of $Q^\bullet$, we know that $\mathrm{Hom}_B(Q^\bullet, B)$ is the complex:

$$\cdots \longrightarrow 0 \longrightarrow \mathrm{Hom}_A(T_1,T) \xrightarrow{\varphi_*} \mathrm{Hom}_A(T_0,T) \longrightarrow 0 \longrightarrow \cdots$$

in $\mathscr{C}^b(B^{\mathrm{op}}\text{-proj})$, where $\varphi_* := \mathrm{Hom}_A(\varphi, T)$, and where the finitely generated projective B^{op}-modules $\mathrm{Hom}_A(T_1,T)$ and $\mathrm{Hom}_A(T_0,T)$ are of degrees -1 and 0, respectively. Note that the conditions (T_2) and (T_3) in Definition 4.1 imply that the sequence

$$0 \longrightarrow \mathrm{Hom}_A(T_1,T) \xrightarrow{\varphi_*} \mathrm{Hom}_A(T_0,T) \longrightarrow T \longrightarrow 0$$

is exact. In other words, the complex $\mathrm{Hom}_B(Q^\bullet, {}_BB)$ is quasi-isomorphic to T_B. Here we use the fact that the functor $\mathrm{Hom}_A(-,T) : \mathrm{add}({}_AT) \to \mathrm{add}(B_B)$ is an equivalence of categories. It follows from the definition of Tor_i^B that

$$H^i\big(\mathrm{Hom}_B(Q^\bullet, B)\otimes_B X\big) \simeq \begin{cases} 0 & \text{if } i > 0, \\ \mathrm{Tor}_{-i}^B(T,X) & \text{if } i \le 0. \end{cases}$$

This means that $\mathrm{Hom}_{\mathscr{D}(B)}(Q^\bullet, X[i]) = 0$ if and only if $\mathrm{Tor}_{-i}^B(T,X) = 0$. Hence

$$\mathscr{E} = \{X \in B\text{-Mod} \mid \mathrm{Hom}_{\mathscr{D}(B)}(Q^\bullet, X[i]) = 0 \text{ for all } i \in \mathbb{Z}\}.$$

Consequently, $\mathscr{E}$ is closed under direct products. Further, since $Q^\bullet$ is a bounded complex of finitely generated projective B-modules, we know that $\mathscr{E}$ is closed under direct sums, too.

(2) This statement follows directly from Proposition 3.3(1). $\square$

The following proposition is crucial to the proof of Theorem 1.1(1).

Proposition 4.6. *The triple* $\big(\mathrm{Tria}(Q^\bullet), \mathrm{Ker}(G), \mathrm{Im}(H)\big)$ *is a* TTF *triple in* $\mathscr{D}(B)$. *Moreover,*

$$\mathrm{Ker}(G) = \{\overline{Y}^\bullet \in \mathscr{D}(B) \mid \overline{Y}^\bullet \simeq Y^\bullet \text{ in } \mathscr{D}(B) \text{ with } Y^i \in \mathscr{E} \text{ for all } i \in \mathbb{Z}\};$$

$$\mathrm{Im}(H) = \{\overline{Z}^\bullet \in \mathscr{D}(B) \mid \overline{Z}^\bullet \simeq Z^\bullet \;\; in \;\; \mathscr{D}(B) \;\; with \; Z^i \in \mathrm{Hom}_A(T, \mathrm{Add}(T)) \; for \;\; all \;\; i \in \mathbb{Z}\},$$

where $\mathrm{Hom}_A(T, \mathrm{Add}(T))$ *stands for the full subcategory of* B-Mod *consisting of all the modules* $\mathrm{Hom}_A(T, T')$ *with* T' *in* $\mathrm{Add}(T)$.

Proof. Recall that we have denoted $\mathrm{Ker}(G)$ by $\mathcal{Y}$, and $\mathrm{Im}(H)$ by $\mathcal{Z}$. The whole proof of this proposition will be divided into three steps.

Step (1). We prove that the pair $(\mathcal{Y}, \mathcal{Z})$ is a torsion pair in $\mathscr{D}(B)$. In fact, for any $Y^\bullet \in \mathcal{Y}$ and $W^\bullet \in \mathscr{D}(A)$, we have $\mathrm{Hom}_{\mathscr{D}(B)}(Y^\bullet, H(W^\bullet)) \simeq \mathrm{Hom}_{\mathscr{D}(A)}(G(Y^\bullet), W^\bullet) = \mathrm{Hom}_{\mathscr{D}(A)}(0, W^\bullet) = 0$ because the pair (G, H) is an adjoint pair of triangle functors by Lemma 4.3. This shows $\mathrm{Hom}_{\mathscr{D}(B)}(\mathcal{Y}, \mathcal{Z}) = 0$. Let $\eta : Id_{\mathscr{D}(B)} \to HG$ be the unit adjunction, and let $\varepsilon : GH \to Id_{\mathscr{D}(A)}$ be the counit adjunction. By Lemma 4.3, we know that ε is invertible. For any $M^\bullet$ in $\mathscr{D}(B)$, the canonical morphism $\eta_{M^\bullet} : M^\bullet \to HG(M^\bullet)$ can be extended to a triangle in $\mathscr{D}(B)$:

$$M^\bullet \xrightarrow{\eta_{M^\bullet}} HG(M^\bullet) \longrightarrow N^\bullet \longrightarrow M^\bullet[1].$$

By applying the functor G to the above triangle, we obtain a triangle in $\mathscr{D}(A)$:

$$G(M^\bullet) \xrightarrow{G(\eta_{M^\bullet})} GHG(M^\bullet) \longrightarrow G(N^\bullet) \longrightarrow G(M^\bullet)[1].$$

Since ε is invertible, we see that $G(\eta_{M^\bullet})$ is an isomorphism. This shows $G(N^\bullet) = 0$, that is, $N^\bullet \in \mathcal{Y}$. Since $\mathcal{Y}$ is a triangulated subcategory of $\mathscr{D}(B)$, we have $N^\bullet[-1] \in \mathcal{Y}$. Thus the following triangle

$$(*) \quad N^\bullet[-1] \longrightarrow M^\bullet \xrightarrow{\eta_{M^\bullet}} HG(M^\bullet) \longrightarrow N^\bullet$$

in $\mathscr{D}(B)$ with $HG(M^\bullet) \in \mathcal{Z}$ shows that the third condition of Definition 2.5 is satisfied. Hence the pair $(\mathcal{Y}, \mathcal{Z})$ is a torsion pair in $\mathscr{D}(B)$ by Definition 2.5. Since $\mathcal{Y}$ is a triangulated category, the torsion pair $(\mathcal{Y}, \mathcal{Z})$ is hereditary.

Step (2). We calculate the categories $\mathcal{Y}$ and $\mathcal{Z}$. Before starting our calculations, we mention the following result in [40, Theorem 10.5.9, Corollary 10.5.11]:

For every complex $X^\bullet$ in $\mathscr{D}(B)$, there exists a quasi-isomorphism $\overline{X}^\bullet \to X^\bullet$ with $\overline{X}^\bullet$ a complex of $({}_AT \otimes_B -)$-acyclic B-modules such that $G(X^\bullet) \simeq T \otimes_B \overline{X}^\bullet$. Here, a B-module N is said to be $({}_AT \otimes_B -)$-acyclic if $\mathrm{Tor}_i^B(T, N) = 0$ for any $i > 0$. Thus the action of the left derived functor G on any complex $U^\bullet$ of $({}_AT \otimes_B -)$-acyclic B-modules is the same as that of the functor ${}_AT \otimes_B -$ which acts in component wise on each term of $U^\bullet$.

A similar statement holds for the right derived functor H.

Now let us first interpret the triangle $(*)$ in terms of objects in $\mathscr{C}(B\text{-Proj})$. For the complex $M^\bullet$, we choose $P^\bullet \in \mathscr{C}(B\text{-Proj})$ such that $P^\bullet$ is quasi-isomorphic to $M^\bullet$. Then $G(M^\bullet) \simeq T \otimes_B P^\bullet$. By Lemma 4.2(3), we have $HG(M^\bullet) = \mathrm{Hom}_A(T, T \otimes_B P^\bullet)$ because the A-module $T \otimes_B P$ is $\mathrm{Hom}_A(T, -)$-acyclic for any projective B-module P. Note that the homomorphism $\eta_{P^\bullet}$ coincides with $(\eta'_{P^n})_{n \in \mathbb{Z}}$, where P^n is the n-th term of the complex $P^\bullet$ and $\eta'_{P^n} : P^n \to \mathrm{Hom}_A(T, T \otimes_B P^n)$ is the unit adjunction morphism for each $n \in \mathbb{Z}$. By Lemma 4.4, there is a short exact sequence of complexes

$$0 \longrightarrow P^\bullet \xrightarrow{\eta_{P^\bullet}} \mathrm{Hom}_A(T, T \otimes_B P^\bullet) \longrightarrow \mathrm{Coker}(\eta_{P^\bullet}) \longrightarrow 0$$

such that $(\mathrm{Coker}(\eta_{P^\bullet}))^i = \mathrm{Coker}(\eta'_{P^i}) \in \mathscr{E}$ for each $i \in \mathbb{Z}$. Thus, we can form the following commutative diagram of triangles in $\mathscr{D}(B)$:

$$\begin{tikzcd}
\mathrm{Coker}(\eta_{P^\bullet})[-1] \arrow[r] \arrow[d, dashed, "\simeq"] & P^\bullet \arrow[r, "\eta_{P^\bullet}"] \arrow[d, "\simeq"] & \mathrm{Hom}_A(T, T\otimes_B P^\bullet) \arrow[r] \arrow[d, equal] & \mathrm{Coker}(\eta_{P^\bullet}) \arrow[d, dashed, "\simeq"] \\
N^\bullet[-1] \arrow[r] & M^\bullet \arrow[r, "\eta_{M^\bullet}"] & \mathbb{R}\mathrm{Hom}_A(T, T\otimes^{\mathbb{L}}_B M^\bullet) \arrow[r] & N^\bullet.
\end{tikzcd}$$

On the one hand, if $M^\bullet \in \mathcal{Y}$, then $T\otimes^{\mathbb{L}}_B M^\bullet = 0$ by definition, and so $M^\bullet \simeq \mathrm{Coker}(\eta_{P^\bullet})[-1]$ in $\mathscr{D}(B)$. On the other hand, if $M^\bullet \simeq Y^\bullet$ in $\mathscr{D}(B)$ for some complex $Y^\bullet$ with $Y^i \in \mathscr{E}$ for each $i \in \mathbb{Z}$, then $T\otimes^{\mathbb{L}}_B M^\bullet \simeq T\otimes^{\mathbb{L}}_B Y^\bullet = T\otimes_B Y^\bullet = 0$ by the above mentioned fact. This means $M^\bullet \in \mathcal{Y}$. Hence the first equality in Proposition 4.6 holds.

To prove the second equality, we observe that, by Lemma 4.2(4), $\mathrm{Hom}_A(T, T\otimes_B \mathrm{Hom}_A(T,T')) \simeq \mathrm{Hom}_A(T,T')$ for any $T' \in \mathrm{Add}(T)$. Let $Z^\bullet$ be a complex in $\mathscr{D}(B)$ such that $Z^i \in \mathrm{Hom}_A(T, \mathrm{Add}(T))$. Then $HG(Z^\bullet) \simeq \mathrm{Hom}_A(T, T\otimes_B Z^\bullet) \simeq Z^\bullet$ in $\mathscr{D}(B)$ because every B-module in $\mathrm{Hom}_A(T,\mathrm{Add}(T))$ is $(T\otimes_B -)$-acyclic by Lemmata 4.2(3) and 4.2(4). This implies $Z^\bullet \in \mathcal{Z}$. Conversely, for any $W^\bullet \in \mathscr{D}(A)$, we can choose a complex $L^\bullet \in \mathscr{C}(B\text{-Proj})$ such that $L^\bullet$ is quasi-isomorphic to $H(W^\bullet)$. By Lemma 4.3, we conclude that $H(W^\bullet) \simeq HG(H(W^\bullet)) \simeq HG(L^\bullet)$ in $\mathscr{D}(B)$. Since $HG(L^\bullet) = H(T\otimes^{\mathbb{L}}_B L^\bullet) = H(T\otimes_B L^\bullet) \simeq \mathrm{Hom}_A(T, T\otimes_B L^\bullet)$, where the last isomorphism follows from Lemma 4.2(3) and the above mentioned fact about the functor H. Clearly, the complex $\mathrm{Hom}_A(T, T\otimes_B L^\bullet)$ has each term in $\mathrm{Hom}_A(T,\mathrm{Add}(T))$. Thus the second equality in Proposition 4.6 holds.

Step (3). We claim that there is a full subcategory $\mathcal{X}$ of $\mathscr{D}(B)$ such that $(\mathcal{X},\mathcal{Y},\mathcal{Z})$ is a TTF triple in $\mathscr{D}(B)$. Furthermore, we have $\mathcal{X} = \mathrm{Tria}(Q^\bullet)$.

Indeed, since $\mathscr{E}$ is closed under direct sums and products by Lemma 4.5, we conclude that $\mathcal{Y}$ is closed under all small coproducts and products. Then the existence of the TTF triple $(\mathcal{X},\mathcal{Y},\mathcal{Z})$ in $\mathscr{D}(B)$ follows straightforward from Lemma 2.10. Moreover, $\mathcal{X} = \mathrm{Ker}(\mathrm{Hom}_{\mathscr{D}(B)}(-,\mathcal{Y}))$ and $\mathcal{Y} = \mathrm{Ker}(\mathrm{Hom}_{\mathscr{D}(B)}(\mathcal{X},-))$. Now we shall prove $\mathcal{X} = \mathrm{Tria}(Q^\bullet)$. First, we show $Q^\bullet \in \mathcal{X}$. This is equivalent to verifying $\mathrm{Hom}_{\mathscr{D}(B)}(Q^\bullet,\mathcal{Y}) = 0$. Let $\mathcal{Y}' := \mathrm{Ker}(\mathrm{Hom}_{\mathscr{D}(B)}(\mathrm{Tria}(Q^\bullet),-))$. By Lemma 2.11, we see that $(\mathrm{Tria}(Q^\bullet), \mathcal{Y}')$ is a torsion pair in $\mathscr{D}(B)$ with

$$\mathcal{Y}' = \{Y^\bullet \in \mathscr{D}(B) \mid \mathrm{Hom}_{\mathscr{D}(B)}(Q^\bullet, Y^\bullet[i]) = 0 \text{ for all } i \in \mathbb{Z}\}.$$

Recall that $\varphi^* := \mathrm{Hom}_A(T,\varphi) : \mathrm{Hom}_A(T,T_0) \to \mathrm{Hom}_A(T,T_1)$ is a homomorphism between finitely generated projective B-modules. We define $\Sigma := \{\varphi^*\}$. Then $\Sigma^\bullet = \{Q^\bullet[1]\}$ (see notations in Section 3). By Lemma 4.5 (1), we have $\Sigma^\perp = \mathscr{E}$. Thus it follows from Proposition 3.3 that

$$\mathcal{Y}' = \mathscr{D}(B)_{\mathscr{E}} := \{Y^\bullet \in \mathscr{D}(B) \mid H^i(Y^\bullet) \in \mathscr{E} \text{ for all } i \in \mathbb{Z}\}.$$

According to Lemma 4.5 (2), $\mathscr{E}$ is an abelian subcategory of B-Mod. This forces $\mathcal{Y} \subseteq \mathcal{Y}'$. In particular, we have $\mathrm{Hom}_{\mathscr{D}(B)}(Q^\bullet,\mathcal{Y}) = 0$, which yields $Q^\bullet \in \mathcal{X}$. Therefore, $\mathrm{Tria}(Q^\bullet) \subseteq \mathcal{X}$ since $\mathcal{X}$ is a full triangulated subcategory of $\mathscr{D}(B)$.

Let $\mathbf{i} : \mathcal{X} \to \mathscr{D}(B)$ and $\mathbf{k} : \mathcal{Z} \to \mathscr{D}(B)$ be the canonical inclusions. Then the functor $\mathbf{i}$ has a right adjoint functor $\mathbf{R} : \mathscr{D}(B) \to \mathcal{X}$. Since $(\mathcal{X},\mathcal{Y},\mathcal{Z})$ is a TTF triple in $\mathscr{D}(B)$, the functor $\mathbf{Rk} : \mathcal{Z} \to \mathcal{X}$ is an equivalence (see the statements after Definition 2.6 in Subsection 2.3). So the composition functor $\mathbf{Rk}H : \mathscr{D}(A) \to \mathcal{X}$ is an equivalence because $H : \mathscr{D}(A) \to \mathcal{Z}$ is an equivalence. Since a functor possessing a right adjoint functor preserves coproducts, we know that the functor $\mathbf{Rk}H$ commutes with coproducts. Note that coproducts depends on the category where coproducts are taken. We know that coproducts in $\mathcal{Z}$ exist since $\mathscr{D}(A)$ admits all small coproducts, but we do not know if these coproducts in $\mathcal{Z}$ coincide with that in $\mathscr{D}(B)$. In general, $\mathcal{Z}$ is not closed under coproducts in $\mathscr{D}(B)$.

Since a torsion class in $\mathscr{D}(B)$ is always closed under coproducts, this means that coproducts in $\mathcal{X}$ exist and coincide with that in $\mathscr{D}(B)$.

Since $H(A) \simeq Q^\bullet \in \mathcal{X}$, we have $\mathbf{Rk}H(A) \simeq \mathbf{R}(Q^\bullet) = Q^\bullet$. Note that $\mathscr{D}(A) = \mathrm{Tria}(A)$ and that the triangle functor $\mathbf{Rk}H : \mathscr{D}(A) \to \mathcal{X}$ is an equivalence under which $\mathrm{Tria}(A)$ has the image $\mathrm{Tria}(Q^\bullet)$ since the functor $\mathbf{Rk}H$ commutes with coproducts. It follows that $\mathcal{X} = \mathrm{Tria}(Q^\bullet)$ and $\mathcal{Y} = \mathcal{Y}'$. Hence $\big(\mathrm{Tria}(Q^\bullet), \mathrm{Ker}(G), \mathrm{Im}(H)\big)$ is a TTF triple in $\mathscr{D}(B)$. $\square$

As a consequence of Proposition 4.6, we give an alternative proof of the fact that finitely generated tilting modules are classical. A known proof of this fact is a combination of [2, Corollary 9.13(5)] together with a result in [10]. We thank Lidia Angeleri-Hügel for pointing out these references.

Corollary 4.7. *Suppose that T is a tilting A-module. If T is finitely generated, then T is classical.*

Proof. In general, the following facts are true for a finitely generated A-module M, with $B := \mathrm{End}_A(M)$:

(1) For any index set δ and $X_i \in A\text{-Mod}$ with $i \in \delta$, the canonical homomorphism $\bigoplus_{i\in\delta} \mathrm{Hom}_A(M, X_i) \to \mathrm{Hom}_A(M, \bigoplus_{i\in\delta} X_i)$, given by $(f_i)_{i\in\delta} \mapsto [m \mapsto (mf_i)_{i\in\delta}]$, is an isomorphism.

(2) The functor $\mathrm{Hom}_A(M, -) : \mathrm{Add}(M) \longrightarrow \mathrm{Add}({}_BB)$ is an equivalence of additive categories.

