Transfer of stable equivalences of Morita type

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Abstract

Let A and B be finite-dimensional k-algebras over a field k such that $A/\mathrm{rad}(A)$ and $B/\mathrm{rad}(B)$ are separable. In this note, we consider how to transfer a stable equivalence of Morita type between A and B to that between eAe and fBf, where e and f are idempotent elements in A and in B, respectively. In particular, if the Auslander algebras of two representation-finite algebras A and B are stably equivalent of Morita type, then A and B themselves are stably equivalent of Morita type. Thus, combining a result with Liu and Xi, we see that two representation-finite algebras A and B over a perfect field are stably equivalent of Morita type if and only if their Auslander algebras are stably equivalent of Morita type. Moreover, since stable equivalence of Morita type preserves n-cluster tilting modules, we extend this result to n-representation-finite algebras and n-Auslander algebras studied by Iyama.

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1 Introduction

Stable equivalence of Morita type introduced by Broué in [2] is one of the fundamental equivalence relations for algebras and groups. It is of considerable interest in the modular representation theory of finite groups, or more generally, of finite-dimensional self-injective algebras. The notion is intimately related to the celebrated conjecture of Broué, which says that certain blocks of group algebras with abelian defect groups should be derived-equivalent (see [2] and [15] for precise formulation); the connection can be seen from Rickard's result that a derived equivalence between self-injective algebras induces a stable equivalence of Morita type [16]. Thus if the conjecture of Broué is true for a class of blocks of group algebras, we get also stable equivalences of Morita type. In fact, the Green correspondence between block algebras provides frequently a stable equivalence of Morita type (see [8]). Recently, it is shown that stable equivalence of Morita type is also of particular interest for general finite-dimensional algebras, for example, it preserves representation dimension [18], representation type [9], Hochschild homological and cohomological groups [12, 17], and the absolute value of Cartan determinant [17]. As is known, stable equivalence of Morita type occurs frequently not only in the block theory of finite groups [10], but also in the representation theory of general finite-dimensional algebras. A plenty of such examples are constructed in [11, 12, 13].

Moreover, it is shown in [12] that, for two finite-dimensional k-algebras A and B over a field k of finite representation-type, if A and B are stably equivalent of Morita type, then their Auslander algebras are also stably equivalent of Morita type. A natural question is whether the converse of this statement is true.

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In this note, we shall consider the general question of how to transfer a stable equivalence of Morita type between two algebras A and B over a field to that between eAe and fBf, where e and f are idempotent elements in A and B, respectively. Thus one may use the Schur functors $\mathrm{Hom}_A(Ae,-)$ and $\mathrm{Hom}_B(Bf,-)$ to transfer properties of the module categories over the given algebras A and B to that over their corner algebras eAe and fBf. Such a situation was successfully used by Green thirty years ago to understand the irreducible representations of symmetric groups (see [14, Chapter 4], for example), and also recently by Diracca and König to investigate the split pairs in [3].

We say that two bimodules ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of Morita type between A and B if M and N are projective as one-sided modules, and there are a projective A-A-bimodule P and a projective B-B-bimodule Q such that $M \otimes_B N \simeq A \oplus P$ and $N \otimes_A M \simeq B \oplus Q$ as bimodules. With these notations in mind, our main result reads as follows:

Theorem 1.1 *Suppose that A and B are finite-dimensional k-algebras over a field k such that both $A/\mathrm{rad}(A)$ and $B/\mathrm{rad}(B)$ are separable. Let ${}_AM_B$ and ${}_BN_A$ be two bimodules defining a stable equivalence of Morita type between A and B. If $e^2 = e \in A$ such that $Pe \in \mathrm{add}(Ae)$, and if f is a sum of the pairwise orthogonal primitive idempotent elements $f_j \in B$ in the decomposition $Ne \simeq \oplus_{j=1}^{m}(Bf_j)^{r_j}$, then the bimodules eMf and fNe define a stable equivalence of Morita type between eAe and fBf.*

From this result, we have the following corollary which supplies a positive answer to our previous question on Auslander algebras. For the unexplained notion in the corollary, we refer the reader to Section 3 below.

