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# COUPLING FOR JUMP PROCESSES*)

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Coupling is probably the most important technique in the subject of interacting particle systems. It is also very useful for other stochastic processes. For discrete time Markov processes, the coupling theory was studied expansively by Dobrushin[8], Griffeath[9], Wasershtein[10] and others (see the conferences in [9]). For continuous time Markov processes, it becomes more complicated. This paper is devoted to discussing the coupling theory for jump Markov processes.

In Section 1 we introduce three basic conditions for a coupling. Then, in Sections 2–4, we discuss the conditions respectively. Finally, Section 5 presents some basic couplings which should be the most useful ones in the subject we study. The main results of the paper are given by Theorems (13), (16), (21), (24), (26), (30), (36) and (37).

In the subsequent paper[6], which is mainly based on this paper, we will give a construction for large classes of Markov processes on product spaces which need not be compact.

## §1 Basic Conditions for Coupling

Let $(E_i, \mathscr{E}_i)$ be an arbitrary measurable space and $(X_t^{(i)})_{t\geqslant 0}$ be a Markov process, $i = 1, 2$. A coupling is simply to construct a Markov process $\big(\widetilde{X}_t\big)_{t\geqslant 0}$, of the two processes $(X_t^{(i)})_{t\geqslant 0}$, $i = 1, 2$ on a common probability space with the product state space $(E, \mathscr{E}) = (E_1 \times E_2, \mathscr{E}_1 \times \mathscr{E}_2)$, which has the property:

(1). marginality.

$$\widetilde{P}^{(x_1,x_2)}\big[\widetilde{X}_t \in A_1 \times E_2\big] = P^{x_1}[X_t^{(1)} \in A_1]$$
$$\widetilde{P}^{(x_1,x_2)}\big[\widetilde{X}_t \in E_1 \times A_2\big] = P^{x_2}[X_t^{(2)} \in A_2]$$
$$x_i \in E_i,\ A_i \in \mathscr{E}_i,\ i = 1,\ 2,\ t \geqslant 0.$$

By using the transition probability function, one can rewrite (1) as:

$$\begin{aligned} &\widetilde{P}(t, (x_1, x_2), A_1 \times E_2) = P_1(t, x_1, A_1)\\ (2)\qquad &\widetilde{P}(t, (x_1, x_2), E_1 \times A_2) = P_2(t, x_2, A_2)\\ &x_i \in E_i,\ A_i \in \mathscr{E}_i,\ i = 1,\ 2,\ t \geqslant 0.\end{aligned}$$

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Throughout the paper, we assume each $(E_i, \mathscr{E}_i)$ is separable. That is, $\{x\} \in \mathscr{E}_i$ for each $x \in E_i$. Also, we restrict ourselves on jump process $P_i(t, x_i, \cdot)$ with totally stable and conservative $q$-pair $(q_i(x_i), q_i(x_i, \cdot))$, which means that

$$
\begin{aligned}
&q_i(x_i) = q_i(x_i, E_i) < \infty,\\
&\frac{\mathrm{d}}{\mathrm{d}t} P_i(t, x_i, B_i)\Big|_{t=0} = q_i(x_i, B_i), -q_i(x_i)\delta(x_i, B_i), \qquad x_i \in E_i,\ B_i \in \mathscr{E}_i,\ i = 1,\ 2
\end{aligned}
$$

where $\delta(x, B) = I_B(x) = 1$, if $x \in B$; $= 0$, if $x \notin B$. We call a $q$-pair regular if it determines a unique jump process $P(t, x, \cdot)$.[1] Thus, a coupling for jump processes requires reasonably the following property:

(3). regularity:. the $q$-pair $(\tilde{q}(\tilde{x}), \tilde{q}(\tilde{x}, \cdot))$ is regular.