(3) If $M \in \mathrm{Add}(N)$ for some A-module N, then $M \in \mathrm{add}(N)$.

The proofs of (1) and (3) are standard. It is easy to see that (2) follows from (1) together with the natural isomorphism $\mathrm{Hom}_A(M, U) \otimes_B - \longrightarrow \mathrm{Hom}_R(M, U \otimes_B -)$ of the functors from $\mathrm{Add}({}_BB)$ to itself, where U is an A-B-bimodule.

We shall first prove that the tilting A-module T is good. Indeed, it follows from $(T3)$ in Definition 4.1 that there exists an exact sequence

$$0 \longrightarrow {}_AA \xrightarrow{f_0} T_0 \xrightarrow{f_1} T_1 \longrightarrow 0$$

of A-modules such that $T_i \in \mathrm{Add}(T)$ for $i = 0, 1$. Without loss of generality, we can assume that $T_0 = T^{(\alpha)}$ for some index set α. Clearly, there exists a finite subset β of α such that $(1)f_0 \in T^{(\beta)}$ which is a direct summand of $T^{(\alpha)}$. This implies $\mathrm{Im}(f_0) \subseteq T^{(\beta)}$. Consequently, we have $f_0 = (f', 0) : A \to T^{(\alpha)} = T^{(\beta)} \oplus T^{(\alpha\setminus\beta)}$, where $f' : A \to T^{(\beta)}$ is the right multiplication map defined by $(1)f_0$. Thus $T_1 \simeq \mathrm{Coker}(f') \oplus T^{(\alpha\setminus\beta)}$ as A-modules, and therefore $\mathrm{Coker}(f') \in \mathrm{Add}(T)$. Further, $\mathrm{Coker}(f')$ is finitely generated since T is finitely generated. By (3), we have $\mathrm{Coker}(f') \in \mathrm{add}(T)$. As a result, there exists an exact sequence $0 \to A \to T^{(\beta)} \to \mathrm{Coker}(f') \to 0$ of A-modules such that both $T^{(\beta)}$ and $\mathrm{Coker}(f')$ belong to $\mathrm{add}(T)$. Thus T is a good tilting A-module.

By Lemma 4.3, the total right derived functor $H : \mathscr{D}(A) \to \mathscr{D}(B)$ with $B := \mathrm{End}_A(T)$ is fully faithful. Thus, to show that T is a classical tilting A-module, it suffices to show $\mathrm{Im}(H) = \mathscr{D}(B)$. In fact, since T is finitely generated, we know that $\mathrm{Hom}_A(T, -) : \mathrm{Add}(T) \to \mathrm{Add}({}_BB)$ is an equivalence by (2). Let $\mathrm{Hom}_A(T, \mathrm{Add}(T))$ stand for the full subcategory of B-Mod consisting of all the modules $\mathrm{Hom}_A(T, T')$ with T' in $\mathrm{Add}(T)$. Then $\mathrm{Hom}_A(T, \mathrm{Add}(T)) = \mathrm{Add}({}_BB)$ by (1). Note that, for each complex $Y^\bullet$ in $\mathscr{D}(B)$, there is a complex $P^\bullet$ in $\mathscr{C}(\mathrm{Add}({}_BB))$ such that $P^\bullet$ is quasi-isomorphic to $Y^\bullet$. Thus we conclude from Proposition 4.6 that $\mathrm{Im}(H) = \mathscr{D}(B)$. This shows that T is classical, finishing the proof. $\square$

With the above preparations, now we prove Theorem 1.1 (1).

Proof of Theorem 1.1 (1). By Proposition 4.6, we know that the triple $(\mathrm{Tria}(Q^\bullet), \mathrm{Ker}(G), \mathrm{Im}(H))$ is a TTF triple in $\mathscr{D}(B)$. Moreover, $\mathscr{D}(A)$ and $\mathrm{Tria}(Q^\bullet)$ are equivalent as triangulated categories. According to the correspondence between recollements and TTF triples in Lemma 2.7(2), we can form the following recollement

$$\begin{tikzcd} \mathrm{Ker}(G) \arrow[r, "\mathbf{j}"] & \mathscr{D}(B) \arrow[l, bend right, "\mathbf{L}"'] \arrow[l, bend left] \arrow[r] & \mathscr{D}(A) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd},$$

where $\mathbf{j}$ is the canonical embedding and $\mathbf{L}$ is the left adjoint of $\mathbf{j}$. Recall that $\varphi^* := \mathrm{Hom}_A(T, \varphi)$ is the homomorphism between the finitely generated projective B-modules $\mathrm{Hom}_A(T, T_0)$ and $\mathrm{Hom}_A(T, T_1)$. As in

Section 3, we define $\Sigma := \{\varphi^*\}$. By Lemma 4.5(1), we have $\Sigma^{\perp} = \mathscr{E}$. By Step (3) in the proof of Proposition 4.6, we have $\mathrm{Ker}(G) = \mathscr{D}(B)_{\Sigma^{\perp}}$. Let $\lambda : B \to B_{\Sigma}$ be the universal localization of B at Σ. Since $\mathbf{L}$ is a functor from $\mathscr{D}(B)$ to $\mathrm{Ker}(G)$, we have $\mathbf{L}(B) \in \mathrm{Ker}(G)$, and therefore it satisfies the condition (5) of Proposition 3.6, according to Proposition 4.6. Thus, by Proposition 3.6, we know that $\lambda_* : \mathscr{D}(B_{\Sigma}) \xrightarrow{\sim} \mathscr{D}(B)_{\Sigma^{\perp}}$ is an equivalence of triangulated categories, and that the homomorphism λ is a homological ring epimorphism. Set $C := B_{\Sigma}$. Then $\mathrm{Ker}(G)$ and $\mathscr{D}(C)$ are equivalent as triangulated categories. Consequently, we can get the following recollement from the above one:

$$\begin{tikzcd} \mathscr{D}(C) \arrow[r] & \mathscr{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(A) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}.$$

In the following, we shall explicitly describe the six triangle functors arising in the above recollement.

Here, we follow the notations used in Definition 2.4, and take $\mathcal{D} = \mathscr{D}(B), \mathcal{D}' = \mathscr{D}(A)$ and $\mathcal{D}'' = \mathscr{D}(C)$. Then we can define $i^* = C \otimes_B^{\mathbb{L}} -$, $i_* = \lambda_*$ and $i^! = \mathbb{R}\mathrm{Hom}_B(C, -)$. As for the other three functors, we put $j_! = \mathbf{iRk}H$, $j^! = G$ and $j_* = H$ up to natural isomorphism. Let $\mathbf{U} : \mathscr{D}(B) \to \mathcal{Z}$ be a left adjoint of the inclusion $\mathbf{k} : \mathcal{Z} \to \mathscr{D}(B)$. By Lemma 2.7 and the proof of Proposition 4.6, we get the following diagram of functors

$$\begin{tikzcd} \mathscr{D}(B) \arrow[r, "\mathbf{R}"] & \mathcal{X} \arrow[l, bend right, "\mathbf{i}"'] \arrow[l, bend left, "\mathbf{kUi}"] & \mathscr{D}(A) \arrow[l, "\mathbf{Rk}H"'] \end{tikzcd}$$

with the properties:

(i) $(\mathbf{i}, \mathbf{R})$ and $(\mathbf{R}, \mathbf{kUi})$ are adjoint pairs,

(ii) $\mathbf{Rk}H$ is an equivalence of triangulated categories.

This implies that $j_! = \mathbf{iRk}H$ and $j_* = (\mathbf{kUi})(\mathbf{Rk}H)$. Note that the composition functor $\mathbf{UiRk} : \mathcal{Z} \to \mathcal{Z}$ of the functors $\mathbf{Ui}$ and $\mathbf{Rk}$ is natural isomorphic to the identity functor $1_{\mathcal{Z}}$ by the property (3) of a TTF triple (see Subsection 2.3). Consequently, we can choose $j_* = H$. Since (G, H) is an adjoint pair of functors, we can choose $j^* = G$. Thus the proof of the first part of Theorem 1.1 is completed. $\square$

Remarks. (1) In the proof of Theorem 1.1(1), we have $\Sigma = \{\varphi^*\}$, $\Sigma^{\perp} = \mathscr{E}$ and $\mathrm{Hom}_{\mathscr{D}(B)}(Q^\bullet, Q^\bullet[1]) = 0$. This means that the homomorphism φ^* satisfies the assumptions in Corollary 3.11. Therefore we can explicitly construct a left adjoint functor $l : B\text{-Mod} \to \mathscr{E}$ of the inclusion $j : \mathscr{E} \to B\text{-Mod}$. In particular, we know that C is isomorphic to the endomorphism ring $\mathrm{End}_B(l(B))$ of $l(B)$.

(2) The ring C equals zero if and only if T is a classical tilting module. In fact, $C = 0$ if and only if $\mathrm{Ker}(G) = 0$ if and only if G is an equivalence if and only if T is classical.

(3) From the proof of Theorem 1.1(1), we know that a good tilting module T has the property: the functor G admits a fully faithful left adjoint $j_!$. In the next section, we shall show that this property guarantees that the tilting module T is good.

Finally, we point out that, by Lemma 3.2, the ring C in Theorem 1.1 is isomorphic to the universal localization of B at the ψ in Lemma 4.2.

5 Existence of recollements implies goodness of tilting modules

In this section, we shall prove the second part of Theorem 1.1, which is a converse of the first part in some sense. Our proof depends on the property that the total left derived functor G admits a fully faithful left adjoint $j_!$.

Proof of Theorem 1.1 (2).

Let T be a tilting A-module and B the endomorphism ring of T. Recall that G and H stand for the triangle functors $T \otimes_B^{\mathbb{L}} - : \mathscr{D}(B) \to \mathscr{D}(A)$ and $\mathbb{R}\mathrm{Hom}_A(T,-) : \mathscr{D}(A) \to \mathscr{D}(B)$, respectively. Suppose that G admits a fully faithful left adjoint $j_! : \mathscr{D}(A) \to \mathscr{D}(B)$. We want to show that T is a good tilting module.

To prove that T is good, it suffices to find a short exact sequence of A-modules,

$$0 \longrightarrow A \longrightarrow T_0 \longrightarrow T_1 \longrightarrow 0,$$

such that $T_i \in \mathrm{add}(T)$ for $i = 0, 1$.

First, we observe some consequences of the assumption that $j_!$ is fully faithful. Set $W^\bullet := j_!(A)$. Since the total left derived functor G commutes with coproducts, we can easily show that the functor $j_!$ preserves compact objects. In particular, the complex $W^\bullet$ is compact in $\mathscr{D}(B)$, which implies $W^\bullet \simeq Q^\bullet$ in $\mathscr{D}(B)$ for some $Q^\bullet \in \mathscr{C}^b(B\text{-proj})$. Since the Hom-functor $\mathrm{Hom}_A(T,-)$ induces an equivalence between $\mathrm{add}(T)$ and B-proj, we can assume that $Q^\bullet = \mathrm{Hom}_A(T, X^\bullet)$, where $X^\bullet \in \mathscr{C}^b(\mathrm{add}(T))$ is of the following form

$$0 \longrightarrow X^s \longrightarrow \cdots \longrightarrow X^i \xrightarrow{d^i} X^{i+1} \longrightarrow \cdots \longrightarrow X^t \longrightarrow 0$$

for $s \leq 0 \leq t$. Since the functor $j_!$ is fully faithful, we conclude from [29, Chapter IV, Section 3, Theorem 1, p.90] that the unit adjunction morphism $\widetilde{\eta} : Id_{\mathscr{D}(A)} \to Gj_!$ is invertible. Thus $A \simeq G(W^\bullet) \simeq G(Q^\bullet)$ in $\mathscr{D}(A)$. Note that $T \otimes_B \mathrm{Hom}_A(T, X^\bullet) \simeq X^\bullet$ in $\mathscr{C}^b(A\text{-mod})$ since $X^i \in \mathrm{add}(T)$ for each $s \leq i \leq t$. Consequently, we have $A \simeq X^\bullet$ in $\mathscr{D}(A)$. It follows that $H^0(X^\bullet) \simeq A$ and $H^i(X^\bullet) = 0$ for any $i \neq 0$.

Second, if $t = 0$, then the homomorphism $X^0 \to H^0(X^\bullet)$ splits, this implies $A \in \mathrm{add}(T)$. Hence T is a good tilting module. Now we assume $t \neq 0$. Then we can decompose $X^\bullet$ into two long exact sequence of A-modules:

$$0 \longrightarrow X^s \xrightarrow{d^s} \cdots \longrightarrow X^{-1} \xrightarrow{d^{-1}} X^0 \xrightarrow{\pi} M \longrightarrow 0,$$

$$0 \longrightarrow A \longrightarrow M \xrightarrow{\mu} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^t} X^t \longrightarrow 0;$$

where $d^0 = \pi\mu$ and M is the image of d^0. We claim $\mathrm{Im}(\mu) \in \mathrm{add}(T)$. In fact, we have a long exact sequence

$$0 \longrightarrow \mathrm{Im}(\mu) \xrightarrow{\nu} X^1 \xrightarrow{d^1} \cdots \xrightarrow{d^t} X^t \longrightarrow 0,$$

where ν is the canonical inclusion. For each $1 \leq i \leq t$, since $X^i \in \mathrm{add}(T)$, we have $\mathrm{Im}(d^i) \in \mathrm{Gen}(T)$. As we know, $T^\perp = \mathrm{Gen}(T)$ for a tilting module T. Consequently, we see that $\mathrm{Ext}_A^1(T, \mathrm{Im}(d^i)) = 0$ for any $1 \leq i \leq t$. Note that $\mathrm{Im}(d^t) = X^t \in \mathrm{add}(T)$. Thus we can easily show $\mathrm{Im}(\mu) \in \mathrm{add}(T)$ by induction on t.

Finally, we shall prove $M \in \mathrm{add}(T)$. If $s = 0$, then $M = X^0 \in \mathrm{add}(T)$. Suppose $s < 0$. Since $\mathrm{Im}(\mu) \in \mathrm{add}(T)$ and the sequence $0 \to A \to M \to \mathrm{Im}(\mu) \to 0$ is exact, we know that $\mathrm{Ext}_A^1(M, T) = 0$ and M has projective dimension at most 1. In addition, $\mathrm{Im}(d^{-1})$ is a quotient module of X^{-1}. It follows that $\mathrm{Ext}_A^1(M, \mathrm{Im}(d^{-1})) = 0$, which implies that the homomorphism π splits. Thus $M \in \mathrm{add}(X^0) \subseteq \mathrm{add}(T)$.

Now we define $T_0 = M$ and $T_1 = \mathrm{Im}(\mu)$. Then the sequence $0 \to A \to T_0 \xrightarrow{\mu} T_1 \to 0$ satisfies $T_i \in \mathrm{add}(T)$ for $i = 0, 1$. Thus T is a good tilting module, and the proof is completed. $\square$

Remark. Suppose that G admits a fully faithful left adjoint $j_! : \mathscr{D}(A) \to \mathscr{D}(B)$. Then there exists a TTF triple $\big(j_!(\mathscr{D}(A)), \mathrm{Ker}(G), H(\mathscr{D}(A))\big)$ in $\mathscr{D}(B)$ (see [14, Chapter I, Proposition 2.11] for details), where $j_!(\mathscr{D}(A))$ and $H(\mathscr{D}(A))$ denote the images of $j_!$ and H, respectively. By Lemma 2.7, we know that the derived category $\mathscr{D}(B)$ is a recollement of the derived category $\mathscr{D}(A)$ and the triangulated category $\mathrm{Ker}(G)$. Since T is good by Theorem 1.1(2), it follows from Theorem 1.1(1) that $\mathrm{Ker}(G)$ is triangle equivalent to the derived category $\mathscr{D}(C)$ of a ring C. Thus we get a recollement of derived module categories as in Theorem 1.1(1).

6 Applications to tilting modules arising from ring epimorphisms

In this section we apply our main result Theorem 1.1 to tilting modules arising from ring epimorphisms. In this case we shall describe the universal localization rings appearing in the main result by coproducts defined by Cohn in [18]. In fact, our discussion in this section will be implemented in the general setup of injective homological ring epimorphisms between arbitrary rings, which is weaker than the condition of being tilting modules.

We start with recalling of some definitions.

Let R_0 be a ring with identity. An R_0-ring is a ring R together with a ring homomorphism $\lambda_R : R_0 \to R$. An R_0-homomorphism from an R_0-ring R to another R_0-ring S is a ring homomorphism $f : R \to S$ such that $\lambda_S = \lambda_R f$. If R_0 is commutative and the image of $\lambda_R : R_0 \to R$ is contained in the center $Z(R)$ of R, then we say that R is an R_0-algebra.

Recall that the coproduct of a family $\{R_i \mid i \in I\}$ of R_0-rings with I an index set is an R_0-ring R together with a family $\{\rho_i : R_i \to R \mid i \in I\}$ of R_0-homomorphisms such that, for any R_0-ring S with a family of R_0-homomorphisms $\{\tau_i : R_i \to S \mid i \in I\}$, there is a unique R_0-homomorphism $\delta : R \to S$ such that $\tau_i = \rho_i\delta$ for all $i \in I$.

It is well-known that the coproduct of a family $\{R_i \mid i \in I\}$ of R_0-rings exists. In this case, we denote their coproduct by $\sqcup_{R_0} R_i$. For example, the coproduct of the polynomial rings $k[x]$ and $k[y]$ over a field k is the free ring $k\langle x, y\rangle$ in two variables x and y over k. Note that $R_0 \sqcup_{R_0} S = S = S \sqcup_{R_0} R_0$ for every R_0-ring S.

Let R_i be an R_0-ring for $i = 1, 2$. We denote by B the matrix ring $\begin{pmatrix} R_1 & R_1 \otimes_{R_0} R_2 \\ 0 & R_2 \end{pmatrix}$. Let $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in B$, and let $\varphi : Be_1 \to Be_2$ be the map sending $\begin{pmatrix} r_1 \\ 0 \end{pmatrix}$ to $\begin{pmatrix} r_1 \otimes 1 \\ 0 \end{pmatrix}$ for $r_1 \in R_1$. Let $\rho_i : R_i \to R_1 \sqcup_{R_0} R_2$ be the canonical R_0-homomorphism for $i = 1, 2$.

The following lemma reveals a relationship between coproducts and unversal localizations.

Lemma 6.1. [37, Theorem 4.10, p. 59] *The universal localization B_φ of B at φ is equal to $M_2(R_1 \sqcup_{R_0} R_2)$, the 2×2 matrix ring over the coproduct $R_1 \sqcup_{R_0} R_2$ of R_1 and R_2 over R_0. Furthermore, the corresponding ring homomorphism $\lambda_\varphi : B \to B_\varphi$ is given by $\begin{pmatrix} r_1 & x_1 \otimes x_2 \\ 0 & r_2 \end{pmatrix} \mapsto \begin{pmatrix} (r_1)\rho_1 & (x_1)\rho_1(x_2)\rho_2 \\ 0 & (r_2)\rho_2 \end{pmatrix}$ for $r_i, x_i \in R_i$ with $i = 1, 2$.*

The next result says, in some sense, that taking coproducts of rings preserves universal localizations.