Corollary 1.2 *Suppose that A and B are finite-dimensional k-algebras over a perfect field k. Assume that A and B are n-representation-finite. If the n-Auslander algebras of A and B are stably equivalent of Morita type, then A and B themselves are stably equivalent of Morita type.*

As is known, Auslander algebras are of relatively simple homological property (in fact, they are of global dimension at most 2), thus we may investigate representation-finite algebras of higher homological dimensions via Auslander algebras through stable equivalences of Morita type. For instance, given two representation-finite algebras A and B, if their Auslander algebras are stably equivalent of Morita type, then it follows from Theorem 1.1 that A and B have the same global, finitistic and dominant dimension, and that their Hochschild cohomology and homology groups are isomorphic.

The proof of Theorem 1.1 is given in the next section.

2 Proof of the main result

Throughout this note, k denotes a fixed field. Given a finite-dimensional k-algebra A, we denote by A-mod the category of all finitely generated left A-modules. If $M \in A$-mod, we denote by $\mathrm{add}(M)$ the full subcategory of A-mod consisting of all modules X which are direct summands of finite sums of copies of M. By an algebra we mean a finite-dimensional k-algebra, and by a module we mean a left module. The global and dominant dimensions of an algebra A are denoted by $\mathrm{gl.dim}(A)$ and $\mathrm{dom.dim}(A)$, respectively. The composition of two homomorphisms $f: X \to Y$ and $g: Y \to Z$ is denoted by $fg: X \to Z$, and the usual k-duality is denoted by $D := \mathrm{Hom}_k(-,k)$. Let us recall the definition of a stable equivalence of Morita type.

Definition 2.1 [2] *Suppose that A and B are two (arbitrary) k-algebras. We say that A and B are stably equivalent of Morita type if there exist an A-B-bimodule ${}_AM_B$ and a B-A-bimodule ${}_BN_A$ such that*

(1) *M and N are projective as one-sided modules, and*

(2) *$M\otimes N \simeq A\oplus P$ as A-A-bimodules for some projective A-A-bimodule P,and $N\otimes M \simeq B\oplus Q$ as B-B-bimodules for some projective B-B-bimodule Q.*

Note that if A and B are stably equivalent of Morita type, then their opposite algebras A^{op} and B^{op} are also stably equivalent of Morita type.

Let A be a representation-finite algebra. An A-module X is called an additive generator for A-mod if $\mathrm{add}(X) = A$-mod, that is, every indecomposable A-module is isomorphic to a direct summand of X. Let X be an additive generator for A-mod. The endomorphism algebra $\Lambda = \mathrm{End}_A(X)$ of X is called the Auslander algebra of A. (This is unique up to Morita equivalence.) Auslander algebras can be described by two homological properties: An algebra A is an Auslander algebra if (1) $\mathrm{gl.dim}(A) \leqslant 2$; and (2) if $0 \to A \to I_0 \to I_1 \to I_2 \to 0$ is a minimal injective resolution of A, then I_0 and I_1 are projective. If an algebra A satisfies the condition (2), we say that A is of dominant dimension at least 2, denoted by $\mathrm{dom.dim}(A) \geq 2$.

An A-module $X \in A$-mod is called a generator for A-mod if $\mathrm{add}({}_AA) \subseteq \mathrm{add}(X)$; a cogenerator for A-mod if $\mathrm{add}(D(A_A)) \subseteq \mathrm{add}(X)$, and a generator-cogenerator if it is both a generator and a cogenerator for A-mod. Clearly, for a representation-finite algebra A, an additive generator for A-mod is a generator-cogenerator for A-mod.

In the following, we shall introduce some notations. Assume that A is a k-algebra.

Let T be an arbitrary A-module, and suppose B is the endomorphism algebra of T. We consider the following subcategories related to T.