Sometimes, a coupling is used to compare an order relation of two copies of the same jump process with different starting points. In this case, $E_1 = E_2 = E$, $\mathscr{E}_1 = \mathscr{E}_2 = \mathscr{E}$ and $E$ is endowed with a semi-order "$\leqslant$". One wants to know whether the process $(X_t)_{t \geqslant 0}$ has

(4). order-preservation.

$$x_1 \leqslant x_2 \Longrightarrow \widetilde{P}^{(x_1, x_2)}\big[X_t^{(1)} \leqslant X_t^{(2)}\big] = 1, \qquad t \geqslant 0,\ (x_1, x_2) \in \widetilde{E}.$$

A function $f$ on $E$ is said monotone, if

$$x_1 \leqslant x_2 \Longrightarrow f(x_1) \leqslant f(x_2), \qquad (x_1, x_2) \in \widetilde{E}. \tag{5}$$

Now, if (2)—(4) are satisfied, then for each nonnegative monotone function $f$, we have

$$x_1 \leqslant x_2 \Longrightarrow P_t^{(1)} f(x_1) \leqslant P_t^{(2)} f(x_2), \qquad (x_1, x_2) \in \widetilde{E},\ t \geqslant 0. \tag{6}$$

where

$$P_t^{(i)} f(x) = \int P_i(t, x, \mathrm{d}y) f(y), \qquad i = 1,\ 2.$$

The conditions (2), (3) and (4) are usually needed for a coupling. However, these conditions are indeed not explicit, they depend on the unknown process $\widetilde{P}(t, \tilde{x}, \cdot)$. The explicit condition should be described by the given $q$-pairs $(q_i(x_i), q_i(x_i, \cdot))$ $(i = 1, 2)$ only, and this point is just what we are going to do in the next three sections.

## §2. Marginality

Let $\widetilde{P}(t, \tilde{x}, \tilde{A})$ be a jump process with $q$-pair $\big(\tilde{q}(\tilde{x}), \tilde{q}(\tilde{x}, \cdot)\big)$, then by the conservative assumption, one can see that

$$\lim_{t \downarrow 0} \frac{\widetilde{P}(t, \tilde{x}, \tilde{A}) - \delta(\tilde{x}, \tilde{A})}{t} = \tilde{q}(\tilde{x}, \tilde{A}) - \tilde{q}(\tilde{x}) I_{\tilde{A}}(\tilde{x}), \qquad \tilde{x} \in \widetilde{E},\ \tilde{A} \in \widetilde{\mathscr{E}}.$$

From condition (2), it follows that

$$
\begin{aligned}
q_1(x_1, A_1) - q_1(x_1) I_{A_1}(x_1) &= \lim_{t \downarrow 0} \frac{P_1(t, x_1, A_1) - \delta(x_1, A_1)}{t}\\
&= \lim_{t \downarrow 0} \frac{\widetilde{P}(t, (x_1, x_2), A_1 \times E_2) - \delta(x_1, A_1)}{t}\\
&= \tilde{q}(x_1, x_2, A_1 \times E_2) - \tilde{q}(x_1, x_2) I_{A_1}(x_1), \qquad (x_1, x_2) \in \widetilde{E},\ A_1 \in \mathscr{E}_1.
\end{aligned}
$$

[1]It is also called a $q$-process.

Hence, by the monotone class theorem, we get

$$\begin{aligned}&\int q_1(x_1, \mathrm{d}y_1)f(y_1) - q_1(x_1)f(x_1)\\ &= \int \tilde{q}(x_1, x_2; \mathrm{d}y_1, \mathrm{d}y_2)f(y_1) - \tilde{q}(x_1, x_2)f(x_1), \qquad (x_1, x_2) \in \widetilde{E},\ f \in {}_b\mathscr{E}_1,\end{aligned}$$

where ${}_b\mathscr{E}_1$ is the set of all bounded $\mathscr{E}_1$-measurable functions. Regarding ${}_b\mathscr{E}_1$ as a bivariable function, and using the following operators