Lemma 6.2. *Let R_0 be a ring, Σ a set of homomorphisms between finitely generated projective R_0-modules, and $\lambda_\Sigma : R_0 \to R_1 := (R_0)_\Sigma$ the universal localization of R_0 at Σ. Then, for any R_0-ring R_2, the coproduct $R_1 \sqcup_{R_0} R_2$ is isomorphic to the universal localization $(R_2)_\Delta$ of R_2 at the set $\Delta := \{R_2 \otimes_{R_0} f \mid f \in \Sigma\}$.*

Proof. Let $R := (R_2)_\Delta$ and $\lambda_\Delta : R_2 \to R$ the universal localization of R_2 at Δ. Suppose that $\lambda_{R_2} : R_0 \to R_2$ is the ring homomorphism defining the R_0-ring R_2. Then R is an R_0- ring via the composition $\lambda_{R_2}\lambda_\Delta$ of λ_{R_2} with λ_Δ. Moreover, we shall prove that there is a unique R_0-ring homomorphism $\nu : R_1 \to R$, that is, a ring homomorphism ν with $\lambda_{R_2}\lambda_\Delta = \lambda_\Sigma\nu$. In fact, for any $f : P_1 \to P_0$ in Σ, the map $R \otimes_{R_0} f : R \otimes_{R_0} P_1 \to R \otimes_{R_0} P_0$ of R-modules is an isomorphism because $R \otimes_{R_0} f \simeq R \otimes_{R_2} (R_2 \otimes_{R_0} f)$ and the latter is an isomorphism. Thus, by the universal property of λ_Σ, there is a unique ring homomorphism $\nu : R_1 \to R$ such that $\lambda_{R_2}\lambda_\Delta = \lambda_\Sigma\nu$, as desired.

Now, we show that R together with the two ring homomorphisms λ_Δ and ν satisfies the definition of coproducts, and therefore $R_1 \sqcup_{R_0} R_2$ is isomorphic to R.

Indeed, suppose that S is an arbitrary R_0-ring with two R_0-homomorphisms $\tau_i : R_i \to S$ for $i = 1, 2$. Then $\lambda_\Sigma\tau_1 = \lambda_{R_2}\tau_2$. Further, since we have

$$S \otimes_{R_2} (R_2 \otimes_{R_0} h) \simeq S \otimes_{R_0} h \simeq S \otimes_{R_1} (R_1 \otimes_{R_0} h),$$

and since $R_1 \otimes_{R_0} h$ is an isomorphism for any $h \in \Sigma$, we infer that $S \otimes_{R_2} (R_2 \otimes_{R_0} h)$ is an isomorphism for any $h \in \Sigma$. It follows from the property of universal localizations that there is a unique ring homomorphism $\delta : R \to S$ such that $\tau_2 = \lambda_\Delta \delta$. Clearly, $\lambda_\Sigma \tau_1 = \lambda_\Sigma \nu\delta$, and $\tau_1 = \nu\delta$ since λ_Σ is a ring epimorphism. Note that δ is also an R_0-ring homomorphism. Thus, $\delta : R \to S$ is actually a unique R_0-homomorphism such that $\tau_1 = \nu\delta$ and $\tau_2 = \lambda_\Delta \delta$. This shows that R is isomorphic to the coproduct $R_1 \sqcup_{R_0} R_2$ of R_1 and R_2 over R_0. $\square$

Sometimes, coproducts can be interpreted as tensor products of rings.

Lemma 6.3. *Let R_0 be a commutative ring, and let R_i be an R_0-algebra for $i = 1,2$. If one of the homomorphisms $\lambda_{R_1} : R_0 \to R_1$ and $\lambda_{R_2} : R_0 \to R_2$ is a ring epimorphism, then the coproduct $R_1 \sqcup_{R_0} R_2$ is isomorphic to the tensor product $R_1 \otimes_{R_0} R_2$.*

Proof. It is known that the tensor product $R_1 \otimes_{R_0} R_2$ of two rings R_1 and R_2 over R_0 has the following universal property: If $f_i : R_i \to R$ is a homomorphism of R_0-rings for $i = 1,2$, such that $(r_2)f_2(r_1)f_1 = (r_1)f_1(r_2)f_2$ for all $r_i \in R_i$ with $i = 1,2$, then there is a unique ring homomorphism $f : R_1 \otimes_{R_0} R_2 \to R$ of R_0-rings that satisfies $(x_1 \otimes x_2)f = (x_1)f_1(x_2)f_2$ for $x_i \in R_i$ with $i = 1,2$. In particular, if $\lambda_1 : R_1 \to R_1 \otimes_{R_0} R_2$ is the map given by $r_1 \mapsto r_1 \otimes 1$ for $r_1 \in R_1$, and if $\lambda_2 : R_2 \to R_1 \otimes_{R_0} R_2$ is the one given by $r_2 \mapsto 1 \otimes r_2$ for $r_2 \in R_2$, then $f_i = \lambda_i f$ for $i = 1,2$.

To prove Lemma 6.3, it suffices to show that, for any R_0-homomorphisms $f_i : R_i \to R$ for $i = 1,2$, the condition $(r_2)f_2(r_1)f_1 = (r_1)f_1(r_2)f_2$ holds true for all $r_i \in R_i$ with $i = 1,2$.

Assume that $\lambda_{R_1} : R_0 \to R_1$ is a ring epimorphism. For any element $y \in R_2$, we define two ring homomorphisms $\theta_1 : R_1 \to \mathrm{M}_2(\mathrm{R})$ and $\theta_2 : R_1 \to \mathrm{M}_2(\mathrm{R})$ as follows:

$$(x)\theta_1 = \begin{pmatrix} (x)f_1 & 0 \\ 0 & (x)f_1 \end{pmatrix}$$

and

$$(x)\theta_2 = \begin{pmatrix} 1 & 0 \\ (y)f_2 & 1 \end{pmatrix} \begin{pmatrix} (x)f_1 & 0 \\ 0 & (x)f_1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -(y)f_2 & 1 \end{pmatrix} = \begin{pmatrix} (x)f_1 & 0 \\ (y)f_2(x)f_1 - (x)f_1(y)f_2 & (x)f_1 \end{pmatrix}$$

for $x \in R_1$. Now, we verify $\lambda_{R_1}\theta_1 = \lambda_{R_1}\theta_2$. This is equivalent to showing that, if $x = (r)\lambda_{R_1}$ with $r \in R_0$, then $(y)f_2\,(x)f_1 = (x)f_1(y)f_2$. In fact, we always have

$$(y)f_2\,(x)f_1 = (y)f_2\big((r)\lambda_{R_1}\big)f_1 = (y)f_2\big((r)\lambda_{R_2}\big)f_2 = \big(y(r)\lambda_{R_2}\big)f_2,$$

$$(x)f_1(y)f_2 = \big((r)\lambda_{R_1}\big)f_1(y)f_2 = \big((r)\lambda_{R_2}\big)f_2(y)f_2 = \big((r)\lambda_{R_2}y\big)f_2.$$

Since R_2 is an R_0-algebra, it follows from $\mathrm{Im}(\lambda_{R_2}) \subseteq Z(R_2)$ that $y(r)\lambda_{R_2} = (r)\lambda_{R_2}y$, and so $(y)f_2(x)f_1 = (x)f_1(y)f_2$ whenever $x = (r)\lambda_{R_1}$ with $r \in R_0$. This shows $\lambda_{R_1}\theta_1 = \lambda_{R_1}\theta_2$ and $\theta_1 = \theta_2$ since $\lambda_{R_1} : R_0 \to R_1$ is a ring epimorphism. Thus $(y)f_2(x)f_1 = (x)f_1(y)f_2$ for any $x \in R_1$. Note that y is an arbitrary element of R_2. Hence $(y)f_2(x)f_1 = (x)f_1(y)f_2$ for any $x \in R_1$ and $y \in R_2$.

As a result, the tensor product $R_1 \otimes_{R_0} R_2$ together with the two ring homomorphisms λ_i satisfies the definition of coproducts, and we therefore have proved that the coproduct $R_1 \sqcup_{R_0} R_2$ is isomorphic to the tensor product $R_1 \otimes_{R_0} R_2$. Similarly, we can prove Lemma 6.3 under the assumption that $\lambda_{R_2} : R_0 \to R_2$ is a ring epimorphism. $\square$

From now on, $\lambda : R \to S$ denotes an injective ring homomorphism from R to S. We define B to be the endomorphism ring of the R-module $S \oplus S/R$, and S' the endomorphism ring of the R-module S/R. Let π stands for the canonical surjective map $S \to S/R$ of R-modules. Then we have an exact sequence of R-modules:

$$(*) \quad 0 \longrightarrow R \longrightarrow S \xrightarrow{\pi} S/R \longrightarrow 0.$$

In the next two lemmas, we collect some facts on ring epimorphisms.

Lemma 6.4. *Let $\lambda : R \to S$ be an injective ring epimorphism with* $\mathrm{Tor}_1^R(S,S)=0$. *Then,*

(1) *an R-module X belongs to S-*Mod *if and only if* $\mathrm{Ext}^i_R(S/R,X)=0$ *for $i=0,1$.*

(2) *Let $T := S\oplus S/R$. Then*

$$\mathrm{End}_R(T)\simeq \begin{pmatrix} S & \mathrm{Hom}_R(S,S/R)\\ 0 & \mathrm{End}_R(S/R)\end{pmatrix}.$$

Moreover, if e_1 and e_2 are the idempotent elements in $\mathrm{End}_R(T)$ corresponding to the summands S and S/R, respectively, then the homomorphism $\pi^* : \mathrm{End}_R(T)e_1 \to \mathrm{End}_R(T)e_2$ induced from the canonical surjection $\pi : S\to S/R$ is given by $\begin{pmatrix} s\\ 0\end{pmatrix}\mapsto \begin{pmatrix} (x\mapsto (xs)\pi)\\ 0\end{pmatrix}$ *for $s,x\in S$.*

Proof. (1) follows from [24]. For (2), it follows from (1) that $\mathrm{Hom}_R(S/R,S)=0$. By applying $\mathrm{Hom}_R(-,S)$ to the exact sequence $(*)$, we get $\mathrm{Hom}_R(S,S)\simeq \mathrm{Hom}_R(R,S)\simeq S$. □

Lemma 6.5. *Suppose that $\lambda : R\to S$ is an injective ring epimorphism with* $\mathrm{Tor}_1^R(S,S)=0$.

(1) *The right multiplication map $\mu : R\to S'$ defined by $r\mapsto (x\mapsto xr)$ for $r\in R$ and $x\in S/R$, is a ring homomorphism. Consequently, S' can be regarded as an R-ring via the map μ. Further, μ is an isomorphism if and only if $\mathrm{Ext}^i_R(S,R)=0$ for $i=0,1$.*

(2) *There is an isomorphism $\theta : S\otimes_R S'\simeq \mathrm{Hom}_R(S,S/R)$ of S-S'-bimodules such that $1\otimes 1$ is mapped to the canonical surjection $\pi : S\to S/R$.*

(3) *There is an exact sequence of R-S'-modules:*

$$0\to S'\xrightarrow{\lambda'} S\otimes_R S'\xrightarrow{\pi\otimes S'} (S/R)\otimes_R S'\to 0,$$

where the map λ' is defined by $f\mapsto 1\otimes f$ for any $f\in S'$. Moreover, the evaluation map $\psi : (S/R)\otimes_R S'\to S/R$ defined by $y\otimes g\mapsto (y)g$ for $y\in S/R$ and $g\in S'$, is an isomorphism of R-S'-bimodules.

(4) *If $\lambda : R\to S$ is homological, then $\mathrm{Tor}_i^R(S,S')=0$ for any $i>0$.*

(5) *If R is commutative, then so is S'.*

Proof. (1) It is easy to check that the right multiplication map μ is a ring homomorphism since S/R is an R-R-bimodule. Clearly, μ is injective if and only if $\mathrm{Hom}_R(S,R)=0$. For μ to be surjective, we use the following exact sequence:

$$0\longrightarrow \mathrm{Hom}_R(S,R)\longrightarrow \mathrm{Hom}_R(S,S)\longrightarrow \mathrm{Hom}_R(S,S/R)\longrightarrow \mathrm{Ext}^1_R(S,R)\longrightarrow \mathrm{Ext}^1_R(S,S),$$

where $\mathrm{Ext}^1_R(S,S)=0$ by Lemma 6.4(1). Thus (1) follows.

(2) Recall that a ring homomorphism is an epimorphism if and only if the multiplication map $S\otimes_R S\to S$ is an isomorphism as S-S-bimodules. Since λ is injective, it follows from the exact sequence $(*)$ that we have a long exact sequence of S-R-bimodules:

$$0\longrightarrow \mathrm{Tor}_1^R(S,S)\longrightarrow \mathrm{Tor}_1^R(S,S/R)\longrightarrow S\otimes_R R\xrightarrow{1\otimes\lambda} S\otimes_R S\longrightarrow S\otimes_R (S/R)\longrightarrow 0.$$

Since $\mathrm{Tor}_1^R(S,S)=0$ and $1\otimes_R\lambda$ is an isomorphism of S-R-modules, we have $S\otimes_R(S/R)=0=\mathrm{Tor}_1^R(S,S/R)$.

Now, by applying $\mathrm{Hom}_R(-,S/R)$ to $(*)$, we can get another exact sequence of R-$\mathrm{End}_R(S/R)$-bimodules:

$$(**)\quad 0\longrightarrow \mathrm{Hom}_R(S/R,S/R)\longrightarrow \mathrm{Hom}_R(S,S/R)\longrightarrow \mathrm{Hom}_R(R,S/R).$$

One can check that the last homomorphism in the above sequence $(**)$ is surjective because each element $s+R$ in S/R gives rise to at least one homomorphism from the R-module S to the R-module S/R by $x\mapsto xs+R$ for $x\in S$. This yields the following exact sequence of S-$\mathrm{End}_R(S/R)$-bimodules:

$$0\longrightarrow S\otimes_R \mathrm{Hom}_R(S/R,S/R)\longrightarrow S\otimes_R \mathrm{Hom}_R(S,S/R)\longrightarrow S\otimes_R (S/R)\longrightarrow 0,$$

which shows that $S\otimes_R \mathrm{Hom}_R(S/R,S/R) \xrightarrow{\sim} S\otimes_R \mathrm{Hom}_R(S,S/R)$. Clearly, under this isomorphism the element $1\otimes_R 1$ in $S\otimes \mathrm{Hom}_R(S/R,S/R)$ is sent to $1\otimes\pi$. Since the multiplication map: $S\otimes_R S \longrightarrow S$ is an isomorphism of S-S-bimodules, we see that the multiplication map: $S\otimes_R X\to X$ is an isomorphism for every S-module X. Clearly, $\mathrm{Hom}_R({}_RS_S,S/R)$ is an S-module. So, it follows that $S\otimes_R \mathrm{Hom}_R(S,S/R)\to \mathrm{Hom}_R(S,S/R)$ is an isomorphism under which $1\otimes\pi$ is sent to π. As a result, the map $\theta: S\otimes_R S'\to \mathrm{Hom}_R(S,S/R)$ defined by $s\otimes f\mapsto \big(t\mapsto (ts)(\pi f)\big)$ for $s,t\in S$ and $f\in S'$, is an isomorphism of S-S'-bimodules. Clearly, under this isomorphism, the element $1\otimes 1$ in $S\otimes_R S'$ is sent to π.

(3) Applying $-\otimes_R S'$ to the sequence $(*)$ and identifying $R\otimes_R S'$ with S', we then obtain the following right exact sequence of R-S'-bimodules:

$$(\spadesuit)\quad S' \xrightarrow{\lambda'} S\otimes_R S' \xrightarrow{\pi\otimes S'} (S/R)\otimes_R S'\to 0,$$

where the map λ' is defined by $f\mapsto 1\otimes f$ for any $f\in S'$. Combining this sequence with $(**)$, one can check that the following diagram of R-S'-bimodules is exact and commutative:

$$\begin{tikzcd}
 & S' \arrow[r, "\lambda'"] \arrow[d, equal] & S\otimes_R S' \arrow[r, "\pi\otimes S'"] \arrow[d, "\theta"] & (S/R)\otimes_R S' \arrow[r] \arrow[d, "\psi"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(S/R,S/R) \arrow[r, "\pi_*"] & \mathrm{Hom}_R(S,S/R) \arrow[r] & S/R \arrow[r] & 0,
\end{tikzcd}$$

where ψ is the evaluation map, and where $\mathrm{Hom}_R(R,S/R)$ is identified with S/R as R-S'-bimodules. Since θ is an isomorphism, we infer that λ' is injective, and that ψ is an isomorphism of R-S'-bimodules.

(4) Suppose that λ is an injective homological ring epimorphism. Then $\mathrm{Tor}_i^R(S,S)=0$ for $i>0$. Recall that we have proved that $S\otimes_R (S/R)=0=\mathrm{Tor}_1^R(S,S/R)$ in (2). Thus, by applying the tensor functor $S\otimes_R -$ to the canonical sequence $(*)$, we conclude that $\mathrm{Tor}_i^R(S,S/R)=0$. By (3), we know that $(S/R)\otimes_R S'\simeq S/R$ as left R-modules. Thus $\mathrm{Tor}_i^R(S,(S/R)\otimes_R S')=0$. Since $S\otimes_R S'$ is a left S-module, it follows from Lemma 2.2(2) that $\mathrm{Tor}_i^R(S,S\otimes_R S')=0$. Now, applying the tensor functor $S\otimes_R -$ to the exact sequence $(\spadesuit)$, we obtain $\mathrm{Tor}_i^R(S,S')=0$.

(5) Since R is commutative, the tensor product $S\otimes_R S'$ of S and S' over R is a ring, which is well-defined. By Lemma 6.5(3), there exists an exact sequence of R-S'-modules:

$$0\to S' \xrightarrow{\lambda'} S\otimes_R S' \xrightarrow{\pi\otimes S'} (S/R)\otimes_R S'\to 0.$$

Since λ' is a ring homomorphism, the ring $S\otimes_R S'$ can be considered an S'-S'-bimodule via λ', and therefore, $(S/R)\otimes_R S'$ can also be regarded as an S'-S'-bimodule. In addition, by Lemma 6.5(3), the evaluation map $\psi: (S/R)\otimes_R S'\to S/R$, defined by $y\otimes g\mapsto (y)g$ for any $y\in S/R$ and $g\in S'$, is an isomorphism of R-S'-bimodules. Since the image of $(y)g\otimes 1$ under ψ is also equal to $(y)g$, we have $(y)g\otimes 1=y\otimes g$ in $(S/R)\otimes_R S'$. Consequently, for any $f,g\in S'$ and $y\in S/R$, we get $y\otimes fg=f(y\otimes g)=f((y)g\otimes 1)=(y)g\otimes f$ in $(S/R)\otimes_R S'$, where the first and third equalities follow from the left S'-module structure of $(S/R)\otimes_R S'$. This yields that $(y)fg=\big(y\otimes fg\big)\psi=\big((y)g\otimes f\big)\psi=(y)gf$ in S/R. Thus $fg=gf$. Since f and g are arbitrary elements in S', we see that S' is a commutative ring. $\square$

As a consequence of Theorem 1.1, we have the following corollary.

Corollary 6.6. (1) *Let $\lambda: R\to S$ be an injective ring epimorphism such that $\mathrm{Tor}_1^R(S,S)=0$. If ${}_RS$ has projective dimension at most one, then there is a recollement of derived module categories:*

$$\begin{tikzcd}
\mathscr{D}(S\sqcup_R S') \arrow[r] & \mathscr{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd},$$

where $S \sqcup_R S'$ is the coproduct of S and S' over R.