$Gen({}_AT) = \{X \in A\text{-mod} \mid$ there is a surjective homomorphism from T^m to X with $m \geqslant 1\}$.

$Pre({}_AT) = \{X \in A\text{-mod} \mid$ there is an exact sequence $T_1 \to T_0 \to X$ with all $T_i \in \mathrm{add}({}_AT)$ $\}$.

$App({}_AT) = \{X \in A\text{-mod} \mid$ there is a homomorphism $g: T_0 \to X$ with $T_0 \in \mathrm{add}({}_AT)$ such that $\mathrm{Ker}(g) \in Gen({}_AT)$ and g is a right $\mathrm{add}({}_AT)$-approximation of $X\}$.

The following lemma is known, for a proof, we refer to [19, Lemma 2.1].

Lemma 2.2 *Let X be an arbitrary A-module. Recall that $B = \mathrm{End}_A(T)$ and ${}_AT_B$ is a natural bimodule. Then:*

(1) *Let Y be a right B-module. The natural homomorphism δ: $Y \otimes_B \mathrm{Hom}_A(T,X) \to \mathrm{Hom}_B(\mathrm{Hom}_A(X,T),Y)$, given by $y \otimes f \mapsto \delta_{y\otimes f}$ with $\delta_{y\otimes f}(g) = y(fg)$ for $y \in Y, f \in \mathrm{Hom}_A(T,X), g \in \mathrm{Hom}_A(X,T)$, is an isomorphism if $X \in \mathrm{add}({}_AT)$.*

(2) *If $X' \in \mathrm{add}({}_AT)$, or $X \in \mathrm{add}({}_AT)$, then the composition map $\mu : \mathrm{Hom}_A(X',T) \otimes_B \mathrm{Hom}_A(T,X) \to \mathrm{Hom}_A(X',X)$ given by $f \otimes_B g \mapsto fg$ is bijective.*

(3) *Let C be a k-algebra, and suppose ${}_AX_C$ is an A-C-bimodule. If ${}_AX \in Gen({}_AT)$, then the evaluation map $e_X : T \otimes_B \mathrm{Hom}_A(T,X) \to X$ is surjective as A-C-bimodules. If $X \in App({}_AT)$, then e_X is an isomorphism as A-C-bimodules. Conversely, if e_X is bijective as A-modules, then $X \in App({}_AT)$.*

The next lemma is taken from [17, Lemma 2.1].

Lemma 2.3 [*17*] (1) *Let A, B, C and E be k-algebras, and let ${}_AX_B$ and ${}_BY_E$ be bimodules with X_B projective. Put $X^* = \mathrm{Hom}_B(X,B)$. Then the natural homomorphism $\phi : {}_AX \otimes_B Y_E \to \mathrm{Hom}_B({}_BX^*_A, {}_BY_E)$, defined by $f \mapsto (xf)y$ for $x \in X, y \in Y$ and $f \in X^*$, is an isomorphism of A-E-bimodules, where the image of x under f is denoted by xf.*

(2) *In the situation $({}_EP_A, {}_CX_B, {}_AU_B)$, if P_A is projective, or if X_B is projective, then ${}_EP \otimes_A \mathrm{Hom}_B({}_CX_B, {}_AU_B) \simeq \mathrm{Hom}_B({}_CX_B, {}_EP \otimes_A U_B)$ as E-C-bimodules. Dually, in the situation $({}_AP_E, {}_BX_C, {}_BU_A)$, if ${}_AP$ is projective, or if ${}_BX$ is projective, then $\mathrm{Hom}_B({}_BX_C, {}_BU_A) \otimes_A P_E \simeq \mathrm{Hom}_B({}_BX_C, {}_BU \otimes_A P_E)$ as C-E-bimodules.*

The following is a well-known result due to Auslander (for example, see [1, Proposition 5.6, p.214]).