$$\begin{aligned}\Omega_i g_i(x_i) &= \int q_i(x_i, \mathrm{d}y_i)(g_i(y_i) - g_i(x_i)), \qquad g_i \in {}_b\mathscr{E}_i,\ i = 1,\ 2\\ \widetilde{\Omega} f(x_1, x_2) &= \int \tilde{q}(x_1, x_2; \mathrm{d}y_1, \mathrm{d}y_2)(f(y_1, y_2) - f(x_1, x_2)), \quad (x_1, x_2) \in \widetilde{E},\ f \in {}_b\mathscr{E},\end{aligned}$$

one can rewrite the above equality as

$$\begin{aligned}&\widetilde{\Omega} f(\cdot, x_2) = \Omega_1 f \quad \text{independent of } x_2,\ \ f \in {}_b\mathscr{E}_1;\\ &\widetilde{\Omega} f(x_1, \cdot) = \Omega_2 f \quad \text{independent of } x_1,\ \ f \in {}_b\mathscr{E}_2.\end{aligned} \tag{7}$$

In other words, we have proven

**(8). Lemma.** (2)$\Longrightarrow$(7)

Next, we prove that (7) $\Longrightarrow$ (2).

It is known that $q$-pair $(q(x), q(x, \cdot))$ on a separable measurable state space $(E, \mathscr{E})$ determines uniquely the minimal jump process $P^{\min}(t, x, \cdot)$. If we define

$$P^{\min}(\lambda, x, \cdot) = \int_0^\infty e^{-\lambda t} P^{\min}(t, x, \cdot)\mathrm{d}t, \qquad t \geqslant 0,\ x \in E \tag{9}$$

then $P^{\min}(\lambda, \cdot, A)$ is the minimal solution to the equation

$$f(x) = \int \frac{q(x, \mathrm{d}y)}{\lambda + q(x)} f(y) + \frac{\delta(x, A)}{\lambda + q(x)}, \qquad x \in E \tag{10}$$

for each fixed $\lambda > 0$ and $A \in \mathscr{E}$. We also call the Laplace transform $P(\lambda, x, \cdot)$ of a jump process $P(t, x, \cdot)$ a jump process.

**(11). Lemma.** *Suppose that (7) holds, then*

$$\begin{aligned}\widetilde{P}^{\min}(\lambda, (x_1, x_2), A_1 \times E_2) &\leqslant P_1^{\min}(\lambda, x, A_1)\\ \widetilde{P}^{\min}(\lambda, (x_1, x_2), E_1 \times A_2) &\leqslant P_2^{\min}(\lambda, x, A_2)\\ \lambda > 0,\ x_i \in E_i,\ A_i &\in \mathscr{E}_i,\ i = 1,\ 2\end{aligned}$$

*where $P_i^{\min}(\lambda, x, \cdot)\,(i = 1, 2)$ and $\widetilde{P}^{\min}(\lambda, (x_1, x_2), \cdot)$ are the minimal jump processes determined by $(q_i(x_i), q_i(x_i, \cdot))$ and $(\tilde{q}(\tilde{x}), \tilde{q}(\tilde{x}, \cdot))$, respectively. In particular, if $(\tilde{q}(\tilde{x}), \tilde{q}(\tilde{x}, \cdot))$ is regular, then so are the marginals.*

*Proof.* By the comparison theorem [2; Theorem 6], it suffices to show that $h(x_1, x_2) := P_1^{\min}(\lambda, x_1, A_1)$ satisfies the inequality

$$h(x_1, x_2) \geqslant \int \frac{\tilde{q}(x_1, x_2; \mathrm{d}y_1, \mathrm{d}y_2)}{\lambda + q(x_1, x_2)} h(y_1, y_2) + \frac{\delta(x_1, A_1)}{\lambda + \tilde{q}(x_1, x_2)}, \qquad (x_1, x_2) \in \widetilde{E}. \tag{12}$$

This follows from (7) and (10) immediately. $\square$

**(13). Theorem.** Suppose that $(\tilde{q}(\tilde{x}), \tilde{q}(\tilde{x}, \cdot))$ is regular, then (2) $\Longleftrightarrow$ (7).