(2) *Let R be a ring, Σ a left Ore set of regular elements of R, and $S := \Sigma^{-1}R$ the localization of R at Σ. If ${}_RS$ has projective dimension at most one, then the recollement in* (1) *exists.*

Proof. (1) Now, let $R_0 = R$, $R_1 = S$, $R_2 = \mathrm{End}_R(S/R)$, $T := S \oplus S/R$, and $B = \mathrm{End}_R(T)$. It is proved in [6] that, under the above assumptions, the R-module T is a good tilting module. By Lemma 6.5, the map φ in Lemma 6.1 is precisely the map π^* in Lemma 6.4(2) under the identification of θ in Lemma 6.5. Thus the universal localization of B at π^* is isomorphic to the 2×2 matrix ring over the coproduct of S and S' over R by Lemma 6.1. Thus Corollary 6.6(1) follows from Theorem 1.1(1).

(2) follows from (1). $\square$

The tilting module $S \oplus S/R$ in Corollary 6.6 has an equivalent form (see [6, Theorem 2.10]), by which we can restate Corollary 6.6 in the following form. Here we present explicitly the R-ring homomorphisms which will be used for later calculations in Section 8.

Corollary 6.7. *Let R be a ring, and let ${}_RT$ be a tilting R-module with an exact sequence*

$$0 \longrightarrow R \longrightarrow T_0 \longrightarrow T_1 \longrightarrow 0$$

of R-modules such that $T_i \in \mathrm{Add}(T)$ and $\mathrm{Hom}_R(T_1, T_0) = 0$. Set $S := \mathrm{End}_R(T_0)$, $S' := \mathrm{End}_R(T_1)$ and $B := \mathrm{End}_R(T_0 \oplus T_1)$. Then there is the following recollement:

$$\begin{tikzcd} \mathscr{D}(S \sqcup_R S') \arrow[r] & \mathscr{D}(B) \arrow[r] \arrow[l, bend right] \arrow[l, bend left] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd},$$

where $S \sqcup_R S'$ is the coproduct of S and S' over R.

Proof. First of all, we show that S and S' can be regarded as R-rings, namely, we construct two ring homomorphisms $\lambda : R \to S$ and $\mu : R \to S'$ (see Lemma 6.5(1)). For any $r \in R$, we denote by $\rho_r : R \to R$ the right multiplication map by the element r. It follows from $\mathrm{Hom}_R(T_1, T_0) = 0 = \mathrm{Ext}^1_R(T_1, T_0)$ that there exists a unique homomorphism $f : T_0 \to T_0$ and therefore a unique homomorphism $g : T_1 \to T_1$ such that the following exact diagram

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r] \arrow[d, "\rho_r"] & T_0 \arrow[r] \arrow[d, dashed, "f"] & T_1 \arrow[r] \arrow[d, dashed, "g"] & 0 \\ 0 \arrow[r] & R \arrow[r] & T_0 \arrow[r] & T_1 \arrow[r] & 0 \end{tikzcd}$$

commutes. Now, we define $\lambda : R \to S$ and $\mu : R \to S'$ by $r \mapsto f$ and $r \mapsto g$, respectively. One can check directly that λ is injective, and that λ and μ are ring homomorphisms. Furthermore, by the proof of [6, Theorem 2.10], there are isomorphisms $\varphi : T_0 \to S$ and $\psi : T_1 \to S/R$ of R-modules, such that the following exact diagram of R-modules

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r] \arrow[d, equal] & T_0 \arrow[r] \arrow[d, "\varphi", "\wr"'] & T_1 \arrow[r] \arrow[d, "\psi", "\wr"'] & 0 \\ 0 \arrow[r] & R \arrow[r, "\lambda"] & S \arrow[r, "\pi"] & S/R \arrow[r] & 0 \end{tikzcd}$$

is commutative. Now, Corollary 6.7 follows from Corollary 6.6. $\square$.

Remarks. (1) In general, Corollary 6.6(1) supplies us a class of recollements which cannot be obtained from the structure of triangular matrix rings.

If $\mathrm{Ext}^i_R(S, R) = 0$ for $0 \le i \le 1$ in Corollary 6.6(1), then $S' \simeq R$ by Lemma 6.5(1), and therefore $S \sqcup_R S' \simeq S \sqcup_R R = S$. Even in this case, the recollement in Corollary 6.6(1) is not equivalent to the one induced from the triangular matrix ring (see Lemma 6.4(2)) since the former is induced by the homological ring epimorphism

$B \to M_2(S)$ and the latter is induced by $B \to S$, they are non-equivalent ring epimorphisms. See Section 7 below for more examples.

(2) In Corollary 6.6(1), the condition that "${}_RS$ has projective dimension at most one" ensures the category $\mathscr{D}(B)_{\Sigma^\perp}$ in Corollary 3.7 can be replaced by the derived module category of a ring. However, this condition is not necessary for getting such a recollement. In fact, we have the following result.

Let $\lambda : R \to S$ be an injective ring epimorphism such that $\mathrm{Tor}_1^R(S,S) = 0$. Suppose that the right multiplication map $\mu : R \to S'$ defined in Lemma 6.5(1) is an isomorphism of rings. Then the universal localization $\lambda_{\pi^*} : B \to B_{\pi^*}$ of B at π^* is homological. In particular, $\mathscr{D}(B)_{\{\pi^*\}^\perp}$ in Corollary 3.7 can be replaced by $\mathscr{D}(S)$.

Proof. Combining Lemmata 6.1 and 6.4(2) with 6.5(2), we know that $B \simeq \begin{pmatrix} S & S \\ 0 & R \end{pmatrix}$ and $B_{\pi^*} \simeq M_2(S)$ as rings. Under these isomorphisms, the universal localization $\lambda_{\pi^*} : B \to B_{\pi^*}$ is equivalent to the canonical injective ring homomorphism $\chi : R_1 := \begin{pmatrix} S & S \\ 0 & R \end{pmatrix} \to M_2(S)$ induced by the injective homomorphism $\lambda : R \to S$. Let e_{ij}, with $1 \leq i,j \leq 2$, denote the usual matrix units in $M_2(S)$. Then $e_{11}R_1 = e_{11}M_2(S)$, and the following diagram of right $M_2(S)$-modules is commutative:

$$\begin{tikzcd} e_{11}M_2(S) \otimes_{R_1} M_2(S) \arrow[r, "m_1"] \arrow[d, "e_{21}\cdot \otimes 1"'] & e_{11}M_2(S) \arrow[d, "e_{21}\cdot"] \\ e_{22}M_2(S) \otimes_{R_1} M_2(S) \arrow[r, "m_2"] & e_{22}M_2(S), \end{tikzcd}$$

where m_i is the multiplication map for $i = 1,2$. Since the map m_1 and the vertical maps are isomorphisms, we see that m_2 is an isomorphism. Hence χ is an ring epimorphism. Clearly, $M_2(S)$ is projective as a right R_1-module. Thus χ is homological by definition, and consequently, λ_{π^*} is homological, and $\mathscr{D}(S)$ is triangle equivalent to $\mathscr{D}(B)_{\{\pi^*\}^\perp}$ by Proposition 3.6. $\square$

Let us give a concrete example satisfying all conditions in the remark (2).

Let $R = \begin{pmatrix} k & 0 & 0 \\ k[x]/(x^2) & k & 0 \\ k[x]/(x^2) & k[x]/(x^2) & k \end{pmatrix}$, where k is a field and $k[x]$ is the polynomial algebra over k in one variable x, and let S be the 3 by 3 matrix ring $M_3(k[x]/(x^2))$. Then the inclusion λ of R into S is a universal localization of R, and therefore a ring epimorphism. Further, we have $\mathrm{Tor}_1^R(S,S) = 0 \neq \mathrm{Tor}_2^R(S,S)$ (see [32]). Thus λ is not homological. So, ${}_RS$ cannot have projective dimension less than or equal to one. But one can check that μ defined in Lemma 6.5 is an isomorphism.

This example also shows that Proposition 6.8 below may be false if the injective ring epimorphism $\lambda : R \to S$ is not homological.

(3) Under the conditions of Remark (2), one can get another pattern of recollements, namely, we have the following result.

Let $\lambda : R \to S$ be an injective ring epimorphism such that $\mathrm{Tor}_1^R(S,S) = 0$. Suppose that the right multiplication map $\mu : R \to \mathrm{End}_R(S/R)$ defined in Lemma 6.5(1) is an isomorphism of rings. Then there is a recollement of derived module categories of S, B and R:

$$\begin{tikzcd} \mathscr{D}(R) \arrow[r] & \mathscr{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(S) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} .$$

Proof. Note that the sequence $0 \longrightarrow R \xrightarrow{\lambda} S \xrightarrow{\pi} S/R \longrightarrow 0$ is an $\mathrm{add}({}_RS)$-split sequence in R-Mod. By [27, Theorem 3.5], we conclude that B is derived equivalent to the endomorphism ring $\mathrm{End}_R(R \oplus S)$ which is isomorphic to the triangular matrix ring $\begin{pmatrix} R & S \\ 0 & S \end{pmatrix}$. Consequently, we can get the above recollement of derived module categories of S, B and R by the structure of triangular matrix rings. $\square$

A variation of Corollary 6.6(1) is the following proposition in which we relax the condition of being tilting modules, and require ring epimorphisms to be homological.

Let $\lambda : R \to S$ be an injective ring homomorphism between rings R and S. We consider $S' := \mathrm{End}_R(S/R)$ as an R-ring via μ defined in Lemma 6.5. Furthermore, let $\rho : S \to S \sqcup_R S'$ and $\rho' : S' \to S \sqcup_R S'$ be the canonical R-homomorphisms in the definition of coproducts of R-rings.

Proposition 6.8. *If $\lambda : R \to S$ is an injective homological ring epimorphism, then the following assertions are equivalent:*

(1) *The universal localization $\lambda_{\pi^*} : B \to B_{\pi^*}$ of B at π^* is homological.*

(2) *The ring homomorphism $\rho' : S' \to S \sqcup_R S'$ is homological.*

In particular, if one of the above assertions holds, then there is a recollement of derived module categories:

$$\begin{tikzcd} \mathscr{D}(S \sqcup_R S') \arrow[r] & \mathscr{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} .$$

Proof. Recall that S' is an R-ring via the right multiplication map $\mu : R \to S'$, defined by $r \mapsto (x \mapsto xr)$ for $r \in R$ and $x \in S/R$ (see Lemma 6.5(1)). Then it follows from the definition of coproducts of rings that $\lambda\rho = \mu\rho' : R \to S \sqcup_R S'$. We point out that ρ' is a ring epimorphism. In fact, if $f, g : S \sqcup_R S' \to S_1$ are two ring homomorphisms such that $\rho' f = \rho' g$, then $\mu\rho' f = \mu\rho' g$. This means that $\lambda\rho f = \lambda\rho g$ and $\rho f = \rho g$ since λ is a ring epimorphism. It follows from the universal property of coproducts that $g = f$. Thus ρ' is a ring epimorphism.

Step (1). We claim that, for any $S \sqcup_R S'$-module W, if we regard W as a left S'-module via the ring homomorphism ρ', then $(S \otimes_R S') \otimes_{S'} W \simeq W$ as S-modules, and $\mathrm{Tor}_i^{S'}(S \otimes_R S', W) = 0$ for any $i > 0$.

To prove this, we fix a projective resolution $Q^\bullet$ of S_R:

$$\cdots \longrightarrow Q^n \longrightarrow Q^{n-1} \longrightarrow \cdots \longrightarrow Q^1 \longrightarrow Q^0 \longrightarrow S_R \longrightarrow 0$$

with Q^i projective right R-modules. By Lemma 6.5(4), we have $\mathrm{Tor}_j^R(S, S') = 0$ for any $j > 0$. It follows that the complex $Q^\bullet \otimes_R S'$ is a projective resolution of the right S'-module $S \otimes_R S'$. Note that we have the following isomorphisms of complexes of abelian groups:

$$\left(Q^\bullet \otimes_R S'\right) \otimes_{S'} W \simeq Q^\bullet \otimes_R \left(S' \otimes_{S'} W\right) \simeq Q^\bullet \otimes_R W.$$

This implies that $\mathrm{Tor}_i^{S'}(S \otimes_R S', W) \simeq \mathrm{Tor}_i^R(S, W)$ for any $i > 0$. Clearly, W admits an S-module structure via the map ρ. Moreover, it follows from $\lambda\rho = \mu\rho'$ that the R-module structure of W endowed via the ring homomorphism $\mu\rho'$ is the same as the one endowed via the ring homomorphism $\lambda\rho$. Then, by Lemma 2.2, we conclude that $S \otimes_R W \simeq W$ as S-modules, and that $\mathrm{Tor}_i^R(S, W) = 0$ for $i > 0$. Therefore, $\mathrm{Tor}_i^{S'}(S \otimes_R S', W) = 0$ for $i > 0$. Note that $(S \otimes_R S') \otimes_{S'} W \simeq S \otimes_R W$ as S-modules. As a result, we have $({}_S S \otimes_R S') \otimes_{S'} W \simeq {}_S W$. This finishes the proof of Step (1).

Step (2). We shall prove that B_{π^*} is Morita equivalent to $S \sqcup_R S'$.

By Lemmata 6.4(2) and 6.5(2), there are isomorphisms of rings

$$B := \mathrm{End}_R(T) \simeq \begin{pmatrix} S & \mathrm{Hom}_R(S, S/R) \\ 0 & \mathrm{End}_R(S/R) \end{pmatrix} \simeq C := \begin{pmatrix} S & S \otimes_R S' \\ 0 & S' \end{pmatrix},$$

where the second isomorphism sends $\begin{pmatrix} 0 & \pi \\ 0 & 0 \end{pmatrix}$ to $\begin{pmatrix} 0 & 1\otimes 1 \\ 0 & 0 \end{pmatrix}$. Let $e_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in C$, and let $\varphi : Ce_1 \to Ce$ be the map that sends $\begin{pmatrix} s \\ 0 \end{pmatrix}$ to $\begin{pmatrix} s\otimes 1 \\ 0 \end{pmatrix}$ for $s \in S$. Then π^* corresponds to φ under the isomorphism $B \simeq C$. Let $\lambda_\varphi : C \to C_\varphi$ be the universal localization of C at φ. Then $B_{\pi^*} \simeq C_\varphi$. Note that λ_{π^*} is homological if and only if λ_φ is homological.

By Lemma 6.1, we know that $C_\varphi = M_2(S \sqcup_R S')$, the 2×2 matrix ring over $S\sqcup_R S'$, and that the corresponding ring epimorphism $\lambda_\varphi : C \to M_2(S\sqcup_R S')$ is given by $\begin{pmatrix} s & t\otimes f \\ 0 & g \end{pmatrix} \mapsto \begin{pmatrix} (s)\rho & (t)\rho(f)\rho' \\ 0 & (g)\rho' \end{pmatrix}$ for $s,t \in S$ and $f,g \in S'$. Hence, B_{π^*} is Morita equivalent to $S\sqcup_R S'$.

Step (3). We shall prove that the ring homomorphism $\lambda_\varphi : C \to M_2(S\sqcup_R S')$ is homological if and only if so is the ring homomorphism $\rho' : S' \to S\sqcup_R S'$.

Before starting our proof, we mention a general result: if $F : \mathcal{C} \to \mathcal{E}$ is an exact functor between abelian categories $\mathcal{C}$ and $\mathcal{E}$, then F can be extended to a canonical triangle functor $\overline{F} : \mathscr{D}(\mathcal{C}) \to \mathscr{D}(\mathcal{E})$, which sends a complex $X^\bullet := \big(X^i, d_X^i\big)_{i\in\mathbb{Z}}$ over $\mathcal{C}$ to the complex $\overline{F}(X^\bullet) := \big(F(X^i), F(d_X^i)\big)_{i\in\mathbb{Z}}$ over $\mathcal{E}$. This is due to the fact that F preserves quasi-isomorphisms. Since $\overline{F}$ is completely determined by F, we may denote $\overline{F}$ by F.

Set $\Gamma := S\sqcup_R S'$, $\Lambda := M_2(\Gamma)$ and $e' := (e)\lambda_\varphi \in \Lambda$. Then $e' = (e')^2$, $\mathrm{End}_\Lambda(\Lambda e') \simeq \Gamma$ and $\mathrm{End}_C(Ce) \simeq S'$. Observe that $\Lambda e'$ is a projective generator for Λ-Mod. Then, by Morita theory, the tensor functor $e'\Lambda\otimes_\Lambda - :$ Λ-Mod $\longrightarrow \Gamma$-Mod is an equivalence of module categories, which can be extended to a canonical triangle equivalence from $\mathscr{D}(\Lambda)$ to $\mathscr{D}(\Gamma)$.

Note that $eC\otimes_C\Lambda \simeq e\Lambda = e'\Lambda$ as S'-Λ-bimodules, where the left S'-module structure of $e'\Lambda$ is induced by the ring homomorphism $\rho' : S' \to \Gamma$. It follows that the following diagram of functors between module categories

$$\begin{tikzcd} \Lambda\text{-Mod} \arrow[r, "e'\Lambda\otimes_\Lambda -"] \arrow[d, "(\lambda_\varphi)_*"'] & \Gamma\text{-Mod} \arrow[d, "(\rho')_*"] \\ C\text{-Mod} \arrow[r, "eC\otimes_C -"] & S'\text{-Mod} \end{tikzcd}$$

is commutative, where $(\lambda_\varphi)_*$ and $(\rho')_*$ stand for the restriction functors induced by the ring homomorphisms $\lambda_\varphi : C \to \Lambda$ and $\rho' : S' \to \Gamma$, respectively. Since all of the functors appearing in the diagram are exact, we can form the following commutative diagram of functors between derived module categories:

$$\begin{tikzcd} \mathscr{D}(\Lambda) \arrow[r, "e'\Lambda\otimes_\Lambda -"] \arrow[d, "(\lambda_\varphi)_*"'] & \mathscr{D}(\Gamma) \arrow[d, "(\rho')_*"] \\ \mathscr{D}(C) \arrow[r, "eC\otimes_C -"] & \mathscr{D}(S'), \end{tikzcd}$$

where the tensor functor $e'\Lambda\otimes_\Lambda -$ in the upper row is a triangle equivalence.

From the triangular structure of the ring C it follows that there is a recollement of derived module categories:

$$\begin{tikzcd} \mathscr{D}(S) \arrow[r] & \mathscr{D}(C) \arrow[l, bend right] \arrow[l, bend left] \arrow[r, "eC\otimes_C -"] & \mathscr{D}(S') \arrow[l, bend right, "Ce\otimes^{\mathbb{L}}_{S'} -"'] \arrow[l, bend left] \end{tikzcd} .$$

In particular, the pair $\big(\mathrm{Tria}(Ce), \mathrm{Tria}(Ce_1)\big)$ is a torsion pair in $\mathscr{D}(C)$, and the functor $eC\otimes_C -$ induces a triangle equivalence between $\mathrm{Tria}(Ce)$ and $\mathscr{D}(S')$.