Lemma 2.4 *Let Λ be an Artin algebra such that $\mathrm{gl.dim}(\Lambda) \leqslant 2 \leqslant \mathrm{dom.dim}(\Lambda)$. Let U be a Λ-module such that $\mathrm{add}(U)$ is the full subcategory of Λ-mod consisting of all projective-injective Λ-modules. Then*

(1) *$A := \mathrm{End}_\Lambda(U)$ is representation-finite.*

(2) *Λ is Morita equivalent to $\mathrm{End}_A(X)$, where X is an additive generator for A-mod.*

For our proof of Theorem 1.1, we also need the following lemma in [4, Theorem 2.7, Corollary 3.1, Lemma 3.2], see also [17, Lemma 3.3].

Lemma 2.5 [*4*] *Suppose that A and B are finite-dimensional k-algebras over a field k such that A and B have no separable direct summands and that $A/\mathrm{rad}(A)$ and $B/\mathrm{rad}(B)$ are separable. Assume that ${}_AM_B$ and ${}_BN_A$ are indecomposable bimodules that define a stable equivalence of Morita type between A and B. Then:*

(1) There are isomorphisms of bimodule: $N \cong \mathrm{Hom}_A(M, A) \cong \mathrm{Hom}_B(M, B)$ and $M \cong \mathrm{Hom}_A(N, A) \cong \mathrm{Hom}_B(N, B)$.

(2) Both $(N \otimes_A -, M \otimes_B -)$ and $(M \otimes_B -, N \otimes_A -)$ are adjoint pairs of functors.

(3) If ${}_AI$ is injective, then so is $N \otimes_A I$.

It follows from Lemma 2.5 that the following result is true. Note that the last statement in Lemma 2.6 below follows from [17, Lemma 4.5].

Lemma 2.6 *Suppose that A and B are finite-dimensional k-algebras over a field k such that $A/\mathrm{rad}(A)$ and $B/\mathrm{rad}(B)$ are separable. Assume that $\{e_1, \cdots, e_n\}$ and $\{f_1, \cdots, f_m\}$ are complete sets of pairwise orthogonal primitive idempotents in A and B, respectively. Let e be the sum of all those e_i for which Ae_i is projective-injective, and let f be the sum of all those f_j for which Bf_j is projective-injective. If M and N are indecomposable bimodules that define a stable equivalence of Morita type between A and B, then $Ne \simeq N \otimes_A Ae \in \mathrm{add}(Bf), Mf \simeq M \otimes_B Bf \in \mathrm{add}(Ae)$, and $Pe \in \mathrm{add}(Ae)$.*

Proof of Theorem 1.1:

Suppose that A and B are two algebras over a field k such that $A/\mathrm{rad}(A)$ and $B/\mathrm{rad}(B)$ are separable, and suppose that ${}_AM_B$ and ${}_BN_A$ define a stable equivalence of Morita type between A and B. We may assume that both A and B have no separable summands since the direct sum of A with a separable k-algebra is always stably equivalent of Morita type to A itself. Furthermore, by [11, Lemma 2.2], we may assume that M and N are indecomposable as bimodules. Then $(M \otimes_B -, N \otimes_A -)$ and $(N \otimes_A -, M \otimes_B -)$ are adjoint pairs by Lemma 2.5.

To prove Theorem 1.1, we shall show that the bimodules eMf and fNe satisfy the conditions of a stable equivalence of Morita type between eAe and fBf.

(1) fNe is projective as both an fBf-module and a right eAe-module.