*Proof.* Since Lemma (8), it is enough to prove that (7)$\Longrightarrow$(2). By Lemma (11) and the assumption, one can see that

$$\begin{aligned} \widetilde{P}(\lambda, (x_1, x_2), A_1 \times E_2) \leqslant P_1(\lambda, x_1, A_1) \\ \lambda > 0,\ x_i \in E_i,\ i = 1,\ 2,\ A_1 \in \mathscr{E}_1. \end{aligned} \tag{14}$$

If

$$P(\lambda, (x_1, x_2), A_1 \times E_2) < P_1(\lambda, x_1, A_1) \tag{15}$$

for some $\lambda > 0$, $(x_1, x_2) \in \widetilde{E}$ and $A_1 \in \mathscr{E}$, then

$$\begin{aligned} 1 &= \lambda \widetilde{P}(\lambda, (x_1, x_2), A_1 \times E_2) + \lambda \widetilde{P}(\lambda, (x_1, x_2), A_1^c \times E_2) \\ &< \lambda P_1(\lambda, x_1, A_1) + \lambda P_1(\lambda, x_1, A_1^c) = \lambda P_1(\lambda, x_1, E_1) \leqslant 1. \end{aligned}$$

This is impossible. □

## §3. Regularity

The uniqueness criteria for general $q$-process were obtained by Chen and Zheng[7]. In this section, we first present some sufficient conditions for uniqueness which are usually more practical. Then we study the relationship between the regularity of the coupled $q$-process and the regularities of its marginal $q$-processes.

**(16). Theorem.** Suppose that there exist a sequence $\{E_n\} \subset \mathscr{E}$ and an $\varphi \in \mathscr{E}_+$,[2] such that[3]

$$E_n \uparrow E, \quad \text{as } n \uparrow \infty; \qquad \sup_{x \in E_n} \varphi(x) < \infty, \tag{17}$$

$$\lim_{n \to \infty} \inf_{x \notin E_n} \varphi(x) = \infty; \tag{18}$$

and there also exists a $c \in \mathbb{R}$ such that

$$\int q(x, \mathrm{d}y) \varphi(y) \leqslant (c + q(x)) \varphi(x), \qquad x \in E \tag{19}$$

then the $q$-process is unique, i. e., the $q$-pair $(q(x), q(x, \cdot))$ is regular.

*Proof.* Without loss of generality, we may assume that $c \geqslant 0$.

(a). Since for each $\lambda > 0$, $\int P^{\min}(\lambda, \cdot, \mathrm{d}y) \varphi(y)$ is the minimal nonnegative solution to the equation

$$f = \int \frac{q(\cdot, \mathrm{d}y)}{\lambda + q(\cdot)} f(y) + \frac{\varphi(\cdot)}{\lambda + q(\cdot)},$$

and by condition (19),

$$\frac{\varphi}{\lambda - c} = \int \frac{q(\cdot, \mathrm{d}y)}{\lambda + q} \frac{\varphi(y)}{\lambda - c} + \frac{\varphi}{\lambda + q}, \qquad \lambda > c$$

[2] the set of all nonnegative $\mathscr{E}$-measurable functions

[3] For condition (18), the author has a helpful discussion with S. Z. Tang.

it follows from the comparison theorem that

$$\int P^{\min}(\lambda,\cdot,\mathrm{d}y)\varphi(y)\leqslant\frac{\varphi}{\lambda-c}<\infty.$$

(b). Set

$$q_n(x,\mathrm{d}y)=I_{E_n}(x)q(x,\mathrm{d}y),\quad q_n(x)=q_n(x,E),\qquad x\in E,\ n\geqslant 1 \tag{20}$$

then $(q_n(x),q_n(x,\cdot))$ is a regular bounded $q$-pair for each $n\geqslant 1$. Clearly, the $q$-pair $(q_n(x),q_n(x,\cdot))$ also satisfies condition (19), therefore, by (a), one can see that

$$\int P_n(\lambda,x,\mathrm{d}y)\varphi(y)\leqslant\frac{\varphi(x)}{\lambda-c}<\infty,\qquad x\in E,\ \lambda>c,\ n\geqslant 1.$$