We claim that the image of the restriction functor $(\lambda_\varphi)_*$ belongs to $\mathrm{Tria}(Ce)$. This implies that, for any complexes $X^\bullet, Y^\bullet \in \mathrm{Im}\big((\lambda_\varphi)_*\big)$, we have

$$\mathrm{Hom}_{\mathscr{D}(C)}(X^\bullet, Y^\bullet) \xrightarrow{\sim} \mathrm{Hom}_{\mathscr{D}(S')}(eX^\bullet, eY^\bullet),$$

where the functor $eC\otimes_C -$ is identified with the left multiplication functor by e. Clearly, $(\lambda_\varphi)_*$ commutes with small coproducts since it admits a right adjoint. In addition, we have $\mathscr{D}(\Lambda)=\mathrm{Tria}(\Lambda e')$. Therefore, to prove the claim, it suffices to prove $\Lambda e'\in\mathrm{Tria}(Ce)$. This is equivalent to showing that $Ce\otimes^{\mathbb{L}}_{S'}e\Lambda e'\simeq\Lambda e'$ in $\mathscr{D}(C)$.

Indeed, set $M:=S\otimes_R S'$, and write C-modules in the form of triples (X,Y,h) with $X\in S'$-Mod, $Y\in S$-Mod and $h: M\otimes_{S'}X\to Y$ a homomorphism of S-modules. The morphisms between two modules (X,Y,h) and (X',Y',h') are pairs of morphisms (α,β), where $\alpha: X\to X'$ and $\beta: Y\to Y'$ are homomorphisms in S'-Mod and S-Mod, respectively, such that $h\beta=(M\otimes_{S'}\alpha)h'$.

With these interpretations, we rewrite $\Lambda e'=(\Gamma,\Gamma,u)\in C$-Mod, where $u: M\otimes_{S'}\Gamma\to\Gamma$ is defined by $(s\otimes f)\otimes\gamma\mapsto(s)\rho(f)\rho'\gamma$ for $s\in S, f\in S'$ and $\gamma\in\Gamma$. Then $e\Lambda e'\simeq\Gamma$ as left S'-modules. Clearly, $Ce\simeq M\oplus S'$ as right S'-modules. Recall that we have proved in Step (1) that u is an isomorphism of S-modules and $\mathrm{Tor}_i^{S'}(M,\Gamma)=0$ for any $i>0$. It follows that $\mathrm{Tor}_i^{S'}(Ce,e\Lambda e')=0$. Then we get the following isomorphisms in $\mathscr{D}(C)$:

$$Ce\otimes^{\mathbb{L}}_{S'}e\Lambda e'\simeq Ce\otimes_{S'}e\Lambda e'\simeq(\Gamma,M\otimes_{S'}\Gamma,1)\simeq\Lambda e'.$$

Thus $\Lambda e'\in\mathrm{Tria}(Ce)$, and therefore the image of the restriction functor $(\lambda_\varphi)_*$ is contained in $\mathrm{Tria}(Ce)$.

With the above preparations, we now can prove that the ring homomorphism $\lambda_\varphi: C\to\Lambda$ is homological if and only if so is the ring homomorphism $\rho': S'\to\Gamma$.

In fact, this can be seen from the following commutative diagram of functors between triangulated categories:

$$\begin{tikzcd}
 & \mathscr{D}(\Lambda) \arrow[r, "e'\Lambda\otimes_\Lambda -", "\simeq"'] \arrow[dl, "(\lambda_\varphi)_*"'] \arrow[d, "(\lambda_\varphi)_*"] & \mathscr{D}(\Gamma) \arrow[d, "(\rho')_*"] \\
\mathscr{D}(C) & \mathrm{Tria}(Ce) \arrow[l, hook'] \arrow[r, "eC\otimes_C -", "\simeq"'] & \mathscr{D}(S'),
\end{tikzcd}$$

which implies that $(\lambda_\varphi)_*$ is fully faithful if and only if so is $(\rho')_*$. It is known that $\lambda_\varphi: C\to\Lambda$ (respectively, $\rho': S'\to\Gamma$) is homological if and only if $(\lambda_\varphi)_*$ (respectively, $(\rho')_*$) is fully faithful.

Thus, we have proved that $\lambda_{\pi^*}: B\to B_{\pi^*}$ is homological if and only if $\rho': S'\to S\sqcup_R S'$ is homological. This finishes the proof of the first part of Proposition 6.8. Clearly, the second part of Proposition 6.8 follows directly from Proposition 3.6. $\square$

Under the assumptions of Corollary 6.6(1), we see that both λ and λ_{π^*} are homological, and therefore Proposition 6.8 implies Corollary 6.6(1). However, for an injective homological ring epimorphism $\lambda: R\to S$, the projective dimension of ${}_RS$ may not be at most one in general (see the example at the end of Corollary 6.10 below). So, from this point of view, Proposition 6.8 may be regarded as a generalization of Corollary 6.6(1).

Combining Lemma 6.2 with Proposition 6.8, we get the following criterion for λ_{π^*} to be homological.

Corollary 6.9. *Let Σ be a set of homomorphisms between finitely generated projective R-modules. Suppose that the universal localization $\lambda_\Sigma: R\to R_\Sigma$ is an injective homological ring homomorphism. Set $S:=R_\Sigma$, $\lambda:=\lambda_\Sigma$, $S':=\mathrm{End}_R(S/R)$ and $\Gamma:=\{S'\otimes_R f\mid f\in\Sigma\}$. Then the universal localization $\lambda_{\pi^*}: B\to B_{\pi^*}$ of B at π^* is homological if and only if the universal localization $\lambda'_\Gamma: S'\to S'_\Gamma$ of S' at Γ is homological. In particular, if one of the above equivalent conditions holds , then there is a recollement of derived module categories:*

$$\begin{tikzcd}
\mathscr{D}(S'_\Gamma) \arrow[r] & \mathscr{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}.$$

As a consequence of Corollary 6.9, we obtain the following result which can be used to show when the universal localization $\lambda_{\pi^*}: B\to B_{\pi^*}$ of B at π^* in Proposition 6.8 is homological in some special cases.

Corollary 6.10. *Let $C \subseteq D$ be an extension of rings. Set $R := \begin{pmatrix} D & D \\ 0 & C \end{pmatrix}$ and $S := M_2(D)$. Let $\lambda : R \to S$ be the canonical injective ring homomorphism. Then the universal localization $\lambda_{\pi^*} : B \to B_{\pi^*}$ of B at π^* is homological if and only if the universal localization $\lambda_{\omega^*} : E \to E_{\omega^*}$ of E at ω^* is homological, where $E := \mathrm{End}_C(D \oplus D/C)$, and the homomorphism $\omega^* : \mathrm{Hom}_C(D \oplus D/C, D) \to \mathrm{Hom}_C(D \oplus D/C, D/C)$ of projective E-modules is induced by the canonical epimorphism $\omega : D \to D/C$.*

Proof. Recall that the right multiplication map $\mu : R \to S'$ is a ring homomorphism (see Lemma6.5(1)). Set $e_1 := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $e_2 := \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $e_{12} := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in R$. Furthermore, let $\varphi : Re_1 \to Re_2$ and $\varphi' : S'(e_1)\mu \to S'(e_2)\mu$ be the right multiplication maps by e_{12} and $(e_{12})\mu$, respectively.

It follows from Lemma 6.1 and $D \sqcup_C C = D$ that the map $\lambda : R \to S$ is the universal localization of R at φ. In particular, λ is a ring epimorphism. Since $S \simeq e_1R \oplus e_1R$ as right R-modules, the embedding λ is always homological. Note that $S' \otimes_R \varphi$ can be identified with φ'. By C0rollary 6.9, the universal localization $\lambda_{\pi^*} : B \to B_{\pi^*}$ of B at π^* is homological if and only if the universal localization $\lambda'_{\varphi'} : S' \to S'_{\varphi'}$ of S' at φ' is homological.

Clearly, $R/Re_1R \simeq C$ as rings. So, every C-module can be regarded as an R-module. In particular, $D \oplus D/C$ can be seen as an R-module. Further, one can check that the map $\alpha : D \oplus D/C \to S/R$ defined by

$$(d, t + C) \mapsto \begin{pmatrix} 0 & 0 \\ d & t \end{pmatrix} + R$$

for $d, t \in D$, is an isomorphism of R-modules. Thus $S' \simeq E$, φ' corresponds to ω^* under this isomorphism, and $S'_{\varphi'} \simeq E_{\omega^*}$. It follows that $\lambda'_{\varphi'} : S' \to S'_{\varphi'}$ is homological if and only if so is $\lambda_{\omega^*} : E \to E_{\omega^*}$. This finishes the proof. $\square$

Remark. In general, it may happen that the special form of the universal localization $\lambda_{\omega^*} : E \to E_{\omega^*}$ of E at ω^*(or equivalently, the universal localization $\lambda_{\pi^*} : B \to B_{\pi^*}$ of B at π^*) in Corollary 6.10 is not homological, through the λ is always homological.

For example, let $C = \left\{ \begin{pmatrix} a & 0 \\ b & a \end{pmatrix} \mid a, b \in k \right\}$ and $D = \begin{pmatrix} k & 0 \\ k & k \end{pmatrix}$ with k a field. Then one can check that the canonical map $\omega : D \to D/C$ is a split epimorphism in C-Mod, and therefore ${}_CD \simeq C \oplus D/C$. Let e be the idempotent of E corresponding the direct summand C of the C-module $D \oplus D/C$. Then $E_{\omega^*} \simeq E/EeE \simeq M_2(k)$. Furthermore, the universal localization $\lambda_{\omega^*} : E \to E_{\omega^*}$ of E at ω^* is equivalent to the canonical projection $\tau : E \to E/EeE$. Since $\mathrm{Ext}^2_E(E/EeE, E/EeE) \neq 0$, we see that τ is not homological. This implies that λ_{ω^*} is not homological, too. Thus $\lambda_{\pi^*} : B \to B_{\pi^*}$ is not homological by Corollary 6.10, that is, the restriction functor $(\lambda_{\pi^*})_* : \mathscr{D}(B_{\pi^*}) \to \mathscr{D}(B)$ is not fully faithful.

7 Commutative rings and recollements of derived module categories

In this section, we shall first discuss recollements of derived module categories arising from injective homological ring epimorphisms $\lambda : R \to S$ between arbitrary commutative rings without the assumption that the modules $S \oplus S/R$ are tilting modules, and then turn to the special case of one-Gorenstein rings. We shall see that, for commutative rings, the universal localizations appearing in the main result Theorem 1.1 will be further strengthened as tensor products. As a consequence, we can produce examples to show that two different stratifications of a derived module category by derived module categories of rings may have different derived composition factors, which answers negatively a question in [5] and shows that the Jordan-Hölder theorem fails for derived module categories with simple derived module categories as composition factors.

Note that if R is a commutative ring and $\lambda : R \to S$ is a ring epimorphism, then S must be commutative. So, in the following, we can assume that both rings R and S are commutative rings.

7.1 General case: Arbitrary commutative rings

The main purpose of this subsection is to prove the following existence result for recollements arising form injective ring epimorphisms between commutative rings. Here we remove the condition of tilting modules.

Proposition 7.1. *Suppose that $\lambda: R \to S$ is an injective homological ring epimorphism between commutative rings R and S. Let B be the endomorphism ring of the R-module $S \oplus S/R$. Then there is a recollement of derived module categories:*

$$\begin{tikzcd} \mathscr{D}(S\otimes_R S') \arrow[r] & \mathscr{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} ,$$

where $S' := \mathrm{End}_R(S/R)$ is commutative, and $S\otimes_R S'$ is the tensor product of S and S' over R.

Proof. Since S is a commutative ring, we see that S' is an R-algebra via the right multiplication map $\mu: R \to S'$, defined by $r \mapsto (x \mapsto xr)$ for $r \in R$ and $x \in S/R$ (see Lemma 6.5(1)). Then the tensor product $S\otimes_R S'$ of S and S' over R makes sense. Moreover, the map $\lambda': S' \to S\otimes_R S'$, defined by $s' \mapsto 1\otimes s'$ for any $s' \in S'$, and the map $\mu': S \to S\otimes_R S'$, defined by $s \mapsto s\otimes 1$ for $s \in S$, are ring homomorphisms. So, $S\otimes_R S'$ is an S'-S'-bimodule via λ'. In addition, it follows from Lemma 6.5(5) that S' is a commutative ring.

Since λ is a ring epimorphism, we know from Lemma 6.3 that the tensor product $S\otimes_R S'$ together with the two ring homomorphisms λ' and μ' satisfies the definition of coproducts. In other words, the coproduct $S\sqcup_R S'$ of R-rings S and S' over R is isomorphic to the tensor product $S\otimes_R S'$.

By Proposition 6.8, to get the recollement of derived module categories in Proposition 7.1, it is sufficient to demonstrate that $\lambda': S' \to S\otimes_R S'$ is homological.

In fact, we have the following composition of a series of isomorphisms of S-S'bimodules:

$$(S\otimes_R S')\otimes_{S'}(S\otimes_R S') \simeq S\otimes_R \big(S'\otimes_{S'}(S\otimes_R S')\big) \simeq S\otimes_R (S\otimes_R S') \simeq (S\otimes_R S)\otimes_R S' \simeq S\otimes_R S'.$$

This composition map is actually the multiplication map from $(S\otimes_R S')\otimes_{S'}(S\otimes_R S')$ to $S\otimes_R S'$. Thus it is an $(S\otimes_R S')$-$(S\otimes_R S')$-bimodule isomorphism. Hence λ' is a ring epimorphism.

It remains to show $\mathrm{Tor}_i^{S'}(S\otimes_R S', S\otimes_R S') = 0$ for all $i > 0$. However, this follows immediately from Step (1) in the proof of Proposition 6.8. Thus, we finish the proof of Proposition 7.1.

As a straightforward consequence of Proposition 7.1, together with a result in [25], we have the following corollary.

Corollary 7.2. *Under the assumptions of Proposition 7.1, the left global dimension of B is finite if and only the global dimensions of R and $S\otimes_R S'$ both are finite.*

7.2 Special case: One-Gorenstein rings

Throughout this subsection, R stands for a commutative ring, $\mathrm{Spec}(R)$ (respectively, $\mathrm{mSpec}(R)$) denotes the set of all prime (respectively, maximal) ideals of R. For each non-negative integer i, we denote by P_i the set of all prime ideals of R with height i.

Let M be an R-module. We denote by $E(M)$ the injective envelope of M, and by $\mathrm{proj.dim}(M), \mathrm{inj.dim}(M)$ and $\mathrm{flat.dim}(M)$ the projective, injective and flat dimensions of ${}_RM$, respectively.

For a multiplication subset Σ of R, we denote by $\Sigma^{-1}R$ the localization of R at Σ, and by $f_\Sigma: R \to \Sigma^{-1}R$ the canonical homological ring epimorphism. In general, the homomorphism f_Σ is not injective. But, if Φ is the multiplicative set of all non-zero divisors of R, then the localization map $f_\Phi: R \to \Phi^{-1}R$ is always injective. In this case, the ring $\Phi^{-1}R$ is called the total quotient ring of R, denoted by Q. In fact, Q is the largest localization of R for which the canonical map is injective, that is, if the map $f_\Sigma: R \to \Sigma^{-1}R$ is injective, then

$\Sigma \subseteq \Phi$, and there is a unique injective ring homomorphism $h : \Sigma^{-1}R \to Q$ such that $f_\Phi = f_\Sigma h$. In addition, if R is noetherian, then P_0 is finite and $\Phi = R \setminus \cup_{\mathfrak{p}\in P_0}\mathfrak{p}$.

As usual, for a prime ideal $\mathfrak{p}$ of R, we always write $R_\mathfrak{p}$ for $(R\setminus\mathfrak{p})^{-1}R$, and $f_\mathfrak{p}$ for $f_{R\setminus\mathfrak{p}}$, and say that $R_\mathfrak{p}$ is the localization of R at $\mathfrak{p}$.

Note that the localization $\mathbb{Z}_\mathfrak{p}$ of $\mathbb{Z}$ at the maximal ideal $\mathfrak{p} = p\mathbb{Z}$ is $\mathbb{Q}_{(p)}$ for every prime $p \in \mathbb{N}$, where $\mathbb{Q}_{(p)}$ is the set of p-integers. Recall that $q = n/m \in \mathbb{Q}$ with m,n a pair of coprime integers is called a p-integer if p does not divide m.

Let $\mathfrak{p}_1,\cdots,\mathfrak{p}_n$ be prime ideals in R, and set $\Sigma := R\setminus \cup_{i=1}^n \mathfrak{p}_i$. Then $\Sigma = \cap_{i=1}^n (R\setminus\mathfrak{p}_i)$ is a multiplicative subset of R, and the prime ideals of the localization $\Sigma^{-1}R$ are in one-to one correspondence with the prime ideals $\mathfrak{p}$ of R with $\mathfrak{p}\cap\Sigma = \emptyset$, that is, with the prime ideals of R contained in $\cup_{i=1}^n P_i$. By prime avoidance theorem, any such prime ideal is contained in one of the $\mathfrak{p}_i$. Hence, $\{\Sigma^{-1}\mathfrak{p}_j \mid 1\le j\le n\}$ contains the maximal ideals of $\Sigma^{-1}R$. If all $\mathfrak{p}_i$ are pairwise incomparable, that is, $\mathfrak{p}_i \not\subseteq \mathfrak{p}_j$ for $i\neq j$, then this is exactly the set of all maximal ideals of $\Sigma^{-1}R$.

Now, let us mention the following homological results about commutative noetherian rings, which are needed for our discussions in this section. For more details, we refer the reader, for instance, to [23, Theorem 3.3.8, Theorem 3.4.1, Lemma 6.7.7] and [33, Corollary 11.2].

Lemma 7.3. *Suppose that R is a noetherian ring. Let $\mathfrak{p},\mathfrak{q} \in Spec(R)$. We define $J_\mathfrak{p} := \varprojlim_i R/\mathfrak{p}^i$. Then,*

(1) $\mathrm{Hom}_R(E(R/\mathfrak{p}), E(R/\mathfrak{q})) \neq 0$ *if and only if* $\mathfrak{p}\subseteq\mathfrak{q}$. *In particular,* $E(R/\mathfrak{p}) \simeq E(R/\mathfrak{q})$ *if and only if* $\mathfrak{p} = \mathfrak{q}$.

(2) *If Σ is a multiplicative subset of R, then, as R-modules,*

$$\Sigma^{-1}E(R/\mathfrak{p}) \simeq \begin{cases} E(R/\mathfrak{p}) & \text{if } \Sigma\cap\mathfrak{p} = \emptyset, \\ 0 & \text{if } \Sigma\cap\mathfrak{p} \neq \emptyset. \end{cases}$$

(3) *If $\mathfrak{p}$ is a maximal ideal of R, then* $\mathrm{End}_R(E(R/\mathfrak{p})) \simeq \varprojlim_i R_\mathfrak{p}/\mathfrak{p}^i R_\mathfrak{p} \simeq J_\mathfrak{p}$.

(4) *Let P be a set of maximal ideals of R. If $\mathfrak{q}$ is a maximal ideal of R, which does not belong to P, then*

$$E(R/\mathfrak{q}) \otimes_R \prod_{\mathfrak{p}\in P} J_\mathfrak{p} = 0.$$

(5) *Define $S_n := \{x \in E(R/\mathfrak{p}) \mid \mathfrak{p}^n x = 0\}$ for each $n > 0$. Then $E(R/\mathfrak{p}) = \cup_n S_n$.*

Of our interest is the class of n-Gorenstein rings. Recall that, for a non-negative integer n, the ring R is called n-Gorenstein if R is noetherian and $\mathrm{inj.dim}(R) \le n$. The following homological properties of n-Gorenstein rings are well-known, for their proofs, we refer to [8, Theorem 1, Theorem 6.2], [23, Theorem 9.1.10, Theorem 9.1.11] and [39, Introduction].