In fact, we have $fNe \simeq fB \otimes_B Ne \simeq \mathrm{Hom}_B(Bf, B) \otimes_B Ne \simeq \mathrm{Hom}_B(Bf, {}_BNe)$ by Lemma 2.3. Since $Ne \in \mathrm{add}(Bf)$ by the definition of f, we see that $\mathrm{Hom}_B(Bf, Ne)$ is projective as an fBf-module, that is, fNe is projective as an fBf-module. To see that fNe is a projective right eAe-module, we notice that $\mathrm{add}(Mf) = \mathrm{add}(M \otimes_B Bf) = \mathrm{add}(M \otimes_B Ne) = \mathrm{add}((M \otimes_B N)e) = \mathrm{add}(Ae \oplus Pe) = \mathrm{add}(Ae)$, here we use the assumption $Pe \in \mathrm{add}(Ae)$. Since $(M \otimes_B -, N \otimes_A -)$ is an adjoint pair, it follows from $\mathrm{Hom}_B(Bf, {}_BN \otimes_A Ae) \simeq \mathrm{Hom}_A(M \otimes_B Bf, Ae) \simeq \mathrm{Hom}_A(Mf, Ae)$ that fNe is projective as a right eAe-module since $Mf \in \mathrm{add}(Ae)$. Thus (1) is proved.

(2) eMf is projective as both an eAe-module and a right fAf-module. The proof of (2) is similar to that of (1), we omit it here.

(3) $fNe \otimes eMf \simeq fBf \oplus fQf$ as bimodules.

Indeed, by the associativity of tensor products, we have the following isomorphisms of fBf-fBf-bimodules:

$$\begin{aligned} fNe \otimes_{eAe} eMf &\simeq fN \otimes_A Ae \otimes_{eAe} eA \otimes_A Mf \\ &\simeq fN \otimes_A Ae \otimes_{eAe} \mathrm{Hom}(Ae, A) \otimes_A Mf \\ &\simeq fN \otimes_A Ae \otimes_{eAe} \mathrm{Hom}(Ae, {}_AMf) \quad (\text{ by Lemma 2.3 }) \\ &\simeq fN \otimes_A Mf \quad (\text{by Lemma 2.2 }). \end{aligned}$$

Since M and N define the stable equivalence of Morita type between A and B, we have $N \otimes_A M \simeq B \oplus Q$ as B-B-bimodules. This implies that $fNe \otimes eMf \simeq fN \otimes Mf \simeq fB \otimes_B N \otimes_A M \otimes_B Bf \simeq fB \otimes_B (B \oplus Q) \otimes_B Bf \simeq fBf \oplus fB \otimes_B Q \otimes_B Bf \simeq fBf \oplus fQf$.

(4) The bimodule fQf in (3) is projective.

In fact, since Q is a projective B-B-bimodule and since $B/\mathrm{rad}(B)$ is separable, the bimodule Q is isomorphic to a direct sum of modules of the form $Q_1 \otimes_k Q_2$, where Q_1 is a projective left

B-module and Q_2 is a projective right B-module. Since $fNe \otimes_{eAe} eMf$ and fBf are projective as left fBf-modules, we infer that $fQ_1 \otimes_k Q_2 f$ is a projective fBf-module. It follows that fQ_1 is a projective fBf-module. Similarly, $Q_2 f$ is a projective right fBf-module. Thus fQf, which is isomorphic to a direct sum of modules of the form $fQ_1 \otimes_k Q_2 f$ with fQ_1 a projective fBf-module and $Q_2 f$ a projective right fBf-module, is projective as an fBf-fBf-bimodule.

(5) Similarly, we can show that $eMf \otimes_{fBf} fNe \simeq eAe \oplus ePe$ and that ePe is a projective eAe-eAe-bimodule.

Indeed, we have

$$\begin{aligned} eMf \otimes_{fBf} fNe &\simeq eM \otimes_B Bf \otimes_{fBf} fB \otimes_B Ne \\ &\simeq eM \otimes_B Bf \otimes_{fBf} \mathrm{Hom}(Bf, B) \otimes_B Ne \\ &\simeq eM \otimes_B Bf \otimes_{fBf} \mathrm{Hom}(Bf, {}_BNe) \quad (\text{ by Lemma 2.3 }) \\ &\simeq eM \otimes_B Ne \quad (\text{by Lemma 2.2 }). \end{aligned}$$

Since M and N define the stable equivalence of Morita type between A and B, we have $M \otimes_B N \simeq A \oplus P$ as A-A-bimodules. This implies that $eMf \otimes_{fBf} fNe \simeq eM \otimes_A Ne \simeq eAe \oplus ePe$. Now, we show that the bimodule ePe is projective.