(c). For $x\in E_n$, we have

$$\begin{aligned}P^{\min}(\lambda,x,E_n)&=\int\frac{q(x,\mathrm{d}y)}{\lambda+q(x)}P^{\min}(\lambda,y,E_n)+\frac{\delta(x,E_n)}{\lambda+q(x)}\\&=\int\frac{q_n(x,\mathrm{d}y)}{\lambda+q_n(x)}P^{\min}(\lambda,y,E_n)+\frac{\delta(x,E_n)}{\lambda+q_n(x)}\end{aligned}$$

and for $x\notin E_n$, we simply have

$$P^{\min}(\lambda,x,E_n)\geqslant 0=\int\frac{q_n(x,\mathrm{d}y)}{\lambda+q_n(x)}P^{\min}(\lambda,y,E_n)+\frac{\delta(x,E_n)}{\lambda+q_n(x)}.$$

Thus, we always have

$$P^{\min}(\lambda,x,E_n)\geqslant\int\frac{q_n(x,\mathrm{d}y)}{\lambda+q_n(x)}P^{\min}(\lambda,y,E_n)+\frac{\delta(x,E_n)}{\lambda+q_n(x)},\quad\lambda>0,\ x\in E,\ n\geqslant 1.$$

Now, the comparison theorem gives us that

$$P^{\min}(\lambda,x,E_n)\geqslant P_n(\lambda,x,E_n),\qquad\lambda>0,\ x\in E,\ n\geqslant 1.$$

(d). By (b) and (c), we get

$$\begin{aligned}\lambda P^{\min}(\lambda,x,E_n)&\geqslant\lambda P_n(\lambda,x,E_n)=1-\lambda P_n(\lambda,x,E_n^c)\\&\geqslant 1-\frac{\lambda\varphi(x)}{(\lambda-c)\inf_{z\notin E_n}\varphi(z)},\qquad\lambda>c,\ x\in E.\end{aligned}$$

and so

$$\lambda P^{\min}(\lambda,x,E)\geqslant\lim_{n\to\infty}\lambda P^{\min}(\lambda,x,E_n)\geqslant 1,\qquad\lambda>c.$$

This completes our proof. □

**(21). Theorem.** For the uniqueness of $q$-processes, each of the following conditions is sufficient:

(i) there exist a constant $c\in\mathbb{R}$ and an $\varphi\in\mathscr{E}$ such that $\varphi\geqslant q$ and[4]

$$\int q(x,\mathrm{d}y)\varphi(y)\leqslant(c+q(x))\varphi(x),\qquad x\in E;$$

(ii) there exists a $\lambda_0>0$ such that

$$\int P^{\min}(\lambda_0,x,\mathrm{d}y)\varphi(y)<\infty,\qquad x\in E;$$

(iii) for each $t\geqslant 0$ and $x\in E$,

$$\int P^{\min}(t,x,\mathrm{d}y)\varphi(y)<\infty.$$

[4]Similar but stronger condition was given by Basis [l].

*Proof.* By the proof (a) of the above theorem, we have (i)$\Longrightarrow$(ii).

Now assume that condition (ii) holds. By the forward Kotmogorov equation[3]:

$$P^{\min}(\lambda, x, A) = \int P^{\min}(\lambda, x, \mathrm{d}y) \int_A \frac{q(y, \mathrm{d}z)}{\lambda + q(z)} + \frac{\delta(x, A)}{\lambda + q(x)}$$

and the monotone class theorem, it follows that

$$\int P^{\min}(\lambda, x, \mathrm{d}y) f(y) = \int P^{\min}(\lambda, x, \mathrm{d}y) \int \frac{q(y, \mathrm{d}z)}{\lambda + q(z)} f(z) + \frac{f(x)}{\lambda + q(x)},$$
$$\lambda > 0,\ x \in E,\ f \in \mathscr{E}_+.$$