Lemma 7.4. *Let n be a non-negative integer. Assume that R is an n-Gorenstein ring.*

(1) *The regular module R has a minimal injective resolution of the form:*

$$0 \longrightarrow R \longrightarrow \bigoplus_{\mathfrak{p}\in P_0} E(R/\mathfrak{p}) \longrightarrow \cdots \longrightarrow \bigoplus_{\mathfrak{p}\in P_n} E(R/\mathfrak{p}) \longrightarrow 0.$$

Moreover, the total quotient ring Q of R is isomorphic to $E(R)$ as an R-module.

(2) *For an R-module M, the following are equivalent:*

(i) $\mathrm{proj.dim}(M) < \infty$;	(ii) $\mathrm{inj.dim}(M) < \infty$;	(iii) $\mathrm{flat.dim}(M) < \infty$;
(iv) $\mathrm{proj.dim}(M) \le n$;	(v) $\mathrm{inj.dim}(M) \le n$;	(vi) $\mathrm{flat.dim}(M) \le n$.

(3) *The R-module*

$$T_{(P_n)} := \bigoplus_{i\le n}\bigoplus_{\mathfrak{p}\in P_i} E(R/\mathfrak{p})$$

is an (infinitely generated) n-tilting module, that is, it is of projective dimension at most n, and satisfies (T2) and (T3) (replaced by a longer exact sequence) in Definition 4.1.

(4) *If Σ is a multiplicative subset of R, then $\Sigma^{-1}R$ is an n-Gorenstein ring.*

From now on, we assume that R **is a** 1**-Gorenstein ring**. Then P_1 consists of all maximal ideals of R which are not minimal prime ideals. By Lemma 7.4, one gets a tilting R-module $T_{(P_1)}$ of projective dimension at most one. This construction of tilting modules from 1-Gorenstein rings can be generalized to obtain the so-called Bass tilting modules, as mentioned in [3]. Now, let us recall the construction.

Let

$$0 \longrightarrow R \xrightarrow{f_\Phi} Q \xrightarrow{\pi} \bigoplus_{\mathfrak{p}\in P_1} E(R/\mathfrak{p}) \longrightarrow 0,$$

be a minimal injective resolution of R, where π is the canonical surjective map which is regarded as a homomorphism of R-modules. Let Δ be a subset of P_1. Then we define

$$R_{(\Delta)} := \pi^{-1}(\bigoplus_{\mathfrak{p}\in\Delta} E(R/\mathfrak{p})) \quad \text{and} \quad T_{(\Delta)} := R_{(\Delta)} \oplus \bigoplus_{\mathfrak{p}\in\Delta} E(R/\mathfrak{p}).$$

Clearly, we get two associated exact sequences of R-modules

$$(a) \qquad 0 \longrightarrow R \longrightarrow R_{(\Delta)} \xrightarrow{\pi} \bigoplus_{\mathfrak{p}\in\Delta} E(R/\mathfrak{p}) \longrightarrow 0;$$

$$(b) \qquad 0 \longrightarrow R_{(\Delta)} \longrightarrow Q \longrightarrow \bigoplus_{\mathfrak{p}\in P_1\setminus\Delta} E(R/\mathfrak{p}) \longrightarrow 0.$$

Note that $R_{(\Delta)}$ is just an R-submodule of Q. It is shown in [3, Section 4] that the R-module $T_{(\Delta)}$ is a tilting module, which is called a Bass tilting module over R. Further, the authors of [39] prove that every tilting module over R is equivalent to a Bass tilting module. Note that the sequence (a) implies that the R-tilting module $T_{(\Delta)}$ is good.

The next lemma describes some properties relevant to Bass tilting modules. Note that the conclusions (1) and (2) of Lemma 7.5 below are mentioned in [38] for Dedekind domains.

Lemma 7.5. *let Δ be a subset of P_1. Assume that each prime ideal belonging to the set $P_1 \setminus \Delta$ contains all zero divisors of R. Then,*

(1) *For each $\mathfrak{p} \in P_1 \setminus \Delta$, the canonical ring homomorphism $f_\mathfrak{p} : R \to R_\mathfrak{p}$ is injective.*

(2) *$R_{(\Delta)} = \bigcap_{\mathfrak{p}\in P_1\setminus\Delta} R_\mathfrak{p}$, which is a flat R-module, where $R_\mathfrak{p}$ is regarded as a subring of the total ring Q of R. Hence $R_{(\Delta)}$ can be regarded as a subring of Q containing R. In particular, the total quotient ring of $R_{(\Delta)}$ also equals Q. (Note that we set $\bigcap_{\mathfrak{p}\in\emptyset} R_\mathfrak{p} = Q$.)*

(3) *The canonical inclusions $\lambda_\Delta : R \to R_{(\Delta)}$ and $\mu_\Delta : R_{(\Delta)} \to Q$ are homological ring epimorphisms.*

(4) *The $R_{(\Delta)}$-module*

$$T'_{(\Delta)} := Q \oplus \bigoplus_{\mathfrak{p}\in P_1\setminus\Delta} E(R/\mathfrak{p})$$

is a good tilting $R_{(\Delta)}$-module. If $R_{(\Delta)}$ is noetherian, then $R_{(\Delta)}$ is 1*-Gorenstein.*

(5)

$$B_\Delta := \mathrm{End}_R(T_{(\Delta)}) \simeq \begin{pmatrix} R_{(\Delta)} & R_{(\Delta)} \otimes_R J_\Delta \\ 0 & J_\Delta \end{pmatrix},$$

where $J_\Delta := \mathrm{End}_R(R_{\Delta)}/R) \simeq \prod_{\mathfrak{p}\in\Delta} J_\mathfrak{p}$.

(6)

$$B'_\Delta := \mathrm{End}_{R_{(\Delta)}}(T'_{(\Delta)}) \simeq \begin{pmatrix} Q & Q \otimes_R J'_\Delta \\ 0 & J'_\Delta \end{pmatrix},$$

where $J'_\Delta := \mathrm{End}_{R_{(\Delta)}}(Q/R_{(\Delta)}) \simeq \prod_{\mathfrak{p}\in P_1\setminus\Delta} J_\mathfrak{p}$.

(7) *The ring homomorphism μ_Δ in (3) induces a ring isomorphism*

$$R_{(\Delta)} \otimes_R J_\Delta \simeq Q \otimes_R J_\Delta.$$

(8) *For any subset P of P_1, the canonical map*

$$\Theta_P : Q \otimes_R \prod_{\mathfrak{p} \in P} J_{\mathfrak{p}} \longrightarrow \prod_{\mathfrak{p} \in P} Q \otimes_R J_{\mathfrak{p}},$$

defined by $q \otimes \left(x_{\mathfrak{p}}\right)_{\mathfrak{p} \in P} \mapsto \left(q \otimes x_{\mathfrak{p}}\right)_{\mathfrak{p} \in P}$ for $q \in Q$ and $x_{\mathfrak{p}} \in J_{\mathfrak{p}}$, is an injective ring homomorphism.

Proof. (1) Note that, for each $r \in \operatorname{Ker}(f_{\mathfrak{p}})$, there exists an element $x \in R \setminus \mathfrak{p}$ such that $rx = 0$. Since $\mathfrak{p}$ contains all zero divisors of R, we know that x is non-zero divisor of R. This implies $r = 0$, and so the map $f_{\mathfrak{p}}$ is injective.

(2) Let $\mathfrak{q} \in P_1 \setminus \Delta$. Since the localization map $f_{\mathfrak{q}} : R \to R_{\mathfrak{q}}$ is injective by (1), there is a unique injective homomorphism $\mu_{\mathfrak{q}} : R_{\mathfrak{q}} \to Q$ such that $f_{\Phi} = f_{\mathfrak{q}}\mu$ by the universal property of the total quotient ring of R. So, we can think of $R_{\mathfrak{q}}$ as a subring of Q. Under this identification, we can speak of the intersection of $R_{\mathfrak{p}}$ defined in (2).

First, we show that if $\Delta = P_1 \setminus \{\mathfrak{p}\}$ for some $\mathfrak{p} \in P_1$, then $R_{(\Delta)} = R_{\mathfrak{p}}$. By Lemma 7.4(1), we have the following exact commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & R \arrow[r, "f_{\mathfrak{p}}"] \arrow[d, equal] & R_{\mathfrak{p}} \arrow[r, "\mu_{\mathfrak{p}}\pi"] \arrow[d, "\mu_{\mathfrak{p}}"] & Y \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R \arrow[r, "f_{\Phi}"] & Q \arrow[r, "\pi"] \arrow[d] & \bigoplus_{\mathfrak{q} \in P_1} E(R/\mathfrak{q}) \arrow[r] \arrow[d, "(g_{\mathfrak{q}})_{\mathfrak{q} \in P_1}"] & 0. \\
 & & E(R/\mathfrak{p}) \arrow[r, equal] \arrow[d] & E(R/\mathfrak{p}) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

where Y denotes the image of the restriction of π to $R_{\mathfrak{p}}$, and where $g_{\mathfrak{q}} : E(R/\mathfrak{q}) \to E(R/\mathfrak{p})$ is a homomorphism of R-modules. Clearly, the localization $Y_{\mathfrak{p}}$ of Y at $\mathfrak{p}$ is zero. Let $\mathfrak{a} \in \Delta$. By Lemma 7.3(1), we know that $\operatorname{Hom}_R(E(R/\mathfrak{a}), E(R/\mathfrak{p})) = 0$ since both $\mathfrak{a}$ and $\mathfrak{p}$ are maximal ideals of R. Consequently, $g_{\mathfrak{a}} = 0$ and $Y = \operatorname{Ker}(g_{\mathfrak{p}}) \oplus \bigoplus_{\mathfrak{q} \in \Delta} E(R/\mathfrak{q})$, where $g := g_{\mathfrak{p}} : E(R/\mathfrak{p}) \to E(R/\mathfrak{p})$ is a surjective homomorphism of R-modules. We claim $\operatorname{Ker}(g) = 0$. In fact, by Lemma 7.3(2), we know that $E(R/\mathfrak{p})) \simeq (E(R/\mathfrak{p}))_{\mathfrak{p}}$ as R-modules. This implies that $\operatorname{Ker}(g) \simeq \operatorname{Ker}(g)_{\mathfrak{p}}$ as R-modules. Then it follows from $Y_{\mathfrak{p}} = 0$ that $\operatorname{Ker}(g)_{\mathfrak{p}} = 0$. Thus $R_{\mathfrak{p}} = R_{(\Delta)}$ under our identification of $R_{\mathfrak{p}}$ in Q.

Second, in the general case, we observe that

$$\Delta = \bigcap_{\mathfrak{b} \in P_1 \setminus \Delta} (P_1 \setminus \{\mathfrak{b}\}) \quad \text{and} \quad \bigoplus_{\mathfrak{p} \in \Delta} E(R/\mathfrak{p}) = \bigcap_{\mathfrak{b} \in P_1 \setminus \Delta} \Big(\bigoplus_{\mathfrak{q} \in P_1 \setminus \{\mathfrak{b}\}} E(R/\mathfrak{q}) \Big).$$

Thus

$$R_{(\Delta)} = \pi^{-1}\big(\bigoplus_{\mathfrak{p} \in \Delta} E(R/\mathfrak{p})\big) = \pi^{-1}\left(\bigcap_{\mathfrak{b} \in P_1 \setminus \Delta} \Big(\bigoplus_{q \in P_1 \setminus \{\mathfrak{b}\}} E(R/\mathfrak{q})\Big) \right) = \bigcap_{\mathfrak{p} \in P_1 \setminus \Delta} R_{\mathfrak{p}}.$$

Third, to prove that $R_{(\Delta)}$ is a flat R-module, we use the exact sequence (b). Since R is noetherian, the arbitrary direct sum of injective R-modules is injective. Thus $\bigoplus_{\mathfrak{p} \in P_1 \setminus \Delta} E(R/\mathfrak{p})$ is injective. Note that ${}_RQ$ is flat. By Lemma 7.4(2), we deduce that $R_{(\Delta)}$ is a flat R-module.

Finally, note that the total quotient ring of $R_{(\Delta)}$ also equals Q.

(3) Recall that Q is the localization of R at the multiplicative set Φ consisting of all non-zero divisors of R. Clearly, the map $f_\Phi : R \to Q$ is a ring epimorphism. Since Q and $R_{(\Delta)}$ are flat R-modules, it follows from Lemma 2.3 that both λ_Δ and μ_Δ are homological ring epimorphisms.

(4) For simplicity, we set $W := T'_{(\Delta)}$. Note that the $R_{(\Delta)}$-module W is injective as an R-modules. Since λ_Δ is a homological ring epimorphism, it follows from Lemma 2.2 that W is an injective $R_{(\Delta)}$-module. In particular, the regular module $R_{(\Delta)}$ has injective dimension at most 1. If $R_{(\Delta)}$ is noetherian, then $R_{(\Delta)}$ is 1-Gorenstein by definition. In this case, it follows directly from [3, Section 4] that the module W is a tilting $R_{(\Delta)}$-module. However, in general, we do not know whether $R_{(\Delta)}$ is noetherian or not. Because of this reason, we have to prove, in the following, that W is a tilting $R_{(\Delta)}$-module. Indeed, since W is an injective R-module and R is 1-Gorenstein, we know from Lemma 7.4(2) that $\mathrm{proj.dim}({}_RW) \le 1$. Note that λ_Δ is a homological ring epimorphism. It follows from Lemma 2.2 that $\mathrm{proj.dim}({}_{R_{(\Delta)}}W) \le \mathrm{proj.dim}({}_RW) \le 1$. Clearly, the module W satisfies the condition (T_3) in Definition 4.1. To see that W satisfies the condition (T_2) in Definition 4.1, we observe that

$$\mathrm{Ext}^i_{R_{(\Delta)}}(W, W^{(\alpha)}) \simeq \mathrm{Ext}^i_R(W, W^{(\alpha)}) = 0$$

for every $i \ge 1$ and every cardinal α, where the first isomorphism follows from Lemma 2.2(3), and the second equality follows from the fact that every direct sum of injective R-modules is injective since R is noetherian. Thus W is a tilting $R_{(\Delta)}$-module. Clearly, the exact sequence (b) implies that W is a good tilting module.

(5) By Lemma 7.3(1) and Lemma 7.3(3), we have $J_\Delta \simeq \mathrm{End}_R(\bigoplus_{\mathfrak{p}\in\Delta} E(R/\mathfrak{p})) \simeq \prod_{\mathfrak{p}\in\Delta} J_\mathfrak{p}$. According to Lemma 6.5(2), we have $\mathrm{Hom}_R\big(R_{(\Delta)}, \bigoplus_{\mathfrak{p}\in\Delta} E(R/\mathfrak{p})\big) \simeq R_{(\Delta)} \otimes_R J_\Delta$ as $R_{(\Delta)}$-J_Δ-bimodules. Now, (5) follows from Lemma 6.4(2) immediately.

(6) We first observe that

$$\mathrm{Hom}_R(X,Y) \simeq \mathrm{Hom}_{R_{(\Delta)}}(X,Y) \text{ and } X \otimes_R Y \simeq X \otimes_{R_{(\Delta)}} Y$$

for any $R_{(\Delta)}$-modules X and Y since $\lambda_\Delta : R \to R_{(\Delta)}$ is a ring epimorphism, and then use Lemma 6.4(2), Lemma 6.5 and Lemma 7.3. We omit the details here.

(7) Note that if R is a commutative noetherian ring and if I is an ideal of R, then (i) the I-adic completion of R is a flat R-module, and (ii) the product of flat R-modules is flat (see [23, Corollary 2.5.15, Theorem 3.2.24]). Hence J_Δ is a flat R-module. In order to prove that $\mu_\Delta \otimes_R J_\Delta : R_{(\Delta)} \otimes_R J_\Delta \to Q \otimes_R J_\Delta$ is an isomorphism, it is sufficient to show $(\bigoplus_{\mathfrak{p}\in P_1\setminus\Delta} E(R/\mathfrak{p})) \otimes_R J_\Delta = 0$. This is equivalent to $E(R/\mathfrak{p}) \otimes_R J_\Delta = 0$ for any $\mathfrak{p} \in P_1 \setminus \Delta$. However, the latter is a direct consequence of Lemma 7.3(4).

(8) Clearly, the map Θ_P is a ring homomorphism. Applying the tensor functors $- \otimes_R \prod_{\mathfrak{p}\in P} J_\mathfrak{p}$ and $- \otimes_R J_\mathfrak{p}$ to the minimal injective coresolution of R, respectively, we can get the following exact commutative diagram of R-modules:

$$\begin{tikzcd}
0 \arrow[r] & R \otimes_R \prod_{\mathfrak{p}\in P} J_\mathfrak{p} \arrow[r] \arrow[d, "\wr"] & Q \otimes_R \prod_{\mathfrak{p}\in P} J_\mathfrak{p} \arrow[r] \arrow[d, "\Theta_P"] & \big(\bigoplus_{\mathfrak{q}\in P_1} E(R/\mathfrak{q})\big) \otimes_R \prod_{\mathfrak{p}\in P} J_\mathfrak{p} \arrow[r] \arrow[d, "\Theta'_P"] & 0 \\
0 \arrow[r] & \prod_{\mathfrak{p}\in P} R \otimes_R J_\mathfrak{p} \arrow[r] & \prod_{\mathfrak{p}\in P} Q \otimes_R J_\mathfrak{p} \arrow[r] & \prod_{\mathfrak{p}\in P} \big(\bigoplus_{\mathfrak{q}\in P_1} E(R/\mathfrak{q})\big) \otimes_R J_\mathfrak{p} \arrow[r] & 0,
\end{tikzcd}$$

where the homomorphism Θ'_P of R-modules is defined in the same way as was done for Θ_P. We claim that Θ'_P is injective. In fact, since tensor functor commutes with direct sums, it follows from Lemma 7.3(4) that we can embed Θ'_P into the following commutative diagram:

$$\begin{tikzcd}
\big(\bigoplus_{\mathfrak{q}\in P_1} E(R/\mathfrak{q})\big) \otimes_R \prod_{\mathfrak{p}\in P} J_\mathfrak{p} \arrow[r, "\sim"] \arrow[d, "\Theta'_P"] & \bigoplus_{\mathfrak{p}\in P} E(R/\mathfrak{p}) \otimes_R J_\mathfrak{p} \arrow[d, "\lambda"] \\
\prod_{\mathfrak{p}\in P} \big(\bigoplus_{\mathfrak{q}\in P_1} E(R/\mathfrak{q})\big) \otimes_R J_\mathfrak{p} \arrow[r, "\sim"] & \prod_{\mathfrak{p}\in P} E(R/\mathfrak{p}) \otimes_R J_\mathfrak{p},
\end{tikzcd}$$

where λ is the canonical inclusion. This shows that Θ'_P is injective, which implies that Θ_P also is injective. $\square$

Note that if R is a local ring or a domain, then the assumption in Lemma 7.5 holds. It is well-known that Dedekind domains are 1-Gorenstein rings. Recall that a commutative ring R is called a Dedekind domain if R is a domain in which every proper ideal in R is the product of a finite numbers of prime ideals. This is equivalent to saying that $R_{\mathfrak{p}}$ is a discrete valuation ring for each prime ideal $\mathfrak{p}$ of R. A typical example of Dedekind domain is the ring $\mathbb{Z}$ of all rational integers.