In fact, since P is a projective A-A-bimodule and since $A/\mathrm{rad}(A)$ is separable, the bimodule P is isomorphic to a direct sum of modules of the form $P_1 \otimes_k P_2$, where P_1 is a projective left A-module and P_2 is a projective right A-module. Since $eMf \otimes_{fBf} fNe$ and eAe are projective as left eAe-modules, we infer that $eP_1 \otimes_k P_2 e$ is a projective eAe-module. It follows that eP_1 is a projective eAe-module. Similarly, $P_2 e$ is a projective right eAe-module. Thus ePe, which is isomorphic to a direct sum of modules of the form $eP_1 \otimes_k P_2 e$ with eP_1 a projective eAe-module and $P_2 e$ a projective right eAe-module, is projective as an eAe-eAe-bimodule.

Thus, by definition, the bimodules eMf and fNe define a stable equivalence of Morita type between eAe and fBf. This finishes the proof of Theorem 1.1. $\square$

Remarks. (1) In Theorem 1.1, if e is an idempotent element in A such that every indecomposable projective-injective A-module is isomorphic to a summand of Ae, then $Pe \in \mathrm{add}(Ae)$. This follows immediately from the proof of [17, Lemma 4.5]. In fact, under the assumption of Theorem 1.1, we infer that $\nu_A^i({}_AP)$ is projective-injective for all $i \geqslant 0$, where ν_A is the Nakayama functor $D\mathrm{Hom}_A(-, {}_AA)$. Hence, if e is an idempotent element in A such that every indecomposable projective-injective A-module X with $\nu_A^j X$ projective-injective for all $j \geqslant 0$ belongs to $\mathrm{add}(Ae)$, then ${}_AP \in \mathrm{add}(Ae)$.

(2) As was pointed out in [4, Section 4], if e is an idempotent in A and if f is an idempotent in B such that $\mathrm{add}(Ae)$ and $\mathrm{add}(Bf)$ are invariant under Nakayama functor, then eAe and fBf are self-injective, and any stable equivalence of Morita type between A and B induces a stable equivalence of Morita type between eAe and fBf. In general, however, our algebras eAe and fBf in Theorem 1.1 may not be self-injective.

As a corollary of Theorem 1.1, we get the following result.

Corollary 2.7 *Suppose A and B be are finite-dimensional k-algebras of finite representation type, and let Λ and Γ be the corresponding Auslander algebras of A and B, respectively. Assume that $\Lambda/\mathrm{rad}(\Lambda)$ and $\Gamma/\mathrm{rad}(\Gamma)$ are separable. Then Λ and Γ are stably equivalent of Morita type if and only if A and B are stably equivalent of Morita type.*

Proof. We know that if X is an additive generator for A-mod with $\Lambda := \mathrm{End}_A(X)$, then $U := \mathrm{Hom}_A(X, D(A_A))$ is a projective-injective Λ-module with $\mathrm{End}_\Lambda(U) \simeq A^{op}$; and every indecomposable projective-injective Λ-module is isomorphic to a direct summand of U. Note that this U satisfies the conditions in Lemma 2.4. If we choose e to be the sum of all idempotents corresponding to the indecomposable injective A-modules, then Lemma 2.6 says that the conditions in Theorem 1.1 on the idempotent $e \in \Lambda$ are satisfied. Note that e defines an idempotent element f in Γ (see Theorem 1.1), and that $\mathrm{add}(\Gamma f)$ contains all projective-injective Γ-modules. With these in mind, the corollary follows from Theorem 1.1 and [13, Theorem 1.1]. $\square$

For an algebra A, we denote by $[A]$ the class of all those algebras B for which there is a stable equivalence of Morita type between B and A. From the above corollary, we have the following result.