In particular, taking $\lambda = \lambda_0,\ f = \lambda_0 + q$, we obtain

$$\int (\lambda_0 + q(y)) P^{\min}(\lambda_0, x, \mathrm{d}y) = \int P^{\min}(\lambda_0, x, \mathrm{d}y) q(y) + 1, \qquad x \in E.$$

Combining this with (ii), we have

$$\lambda_0 P(\lambda_0, x, E) = 1, \qquad x \in E.$$

This certainly implies the uniqueness. The last assertion can be proved in a similar way. □

**(22). Remark.** It is easy to show that the condition (21) (i) implies the assumptions of Theorem (16). To see this, simply take

$$E_n = \{x \in E : q(x) \leqslant n\}, \qquad \varphi(x) = q(x), \qquad x \in E.$$

but the converse fails. The following counterexample is due to J. L. Zheng:

Take $E = \{1, 2, \cdots\}$ and let $q_1, q_2, \cdots$ be the prime numbers in the natural order. Set

$$q_{i,i+1} = q_i, \quad i \in E; \qquad q_{ij} = 0, \quad j \neq i, i+1.$$

This $Q$-matrix $(q_{ij})$ satisfies the assumptions of Theorem (16). To this end, we take $c = 1$, $\varphi_1 = 1$, $\varphi_i = \prod_{k=1}^{i-1}(1 + 1/q_k)$ for $i \geqslant 2$. Since $\prod_{n=1}^{\infty}(1 + 1/q_n)$ and $\sum_{n=1}^{\infty} 1/q_n$ are convergent or divergent simultaneously, it follows that $\lim_{n\to\infty} \varphi_n = \infty$. Therefore, the assumptions are satisfied with $E_n = \{0, 1, 2, \cdots, n\}$, and so the $Q$-process is unique.

Next we show that the condition (21) (i) fails. Indeed, we will show that the condition (21) (ii) fails also. By the backward Kolmogorov equation, one can easily figure out:

$$P_{ik}(\lambda) = 0,\ k < i; \qquad P_{ii}(\lambda) = \frac{l}{\lambda + q_i};$$
$$P_{ij}(\lambda) = \frac{q_i \cdots q_{j-1}}{(\lambda + q_i) \cdots (\lambda + q_j)}, \qquad j > i,$$

hence

$$\sum_{j=1}^{\infty} P_{1j}(\lambda) q_j = \sum_{j=1}^{\infty} \frac{q_i \cdots q_j}{(\lambda + q_i) \cdots (\lambda + q_j)} =: \sum_{j=1}^{\infty} a_j.$$

Because

$$\lim_{j\to\infty} j\left(\frac{a_j}{a_{j+1}} - 1\right) = \lambda \lim_{j\to\infty} \frac{j}{a_{j+1}} = 0, \qquad \lambda > 0$$

one can see that the above series is divergent for each $\lambda > 0$.

**(23). Remark.** We point out here that the Theorem (16) is quite general. In some special case (for example, for generalized birth-death $Q$-processes), the conditions of (16) are also necessary.

Now we turn to discuss the relationship between tile regularities of a coupled process and its marginal processes. The next result was proved in Lemma (11).

**(24). Theorem.** If a coupled $q$-pair $(\tilde{q}(\tilde{x}), \tilde{q}(\tilde{x}, \cdot))$ satisfying (7) is regular, then its marginal $q$-pairs $(q_i(x_i), q_i(x_i, \cdot))$, $i = 1, 2$ are all regular.

Note that there are many choices of coupled $q$-pairs satisfying (7), also, the coupled $q$-pairs are usually more complicated than the given marginal $q$-pairs, it is certainly more interesting to prove that the regularities of the marginal $q$-pairs imply the one of a coupled $q$-pair. Unfortunately, we do not know at the moment how to prove it completely. What we can do now is to present the following result, which is an interesting application of Theorem (16) and quite general:

**(25). Theorem.** If the marginal $q$-pair $(q_i(x_i), q_i(x_i, \cdot))$ $(i = 1, 2)$ satisfy the assumption of Theorem (16), then every coupled $q$-pair satisfying (7) is regular.