To ensure (1) in Lemma 7.5, the general assumption of Lemma 7.5 cannot be dropped. For example, if R is a local 1-Gorenstein ring, then the direct sum $S := R \oplus R$ of two copies of R is again 1-Gorenstein. If we take $\mathfrak{m}$ to be the unique maximal ideal of R, then the localization of S at the maximal ideal $\mathfrak{p} := (\mathfrak{m}, R)$ is isomorphic to $R_{\mathfrak{m}}$. This shows that the localization map $S \to S_{\mathfrak{p}}$ is not injective. Clearly, the assumption of Lemma 7.5 is not satisfied in this case.

By Lemma 7.5(2), we know that $R_{(\Delta)}$ is always an intersection of localizations. But, in general, it may not be a localization of R at any multiplicative set. For a counterexample, we refer the reader to [34]. A natural question arises: when is $R_{(\Delta)}$ itself a localization of R at some multiplicative set ? The following result provides some partial answers to this question.

Lemma 7.6. *Let Δ be a subset of P_1. Assume that each prime ideal belonging to $P_1 \setminus \Delta$ contains all zero divisors of R. Define $\Sigma := R \setminus \bigcup_{\mathfrak{q} \in P_1 \setminus \Delta} \mathfrak{q}$ and $\Delta_1 := \{\mathfrak{a} \in P_1 \mid \mathfrak{a} \subseteq \bigcup_{\mathfrak{q} \in P_1 \setminus \Delta} \mathfrak{q}\}$. Then,*

(1) $\Sigma^{-1}R = \pi^{-1}\big(\bigoplus_{\mathfrak{p} \in P_1 \setminus \Delta_1} E(R/\mathfrak{p})\big) \subseteq R_{(\Delta)} \subseteq Q$.

(2) $R_{(\Delta)} = \Sigma_1^{-1}R$ *for some multiplicative subset* Σ_1 *of* R *if and only if* $R_{(\Delta)} = \Sigma^{-1}R$ *if and only if* $\Delta_1 = P_1 \setminus \Delta$.

(3) *If* $P_1 \setminus \Delta$ *is a finite set, or if each ideal in* Δ *is principal, then* $R_{(\Delta)} = \Sigma^{-1}R$.

Proof. (1) Clearly, we have $\Sigma \subseteq \Phi$ and Σ is a multiplicative set. Thus the canonical map $f_\Sigma : R \to \Sigma^{-1}R$ is injective, and there is a unique injective ring homomorphism $h : \Sigma^{-1}R \to Q$ such that $f_\Phi = f_\Sigma h$. In this sense, we may regard $\Sigma^{-1}R$ as a subring of the total quotient ring Q containing R. Moreover, the total quotient ring of $\Sigma^{-1}R$ equals Q. Since R is a 1-Gorenstein ring, it follows from Lemma 7.4(4) that $\Sigma^{-1}R$ also is a 1-Gorenstein ring. In addition, it follows from standard commutative algebra that the map $\varphi : \Delta_1 \to \mathrm{Spec}(\Sigma^{-1}R)$ sending $\mathfrak{q}$ to $\Sigma^{-1}\mathfrak{q}$ for $\mathfrak{q} \in \Delta_1$ is a bijection. This shows that we have an exact sequence of R-modules:

$$0 \longrightarrow R_\Sigma \xrightarrow{h} Q \longrightarrow \bigoplus_{\mathfrak{q} \in \Delta_1} E(R/\mathfrak{q}) \longrightarrow 0.$$

By Lemma 7.4, we can further form the following exact commutative diagram of R-modules:

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & R \arrow[r, "f_\Sigma"] \arrow[d, equal] & \Sigma^{-1}R \arrow[r, "h\pi"] \arrow[d, "h"] & Y' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & R \arrow[r, "f_\Phi"] & Q \arrow[r, "\pi"] \arrow[d] & \bigoplus_{\mathfrak{q} \in P_1} E(R/\mathfrak{q}) \arrow[r] \arrow[d, "(g'_\mathfrak{q})_{\mathfrak{q} \in P_1}"] & 0, \\
 & & \bigoplus_{\mathfrak{q} \in \Delta_1} E(R/\mathfrak{q}) \arrow[r, equal] \arrow[d] & \bigoplus_{\mathfrak{q} \in \Delta_1} E(R/\mathfrak{q}) \arrow[d] & \\
 & & 0 & 0 &
\end{tikzcd}$$

where Y' denotes the image of $h\pi$, and where $g'_\mathfrak{q} : E(R/\mathfrak{q}) \to \bigoplus_{\mathfrak{q} \in \Delta_1} E(R/\mathfrak{q})$ is a homomorphism of R-modules. By the same argument as in the proof of Lemma 7.5(2), we can prove $Y' = \bigoplus_{\mathfrak{p} \in P_1 \setminus \Delta_1} E(R/\mathfrak{p})$. Thus $\Sigma^{-1}R = \pi^{-1}(\bigoplus_{\mathfrak{p} \in P_1 \setminus \Delta_1} E(R/\mathfrak{p}))$. By definition, we have $P_1 \setminus \Delta \subseteq \Delta_1$, and so $\Sigma^{-1}R \subseteq R_{(\Delta)}$.

(2) It follows from the definition of $R_{(\Delta)}$ and the statement (1) that $R_{(\Delta)} = \Sigma^{-1}R$ if and only if $\Delta_1 = P_1 \setminus \Delta$. To prove the statement (2), it suffices to show that if $R_{(\Delta)} = \Sigma_1^{-1}R$ for some multiplicative subset Σ_1 of R, then $R_{(\Delta)} = \Sigma^{-1}R$. Now, assume $R_{(\Delta)} = \Sigma_1^{-1}R$. By Lemma 7.4(4), $R_{(\Delta)}$ is a 1-Gorenstein ring. Note that $Q/R_{(\Delta)} \simeq \bigoplus_{\mathfrak{p}\in P_1\setminus\Delta} E(R/\mathfrak{p})$ as R-modules. Then it follows from Lemmata 7.4(1) and 7.3(2) that, for any $\mathfrak{p} \in P_1 \setminus \Delta$, we have $\mathfrak{p} \cap \Sigma_1 = \emptyset$, and so $\Sigma_1 \subseteq R \setminus \mathfrak{p}$. Since $\Sigma := R \setminus \bigcup_{\mathfrak{q}\in P_1\setminus\Delta} \mathfrak{q} = \bigcap_{\mathfrak{q}\in P_1\setminus\Delta}(R\setminus\mathfrak{q})$, we have $\Sigma_1 \subseteq \Sigma$, and so $R_{(\Delta)} = \Sigma_1^{-1}R \subseteq \Sigma^{-1}R$. Thanks to the statement (1), we get $\Sigma^{-1}R \subseteq R_{(\Delta)} \subseteq Q$. Thus $R_{(\Delta)} = \Sigma^{-1}R$.

(3) It suffices to show $\Delta_1 = P_1 \setminus \Delta$. Clearly, we have $P_1 \setminus \Delta \subseteq \Delta_1$ by definition. Now we show $\Delta_1 \subseteq P_1 \setminus \Delta$. In fact, if $\mathfrak{a} \in \Delta_1$, then $\mathfrak{a} \subseteq \bigcup_{\mathfrak{q}\in P_1\setminus\Delta} \mathfrak{q}$. Thus, if, in addition, $P_1 \setminus \Delta$ is finite, then $\mathfrak{a} \subseteq \mathfrak{q}_1$ for some $\mathfrak{q}_1 \in P_1 \setminus \Delta$ by prime avoidance theorem. Since $\mathfrak{a}$ is a maximal ideal of R, it follows that $\mathfrak{a} = \mathfrak{q}_1$. Hence $\Delta_1 = P_1 \setminus \Delta$. If we assume that each ideal in Δ is principal, then $\mathfrak{a}$ must be in $P_1 \setminus \Delta$. In fact, if it is not the case, then $\mathfrak{a} \in \Delta$, and so there exists an $r \in R$ such that $\mathfrak{a} = Rr$. Since $\mathfrak{a} \subseteq \bigcup_{\mathfrak{q}\in P_1\setminus\Delta} \mathfrak{q}$, we know that $r \in \mathfrak{q}$ for some $\mathfrak{q} \in P_1 \setminus \Delta$, and so $\mathfrak{a} \subseteq \mathfrak{q}$. By the maximality of $\mathfrak{a}$, we have $\mathfrak{a} = \mathfrak{q}$. This is impossible because the intersection of Δ with $P_1 \setminus \Delta$ is empty. Hence $\Delta_1 = P_1 \setminus \Delta$.

By (2), we have $R_{(\Delta)} = \Sigma^{-1}R$ for either case. $\square$

Combining Corollary 6.6(1) and Lemma 6.3 with Lemma 7.5(7), we have the following result on recollements of derived module categories of endomorphism rings.

Proposition 7.7. *Let R be a 1-Gorenstein ring, and let Δ be a subset of P_1. Assume that each prime ideal in $P_1 \setminus \Delta$ contains all zero divisors of R. Then we get the following recollements of derived module categories:*

$$\begin{tikzcd}
\mathscr{D}(Q\otimes_R J_\Delta) \arrow[r] & \mathscr{D}(B_\Delta) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left}
\end{tikzcd},$$

$$\begin{tikzcd}
\mathscr{D}(Q\otimes_R J'_\Delta) \arrow[r] & \mathscr{D}(B'_\Delta) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R_{(\Delta)}) \arrow[l, bend right] \arrow[l, bend left]
\end{tikzcd}.$$

Proof. Here we provide a different proof. We consider the injective homological ring epimorphism $\lambda_\Delta : R \to R_{(\Delta)}$ defined in Lemma 7.5(3). Then, we have $R_{(\Delta)} \otimes_R J_\Delta \simeq Q \otimes_R J_\Delta$ by Lemma 7.5(7). Now, the first recollement in Proposition 7.7 follows immediately from Proposition 7.1.

The proof of the existence of the second recollement in Proposition 7.7 can be implemented similarly as we did for the first one. $\square$

In the rest of this subsection, we consider the ring $\mathbb{Z}$, it is a Dedekind domain and, of course, a 1-Gorenstein ring. Clearly, it fulfills the assumption of Proposition 7.7. Moreover, in this case, we can have a much better formulation than Proposition 7.7. Our discussion below uses some basic results on p-adic numbers in algebraic number theory.

Fix a prime number $p \geq 2$. A p-adic integer is a formal infinite series $\sum_{i=0}^{\infty} a_i p^i$, where $0 \leq a_i < p$ for all $i \geq 0$. A p-adic number is a formal infinite series of the form $\sum_{j=-m}^{\infty} a_j p^j$, where $m \in \mathbb{Z}$ and $0 \leq a_j < p$ for all $j \geq -m$. The sets of all p-adic integers and p-adic numbers are denoted by $\mathbb{Z}_p$ and $\mathbb{Q}_p$, respectively. Note that $\mathbb{Z}_p$ is a discrete valuation ring of global dimension 1 with the unique maximal ideal $p\mathbb{Z}_p$, and that $\mathbb{Q}_p$ is a field.

If $f \in \mathbb{Q}$ is a rational number, then we can write $f = \frac{g}{h}p^{-m}$, where $g, h \in \mathbb{Z}$, $(gh, p) = 1$. Since the rational number $\frac{g}{h}$ always belongs to $\mathbb{Z}_p$, that is, there are $0 \leq a_i < p$ for all $i \geq 0$ such that $\frac{g}{h} = \sum_{i=0}^{\infty} a_i p^i$. Consequently, we have

$$f = \sum_{i=0}^{\infty} a_i p^{-m+i} \in \mathbb{Q}_p.$$

In this way, we can regard $\mathbb{Q}$ as a subfield of $\mathbb{Q}_p$. This implies that, for $f \in \mathbb{Q}$, there are at most finitely many prime numbers q such that $f \in \mathbb{Q}_q \setminus \mathbb{Z}_q$, or equivalently, $f \in \mathbb{Z}_q$ for almost all prime number q. It is well-known that $\mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}_p \simeq \mathbb{Q}_p$ by multiplication map since $\mathbb{Q}_p = \{p^m y \mid m \in \mathbb{Z}, y \in \mathbb{Z}_p\}$. Clearly, $\mathbb{Q} \subset \mathbb{Q}_p$ and $\mathbb{Z} \subset \mathbb{Q}_{(p)} \subset \mathbb{Z}_p \subset \mathbb{Q}_p$ for every prime $p \in \mathbb{N} := \{0,1,2,\cdots\}$. It is known that $\mathrm{End}_{\mathbb{Z}}(\mathbb{Q}/\mathbb{Z}) \simeq \prod_p \mathbb{Z}_p$ as rings, where p goes through all prime numbers.

An alternative definition of $\mathbb{Z}_p$ is that $\mathbb{Z}_p$ is the p-adic completion $\varprojlim_i \mathbb{Z}/p^i\mathbb{Z}$ of $\mathbb{Z}$. Another algebraic definition of $\mathbb{Z}_p$ is that $\mathbb{Z}_p$ is isomorphic to the quotient of the formal power series ring $\mathbb{Z}[[X]]$ by the ideal generated by $X-p$. Note that $\mathbb{Q}_p$ is the field of fractions of $\mathbb{Z}_p$. For more details about p-adic numbers, one may refer to [33, Chapter IV, Section 2]. We denote by $\widehat{\mathbb{Z}}$ the product $\prod_p \mathbb{Z}_p$ of all $\mathbb{Z}_p$ with p positive prime numbers. This is a commutative ring.

Now, let Λ be the set of all prime numbers in $\mathbb{N}$, and let I be a subset of Λ. Set $I' := \Lambda \setminus I$, $\Delta := \{\mathfrak{p} = p\mathbb{Z} \mid p \in I\}$ and $\mathbb{Z}_{(I)} := \mathbb{Z}_{(\Delta)}$.

Lemma 7.8. *The following statements hold true for the ring $\mathbb{Z}$ of integers.*

(1) *Let $\Sigma := \mathbb{Z} \setminus \cup_{q\in I'} \mathfrak{q}$. Then $\mathbb{Z}_{(I)} = \Sigma^{-1}\mathbb{Z}$, which is the smallest subring of $\mathbb{Q}$ containing $\frac{1}{p}$ for all $p \in I$.*

(2)*The injective ring homomorphism*

$$\Theta_I : \mathbb{Q}\otimes_{\mathbb{Z}} \prod_{p\in I} \mathbb{Z}_p \longrightarrow \prod_{p\in I} \mathbb{Q}_p$$

defined by $q\otimes (x_p)_{p\in I} \mapsto (qx_p)_{p\in I}$ for $q \in \mathbb{Q}$ and $x_p \in \mathbb{Z}_p$ satisfies that

$$\mathrm{Im}(\Theta_I) = \mathbb{A}_I := \big\{(y_p)_{p\in I} \in \prod_{p\in I} \mathbb{Q}_p \mid y_p \in \mathbb{Z}_p \textit{ for almost all } p \in I\big\}.$$

In particular, if I is a finite set, then $\mathrm{Im}(\Theta_I) = \mathbb{A}_I = \prod_{p\in I} \mathbb{Q}_p$. Note that $\mathbb{A}_I$ is a kind of adéle in global class field theory (see [33, Chapter VI]).

(3) *There are ring isomorphisms:*

$$\mathbb{Q}\otimes_{\mathbb{Z}} \prod_{p\in I} \mathbb{Z}_p \simeq \mathbb{A}_I, \qquad \mathbb{Q}\otimes_{\mathbb{Z}} \prod_{p\in I'} \mathbb{Z}_p \simeq \mathbb{A}_{I'}.$$

Proof. (1) Let $q \in I'$. By Lemma 7.5(2), we have $\mathbb{Z}_{(I)} = \bigcap_{q\in I'} \mathbb{Z}_{\mathfrak{q}}$, where $\mathbb{Z}_{\mathfrak{q}}$ is the localization of $\mathbb{Z}$ at $\mathfrak{q}$ with $\mathfrak{q} = q\mathbb{Z}$. It follows from $\mathbb{Z}_{\mathfrak{q}} = \mathbb{Q}_{(q)}$ that

$$\mathbb{Z}_{(I)} = \bigcap_{q\in I'} \mathbb{Q}_{(q)} = \mathbb{Z}[p^{-1} \mid p \in I] = \Sigma^{-1}\mathbb{Z}.$$

(2) For each prime number p, the canonical ring homomorphism $\mu : \mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}_p \to \mathbb{Q}_p$, defined by $f\otimes x_p \mapsto fx_p$ for any $f \in \mathbb{Q}$, and $x_p \in \mathbb{Z}_p$, is an isomorphism. Moreover, for each $f \in \mathbb{Q}$, there are at most finitely many prime numbers q such that $f \in \mathbb{Q}_q \setminus \mathbb{Z}_q$. In other words, $f \in \mathbb{Z}_q$ for almost all prime number q. This implies $\mathrm{Im}(\Theta_I) = \mathbb{A}_I$.

(3) This follows from (2). $\square$

With help of Lemma 7.8, we can state Proposition 7.7 for $R = \mathbb{Z}$ more explicitly.

Corollary 7.9. *We have the following recollements of derived module categories:*

$$\begin{tikzcd} \mathscr{D}(\mathbb{A}_I) \arrow[r] & \mathscr{D}(B_I) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(\mathbb{Z}) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd},$$

$$\begin{tikzcd} \mathscr{D}(\mathbb{A}_{I'}) \arrow[r] & \mathscr{D}(B_I') \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(\mathbb{Z}_{(I)}) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd},$$

where $B_I := \mathrm{End}_{\mathbb{Z}}(\mathbb{Z}_{(I)} \oplus \mathbb{Z}_{(I)}/\mathbb{Z})$ and $B_I' := \mathrm{End}_{\mathbb{Z}_{(I)}}(\mathbb{Q}\oplus\mathbb{Q}/\mathbb{Z}_{(I)})$.

7.3 Examples

In the following we shall exploit Corollary 7.9 to give a couple of concrete examples of derived module categories that have two different stratifications by derived module categories of rings with different composition factors. This is related to the following problem proposed in [5]:

Problem: Given a ring R, do all stratifications of $\mathscr{D}(R)$ by derived module categories of rings have the same finite number of factors, and are these factors the same for all stratifications, up to ordering and up to derived equivalence?

A negative partial solution to this problem can be seen from Examples 7.10 and 7.11 below.

Let us first recall the definition of a stratification of $\mathscr{D}(R)$ for R a ring in [5].

Let R be a ring. If there are rings R_1 and R_2 such that a recollement

$$(*)\qquad \begin{tikzcd} \mathscr{D}(R_1) \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R_2) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

exists, then R_i or $\mathscr{D}(R_i)$, with $1 \leq i \leq 2$, are called factors of R or $\mathscr{D}(R)$. In this case, we also say that $(*)$ is a recollement of R. The ring R is called derived simple if $\mathscr{D}(R)$ does not admit any non-trivial recollement whose factors are derived categories of rings. It is pointed out in [4] that every Dedekind ring (thus every discrete valuation ring) is derived simple.