Corollary 2.8 *Suppose that k is a perfect field. Let $\mathcal{F}$ be the set of equivalence classes $[A]$ of representation-finite k-algebras A with respect to stable equivalence of Morita type, and let $\mathcal{A}$ be the set of equivalence classes $[\Lambda]$ of Auslander k-algebras Λ with respect to stable equivalence of Morita type. Then there is an one-to-one correspondence between $\mathcal{F}$ and $\mathcal{A}$.*

Another consequence of Theorem 1.1 is the following corollary.

Corollary 2.9 *Suppose that A and B are two k-algebras. Let ${}_AX$ be a generator-cogenerator for A-mod such that $\mathrm{End}_A(X)/\mathrm{rad}(\mathrm{End}_A(X))$ is separable, and let ${}_BY$ be a generator-cogenerator for B-mod such that $\mathrm{End}_B(Y)/\mathrm{rad}(\mathrm{End}_B(Y))$ is separable. If $\mathrm{End}_A(X)$ and $\mathrm{End}_B(Y)$ are stably equivalent of Morita type, then so are A and B. In this case, A and B have the same global, dominant, finitistic and representation dimensions.*

Finally, we remark that if we consider derived equivalence instead of stable equivalence of Morita type in Corollary 2.7, then we know from [5] that a derived equivalence between representation-finite, self-injective algebras A and B implies a derived equivalence between their Auslander algebras. But the converse of this statement is still open. For further information on constructing derived equivalences, we refer the reader to a current paper [6].

3 Higher Auslander algebras

In the following, we point out that Corollary 2.7 holds true for n-representation-finite algebras and n-Auslander algebras studied in [7].

Now we recall some definitions and facts from [7].

Let A be a finite-dimensional k-algebra, and let $n \geqslant 1$ be a natural number. An A-module T is called an n-cluster tilting module if $\mathrm{add}(T) = \{X \in A\text{-mod} \mid \mathrm{Ext}^i_A(X,T) = 0, 1 \leqslant i < n\}$ $= \{X \in A\text{-mod} \mid \mathrm{Ext}^i_A(T,X) = 0, 1 \leqslant i < n\}$. The k-algebra A is called 1-representation-finite if there is an 1-cluster tilting A-module T. This is equivalent to saying that A is representation-finite. For $n \geqslant 2$, the k-algebra A is called n-representation-finite if $\mathrm{gl.dim}(A) \leqslant n$ and there is an n-cluster tilting A-module T.

A k-algebra Λ is called an n-Auslander algebra if there is an n-representation-finite k-algebra A with an n-cluster tilting A-module T such that Λ is Morita equivalent to $\mathrm{End}_A(T)$. Note that, for an n-representation-finite algebra A, its n-Auslander algebra is unique up to Morita equivalence. Homologically, a k-algebra Λ is an n-Auslander algebra if $\mathrm{gl.dim}(\Lambda) \leqslant n+1 \leqslant \mathrm{dom.dim}(\Lambda)$.

Clearly, each n-cluster tilting A-module T is a generator and co-generator for A-mod. Thus the indecomposable projective-injective $\mathrm{End}_A(T)$-modules are of the form $\mathrm{Hom}_A(T,I)$, where I is an indecomposable injective A-module.