A stratification of $\mathscr{D}(R)$ is defined to be a sequence of iterated recollements of the following form: a recollement of R, if it is not derived simple,

$$\begin{tikzcd} \mathscr{D}(R_0) \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R_1) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} ,$$

a recollement of R_0, if it is not derived simple,

$$\begin{tikzcd} \mathscr{D}(R_{00}) \arrow[r] & \mathscr{D}(R_0) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R_{01}) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} ,$$

and a reollement of R_2, if it is not derived simple,

$$\begin{tikzcd} \mathscr{D}(R_{10}) \arrow[r] & \mathscr{D}(R_1) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R_{11}) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}$$

and recollements of R_{ij} with $0 \leq i, j \leq 1$, if they are not derived simple, and so on, until one arrives at derived simple rings at all positions, or continue to infinitum. All the derived simple rings appearing in this procedure are called composition factors of the stratification. The cardinality of the set of all composition factors (counting the multiplicity) is called the length of the stratification. If this procedure stops after finitely many steps, we say that this stratification is finite or of finite length.

The first example below shows two stratifications of a derived module category with infinitely many different derived simple module categories as composition factors

Example 7.10. Let $\mathbb{Z} \hookrightarrow \mathbb{Q}$ be the inclusion. Then $T = \mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z}$ is a tilting $\mathbb{Z}$-module, and

$$B := \mathrm{End}_R(T) = \begin{pmatrix} \mathbb{Q} & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \\ 0 & \widehat{\mathbb{Z}} \end{pmatrix}.$$

Note that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \simeq \mathbb{R}$ as abelian groups, where $\mathbb{R}$ is the field of real numbers.

We take $\Delta := \mathrm{mSpec}(\mathbb{Z})$. By Corollary 7.9 and Lemma 7.8(3), we have a recollement:

$$\begin{tikzcd} \mathscr{D}(\mathbb{Q} \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}) \arrow[r] & \mathscr{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(\mathbb{Z}) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} .$$

Let $e_2 = (1,0,\cdots,) \in \widehat{\mathbb{Z}}$. Then $\widehat{\mathbb{Z}} = \widehat{\mathbb{Z}}e_2 \oplus \widehat{\mathbb{Z}}(1-e_2)$. This is a decomposition of ideals of $\widehat{\mathbb{Z}}$. Thus we have a decomposition of ideals of the ring $\mathbb{Q}\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}}$:

$$\mathbb{Q}\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}} = \mathbb{Q}\otimes_{\mathbb{Z}}\mathbb{Z}_2 \oplus \mathbb{Q}\otimes_{\mathbb{Z}}\prod_{p\geq 3}\mathbb{Z}_p = \mathbb{Q}_2 \oplus \mathbb{Q}\otimes_{\mathbb{Z}}\prod_{p\geq 3}\mathbb{Z}_p.$$

This procedure can be repeated infinitely many times. Then it follows that $\mathscr{D}(\mathbb{Q}\otimes_{\mathbb{Z}}\widehat{\mathbb{Z}})$ has a derived composition series with infinitely many simple factors $\mathscr{D}(\mathbb{Q}_p)$. This shows that $\mathscr{D}(B)$ has a stratification with derived composition factors equivalent to either $\mathscr{D}(\mathbb{Z})$ or $\mathscr{D}(\mathbb{Q}_p)$, both are derived simple, that is, each of them cannot be a middle term in any proper recollement of derived module categories of rings.

Transparently, it follows from the triangular form of B that $\mathscr{D}(B)$ has a stratification with infinitely many composition factors equivalent to either $\mathscr{D}(\mathbb{Q})$ or $\mathscr{D}(\mathbb{Q}_p)$. Clearly, $\mathscr{D}(\mathbb{Z})$ and $\mathscr{D}(\mathbb{Q})$ are not equivalent as triangulated categories since the global dimension of $\mathbb{Z}$ is one and the global dimension of $\mathbb{Q}$ is zero. Thus $\mathscr{D}(B)$ has two stratifications which have different composition factors. This gives negatively an answer to the second question of the above mentioned problem.

In Example 7.10 the two stratifications of the category $\mathscr{D}(B)$ by derived module categories have infinite many composition factors. In the next example we shall see that even one requires finiteness of stratifications of a derived module category, their composition factors still may be different. This is contrary to the well-known Jordan-Hölder theorem which says that any two (finite) composition series of a group have the same list of composition factors (up to the ordering and up to isomorphism).

Example 7.11. (1) Let I be a non-empty finite subset of $\mathrm{mSpec}(\mathbb{Z})$. We consider the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z}_{(I)} \to \bigoplus_{p\in I} E(\mathbb{Z}/p) \to 0$$

of abelian groups. Then $T := \mathbb{Z}_{(I)} \oplus \bigoplus_{p\in I} E(\mathbb{Z}/p\mathbb{Z})$ is a tilting module. On the one hand, by Lemmata 6.4(2) and 7.5(5), we have

$$\mathrm{End}_{\mathbb{Z}}(T) \simeq \begin{pmatrix} \mathbb{Z}_{(I)} & \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}_I, \mathbb{Z}_{(I)}/\mathbb{Z}) \\ 0 & \bigoplus_{p\in I}\mathbb{Z}_p \end{pmatrix}.$$

On the other hand, since I is a finite set, by Corollary 7.9, $\mathrm{End}_{\mathbb{Z}}(T)$ admits a recollement

$$\begin{tikzcd} \mathscr{D}(\bigoplus_{p\in I}\mathbb{Q}_p) \arrow[r] & \mathscr{D}(\mathrm{End}_{\mathbb{Z}}(T)) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(\mathbb{Z}) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}.$$

Thus, $\mathscr{D}(\mathrm{End}_{\mathbb{Z}}(T))$ admits two stratifications, one of which has the composition factors $\mathbb{Z}_{(I)}$ and $\mathbb{Z}_p$ with $p\in I$, and the other has the composition factors $\mathbb{Z}$ and $\mathbb{Q}_p$ with $p \in I$. Since $\mathbb{Z}_{(I)}$ is a localization of $\mathbb{Z}$ by Lemma 7.8, it is of global dimension one. Note that derived equivalences preserve the centers of rings. This shows that all the rings $\mathbb{Z}, \mathbb{Z}_{(I)}, \mathbb{Z}_p$ and $\mathbb{Q}_p$ are pairwise not derived-equivalent. Hence the two stratifications have completely different composition factors.

(2) Let $\mathfrak{p} = p\mathbb{Z} \subset \mathbb{Z}$ with p a prime number in $\mathbb{N}$. We consider the exact sequence of $\mathbb{Z}_{\mathfrak{p}}$-modules:

$$0 \to \mathbb{Z}_{\mathfrak{p}} \to \mathbb{Q} \to E(\mathbb{Z}_{\mathfrak{p}}/p\mathbb{Z}_{\mathfrak{p}}) \to 0.$$

Define $T := \mathbb{Q} \oplus E(\mathbb{Z}_{\mathfrak{p}}/p\mathbb{Z}_{\mathfrak{p}})$. Thus, by Lemma 7.5 and Corollary 7.9, we have

$$\mathrm{End}_{\mathbb{Z}_{\mathfrak{p}}}(T) \simeq \mathrm{End}_{\mathbb{Z}}(T) \simeq \begin{pmatrix} \mathbb{Q} & \mathbb{Q}_p \\ 0 & \mathbb{Z}_p \end{pmatrix},$$

and a recollement:

$$\begin{tikzcd} \mathscr{D}(\mathbb{Q}_p) \arrow[r] & \mathscr{D}(\mathrm{End}_{\mathbb{Z}_{\mathsf{p}}}(T)) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(\mathbb{Z}_{\mathsf{p}}) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} .$$

Note that the ring $\mathrm{End}_{\mathbb{Z}_{\mathsf{p}}}(T)$ is left hereditary, but not left noetherian.

On the one hand, $\mathscr{D}(\mathrm{End}_{\mathbb{Z}_{\mathsf{p}}}(T))$ has clearly a stratification of length 2 with the composition factors $\mathbb{Q}$ and $\mathbb{Z}_p$. On the other hand, it admits another stratification of length 2 with the composition factors $\mathbb{Q}_p$ and $\mathbb{Z}_{\mathsf{p}}$. Note that $\mathbb{Z}_{\mathsf{p}} = \mathbb{Q}_{(p)}$. Since $\mathbb{Z}_p$ and $\mathbb{Q}_p$ are uncountable sets and since derived equivalences preserve the centers of rings, we deduce that neither $\mathbb{Q}$ and $\mathbb{Q}_{(p)}$, nor $\mathbb{Z}_p$ and $\mathbb{Q}_{(p)}$ are derived equivalent. Clearly, the global dimensions of $\mathbb{Z}_p$ and $\mathbb{Q}_{(p)}$ are one. Thus we have proved that the derived category of the ring $\mathrm{End}_{\mathbb{Z}_{\mathsf{p}}}(T)$ has two stratifications of length two without any common composition factors.

Thus, this example shows also that the main result in [5, Theorem 6.1] for hereditary artin algebras cannot be extended to left hereditary rings.

Note that in each example given in this section the sets of composition factors of the two stratifications of the derived module category have the same cardinalities. In the next section we shall see that this phenomenon is not always true.

8 Further examples and open questions

The main purpose of this section is to present examples of derived module categories of rings such that they possess two stratifications (by derived module categories of rings) with different finite lengths. Namely, we consider the following

Question. Is there a ring R such that $\mathscr{D}(R)$ has two stratifications of different finite lengths by derived module categories of rings ?

Thus we solve the whole problem in [5] negatively.

Let k be a field. We denote by $k[x]$ and $k[[x]]$ the polynomial and formal power series algebras over k in one variable x, respectively, and by $k((x))$ the Laurent power series algebra in one variable x, that is, $k((x)) := \{x^{-n}a \mid n \in \mathbb{N}, a \in k[[x]]\}$.

Now, let k be an algebraically closed field, and let R be the Kronecker algebra $\begin{pmatrix} k & k^2 \\ 0 & k \end{pmatrix}$. It is known that R can be given by the following quiver

$$Q:\ 2 \underset{\beta}{\overset{\alpha}{\rightrightarrows}} 1 ,$$

and that R-Mod is equivalent to the category of representations of Q over k.

Let V be a simple regular R-module. For each $m > 0$, we denote by $V[m]$ the module of regular length m on the ray

$$V = V[1] \subset V[2] \subset \cdots \subset V[m] \subset V[m+1] \subset \cdots ,$$

and let $V[\infty] = \varinjlim V[m]$ be the corresponding Prüfer modules. Note that the only regular submodule of $V[\infty]$ of regular length m is $V[m]$ with its canonical inclusion in $V[\infty]$, and that each endomorphism of $V[\infty]$ in R-Mod restricts to an endomorphism of $V[m]$ for any $m > 0$. Thus, $V[\infty]$ admits a unique chain of regular submodules. For more details, we refer to [36, Section 4.5].

From now on, we denote by V the simple regular R-module: $k \underset{1}{\overset{0}{\rightrightarrows}} k$. Let $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ and $e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Since $\mathrm{Hom}_R(Re_1, Re_2) \simeq k^2$, we can identify a homomorphism from Re_1 to Re_2 in R-Mod with an element in k^2. Fix a minimal projective resolution of V:

$$0 \longrightarrow Re_1 \xrightarrow{\partial:=(1,0)} Re_2 \longrightarrow V \longrightarrow 0,$$

and denote by $\lambda : R \to R_V$ the universal localization of R at the set $\Sigma := \{\partial\}$.

It follows from [37, Theorems 4.9, 5.1, and 5.3] that R_V is hereditary, λ is injective, and $R_V \oplus R_V/R$ is a tilting R-module. Moreover, by [7, Proposition 1.8], we get $R_V/R \simeq V[\infty]^2$ as R-modules. Note that $\mathrm{Hom}_R(R_V/R, R_V) = 0$ because R_V/R is a torsion module and R_V is a torsion-free module.

For simplicity of notations, we denote by T the tilting module $R_V \oplus V[\infty]^2$. Now, applying Corollary 6.7 to the module T, we can get the following recollement of derived module categories:

$$(*) \qquad \begin{tikzcd} \mathscr{D}(R_V \sqcup_R S') \arrow[r] & \mathscr{D}(B) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd},$$

where $B := \mathrm{End}_R(T)$, $S' := M_2\big(\mathrm{End}_R(V[\infty])\big)$ and $R_V \sqcup_R S'$ is the coproduct of R_V and S' over R.

In the following, we shall describe the rings B, S' and $R_V \sqcup_R S'$ explicitly.

First, by Lemma 3.1, we can check that $R_V = M_2(k[x])$, the 2×2 matrix algebra over $k[x]$, and the map $\lambda : R \to R_V$ is given by $\begin{pmatrix} a & (c,d) \\ 0 & b \end{pmatrix} \mapsto \begin{pmatrix} a & c+dx \\ 0 & b \end{pmatrix}$ for any $a,b,c,d \in k$. This means $R_V e_1 \simeq R_V e_2$ as R_V-modules, and therefore we have the following ring isomorphisms:

$$(**) \quad B \simeq M_2\Big(\mathrm{End}_R(R_V e_1 \oplus V[\infty])\Big) \simeq M_2\left(\begin{pmatrix} e_1 R_V e_1 & \mathrm{Hom}_R(R_V e_1, V[\infty]) \\ 0 & \mathrm{End}_R(V[\infty]) \end{pmatrix}\right).$$

Second, we claim that $\mathrm{End}_R(V[\infty])$ is isomorphic to $k[[x]]$. In fact, this follows from the following isomorphisms of abelian groups:

$$\mathrm{End}_R(V[\infty]) \simeq \varprojlim \mathrm{Hom}_R(V[m], V[\infty]) \simeq \varprojlim \mathrm{Hom}_R(V[m], V[m]) \simeq \varprojlim k[x]/(x^m) \simeq k[[x]],$$

where the composition of the above isomorphisms gives rise to a ring isomorphism $\omega : \mathrm{End}_R(V[\infty]) \to k[[x]]$. Thus $S' \simeq M_2(k[[x]])$ as rings. In this sense, we can identify S' with $M_2(k[[x]])$ under the isomorphism ω.

Third, a direct calculation shows that the ring homomorphism $\mu : R \to S'$, which appears in the proof of Corollary 6.7, is given by $\begin{pmatrix} a' & (c',d') \\ 0 & b' \end{pmatrix} \mapsto \begin{pmatrix} a' & d'+c'x \\ 0 & b' \end{pmatrix}$ for any $a',b',c',d' \in k$.

Finally, we claim $R_V \sqcup_R S' \simeq M_2\big(k((x))\big)$ as rings.

In fact, we recall that R_V is the universal localization of R at $\Sigma := \{\partial\}$. Define $\varphi := S' \otimes_R \partial : S'e_1 \to S'e_2$. Then it follows from Lemma 6.2 that $R_V \sqcup_R S'$ is isomorphic to the universal localization S'_φ of S' at φ. Since $\mathrm{Hom}_{S'}(S'e_1, S'e_2) \simeq e_1 S' e_2 \simeq k[[x]]$, the map φ corresponds to the matrix element $\begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$ in S'. Now, let $\rho_x : k[[x]] \to k[[x]]$ be the right multiplication map by x. Since S' is Morita equivalent to $k[[x]]$, we conclude from Lemma 3.5 that $S'_\varphi = M_2\big(k[[x]]_{\rho_x}\big)$, where $k[[x]]_{\rho_x}$ is the universal localization of $k[[x]]$ at ρ_x. Since $k[[x]]$ is commutative, the ring $k[[x]]_{\rho_x}$ is isomorphic to the localization $\Theta^{-1}k[[x]]$ of $k[[x]]$ at the multiplicative subset $\Theta := \{x^m \mid m \in \mathbb{N}\}$. Thus $\Theta^{-1}k[[x]]$ is the Laurent power series ring $k((x))$. Therefore, we get the following isomorphisms of rings:

$$R_V \sqcup_R S' \simeq S'_\varphi \simeq M_2\big(k[[x]]_{\rho_x}\big) \simeq M_2\big(\Theta^{-1}k[[x]]\big) \simeq M_2\big(k((x))\big).$$

On the one hand, by setting $C := \mathrm{End}_R(R_V e_1 \oplus V[\infty])$ and using Morita equivalences, the recollement $(*)$ can be rewritten as

$$\begin{tikzcd} \mathscr{D}\big(k((x))\big) \arrow[r] & \mathscr{D}(C) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(R) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd}.$$

On the other hand, since $e_1 R_V e_1 \simeq k[x]$ and $\mathrm{End}_R(V[\infty]) \simeq k[[x]]$, it follows from $(**)$ that the ring C admits another recollement

$$\begin{tikzcd} \mathscr{D}\big(k[x]\big) \arrow[r] & \mathscr{D}(C) \arrow[l, bend right] \arrow[l, bend left] \arrow[r] & \mathscr{D}(k[[x]]) \arrow[l, bend right] \arrow[l, bend left] \end{tikzcd} .$$

Since derived equivalences preserve the centers of rings, all the rings k, $k[x]$, $k[[x]]$ and $k((x))$ are pairwise not derived equivalent. But, they are derived simple. Clearly, $\mathscr{D}(R)$ has a stratification of length 2 with composition factors $\mathscr{D}(k)$ and $\mathscr{D}(k)$. Thus C admits two stratifications, one of which is of length 3 with three composition factors $k((x))$, k and k, and the other is of length 2 with composition factors $k[x]$ and $k[[x]]$. As a result, we have shown that the two stratifications of $\mathscr{D}(C)$ by derived categories of rings are of different lengths and without any common composition factors.

Remarks. (1) For any simple regular R-module V', we can choose an automorphism $\sigma : R \to R$, such that the induced functor $\sigma_* : R\text{-Mod} \to R\text{-Mod}$ by σ is an equivalence and satisfies $\sigma_*(V') \simeq V$. Hence, instead of V, we may use V' to proceed the above procedure, but we will then get the same recollements, up to derived equivalence of each term.

(2) Let $K_0(R)$ be the Grothendieck group of R, that is, the abelian group generated by isomorphism classes $[P]$ of finitely generated projective R-modules P subject to the relation $[P]+[Q]=[P\oplus Q]$, where P and Q are finitely generated projective R-modules. One can check that $K_0\big(k((x))\big) \simeq \mathbb{Z}$ and $K_0(C) \simeq \mathbb{Z}\oplus\mathbb{Z}$. The above example shows that, even if $\mathscr{D}(A_2)$ is a recollement of $\mathscr{D}(A_1)$ and $\mathscr{D}(A_3)$, where A_i are rings for $i=1,2,3$, we cannot get $K_0(A_2) \simeq K_0(A_1)\oplus K_0(A_3)$ in general.

For a general consideration of stratifications of the endomorphism algebras of tilting modules over tame hereditary algebras, we shall discuss it in a forthcoming paper.

Finally, we remark that Theorem 1.1(2) can be extended to n-tilting modules. However, since there is not defined any reasonable torsion theory in module categories for general n-tilting modules, we are not able to extend Theorem 1.1(1) to n-tilting modules. So we mention the following open question.

Question 1. Is Theorem 1.1(1) true for n-good tilting modules ?

Another question related to our examples is:

Question 2. Is there a ring R such that $\mathscr{D}(R)$ has two stratifications by derived module categories of rings, one of which is of finite length, and the other is of infinite length ?

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October 10, 2010