Let A be an n-representation-finite k-algebra with T an n-cluster tilting A-module. Furthermore, we assume that A has no separable direct summands and that $A/\mathrm{rad}(A)$ is separable. If A is stably equivalent of Morita type to an algebra B such that B has no separable direct summand and $B/\mathrm{rad}(B)$ is separable, then B is n-representation-finite. In fact, if two indecomposable bimodules ${}_AM_B$ and ${}_BN_A$ define the stable equivalence of Morita type between A and B, then $N \otimes_A T$ is an n-cluster tilting B-module: Since this stable equivalence of Morita type is of adjoint type by Lemma 2.5, we see that $\mathrm{Ext}^i_A(N \otimes_B T, N \otimes_A T) \simeq \mathrm{Ext}^i_A(T, M \otimes_B N \otimes_A T) \simeq \mathrm{Ext}^i_A(T, T \oplus P \otimes_A T) = \mathrm{Ext}^i_A(T,T) \oplus \mathrm{Ext}^i_A(T, P \otimes_A T) = 0$ for $1 \leqslant i < n$ since $P \otimes_A T$ is a projective-injective A-module. This shows that $\mathrm{add}(N \otimes_A T)$ is contained in both $\{X \in B\text{-mod} \mid \mathrm{Ext}^i_B(X, N \otimes_A T) = 0, 1 \leqslant i < n\}$ and $\{X \in B\text{-mod}$ $\mid \mathrm{Ext}^i_B(N \otimes_A T, X) = 0, 1 \leqslant i < n\}$. Now, let $Y \in B$-mod such that $\mathrm{Ext}^j_B(N \otimes_A T, Y) = 0$ for $1 \leqslant j < n$. Then $0 = \mathrm{Ext}^j_B(N \otimes_A T, Y) = \mathrm{Ext}^j_A(T, M \otimes_B Y)$ for $1 \leqslant j < n$, and therefore $M \otimes_B Y \in \mathrm{add}(T)$. This implies that $Y \in \mathrm{add}(N \otimes_A T)$. Similarly, we show that

$\mathrm{add}(N \otimes_A T) = \{Y \in B\text{-mod} \mid \mathrm{Ext}^i_B(Y, N \otimes_A T) = 0, 1 \leqslant i < n\}$. Hence $N \otimes_A T$ is an n-cluster tilting B-module.

Thus, n-representation-finite k-algebras A with $A/\mathrm{rad}(A)$ separable are closed under stable equivalences of Morita type.

As in the case of Corollary 2.7, the following is a consequence of Theorem 1.1.

Theorem 3.1 *Suppose that A and B are finite-dimensional k-algebras such that both are n-representation-finite. Let Λ and Γ be the corresponding n-Auslander algebras of A and B, respectively. Assume that both $\Lambda/\mathrm{rad}(\Lambda)$ and $\Gamma/\mathrm{rad}(\Gamma)$ are separable. Then Λ and Γ are stably equivalent of Morita type if and only if A and B are stably equivalent of Morita type.*

Proof. For $n = 1$, we have done by Corollary 2.7. Let $n \geqslant 2$. Suppose ${}_AT$ is an n-cluster tilting A-module such that $\mathrm{End}_A(T) = \Lambda$, and suppose ${}_BS$ is an n-cluster tilting B-module such that $\mathrm{End}_B(S) = \Gamma$. If ${}_AM_B$ and ${}_BN_A$ are two indecomposable bimodules defining a stable equivalence of Morita type between A and B, then, by the above discussion, we know that Γ is Morita equivalent to $\mathrm{End}_B(N \otimes_A T)$. Now, we use [13, Theorem 1.1, or Theorem 1.3] which states that if R is an A-module with $\mathrm{add}({}_AA) \subseteq \mathrm{add}(R)$, then $\mathrm{End}_A(R)$ and $\mathrm{End}_B(N \otimes_A R)$ are stably equivalent of Morita type.

Conversely, suppose that two bimodules ${}_\Lambda X_\Gamma$ and ${}_\Gamma Y_\Lambda$ define a stable equivalence of Morita type between Λ and Γ. Note that $\mathrm{add}(T)$ contains both $\mathrm{add}({}_AA)$ and $\mathrm{add}(D(A_A))$. Let e be an idempotent in Λ such that $\mathrm{add}(\Lambda e)$ is just the category of projective-injective Λ-modules. Then $e\Lambda e$ is Morita equivalent to A^{op}. As in Theorem 1.1, we have an idempotent f in Γ such that $f\Gamma f$ is Morita equivalent to B^{op}. Thus a stable equivalence of Morita type between Λ and Γ implies a stable equivalence of Morita type between A^{op} and B^{op} by Theorem 1.1, and therefore a stable equivalence of Morita type between A and B. □

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