*Proof.* For $i = 1, 2$, we use $E_i^{(n)}$, $\varphi_i$ and $c_i$ to denote, respectively, the subsets, function and constant in the assumptions of Theorem (16), corresponding to the $q$-pair $(q_i(x_i), q_i(x_i, \cdot))$. Put

$$\widetilde{E}_n = E_1^{(n)} \times E_2^{(n)}, \qquad n \geqslant 1$$

$$\tilde{\varphi}(x_1, x_2) = \varphi_1(x_1) + \varphi_2(x_2), \qquad (x_1, x_2) \in \widetilde{E}.$$

Then $\{\widetilde{E}_n\}_1^\infty \subset \widetilde{\mathscr{E}}$ and $\widetilde{E}_n \uparrow \widetilde{E}$. By (7), one can see that

$$\tilde{q}(x_1, x_2) \leqslant q_1(x_1) + q_2(x_2), \qquad (x_1, x_2) \in \widetilde{E} \tag{26}$$

and so

$$\sup_{(x_1,x_2)\in\widetilde{E}_n} \tilde{q}(x_1, x_2) < \infty, \qquad n \geqslant 1.$$

On the other hand,

$$\lim_{n\to\infty} \inf_{(x_1,x_2)\notin\widetilde{E}_n} \tilde{\varphi}(\tilde{x}) \geqslant \Big(\lim_{n\to\infty} \inf_{x_1\notin E_n^{(1)}} \varphi_1(x_1)\Big) \bigwedge \Big(\lim_{n\to\infty} \inf_{x_2\notin E_n^{(2)}} \varphi_2(x_2)\Big) = \infty.$$

Finally, using the assumptions:

$$\int q_i(x_i, \mathrm{d}y_i)\varphi_i(y_i) \leqslant (c_i + q_i(x_i))\varphi_i(x_i), \qquad x_i \in E_i,\ i = 1,\ 2$$

and condition (7), it follows that

$$\int \tilde{q}(x_1, x_2; \mathrm{d}y_1, \mathrm{d}y_2)\tilde{\varphi}(y_1, y_2) \leqslant (c_1 \vee c_2 + \tilde{q}(x_1, x_2))\tilde{\varphi}(x_1, x_2), \qquad (x_1, x_2) \in \widetilde{E}.$$

Therefore the $q$-pair $(\tilde{q}(\tilde{x}), \tilde{q}(\tilde{x}, \cdot))$ also satisfies the assumptions of Theorem (16). $\square$

**(27). Corollary.** If the marginal $q$-pairs satisfy simultaneously one of the conditions of Theorem (21), then every coupled $q$-pair satisfying (7) is regular.

## §4. Order-Preservation

In this section, we assume that $E_1 = E_2 = E$, $\mathscr{E}_1 = \mathscr{E}_2 = \mathscr{E}$, that $E$ is endowed a semi–order "$\leqslant$", and the subset $\{(x, y) \in \widetilde{E} : x \leqslant y\} =: \widetilde{F}$ is $\widetilde{\mathscr{E}}$-measurable. We also assume that the coupled $q$-pair is regular.

We can rewrite the condition (4) as follows:

(28). Order-preservation.

$$\widetilde{P}(t,(x_1,x_2),\widetilde{F})=1,\qquad t\geqslant 0,\ (x_1,x_2)\in\widetilde{F}.$$

By differentiation, the above condition gives

$$\tilde{q}\big(x_1,x_2;\widetilde{F}^c\big)=0,\qquad (x_1,x_2)\in\widetilde{F}. \tag{29}$$

Indeed, we have

**(30). Theorem.** (28)$\Longleftrightarrow$(29).

*Proof.* We have seen that (28)$\Longrightarrow$(29). Now assume that (29) holds. Note that

$$\widetilde{P}^{(0)}\big(\lambda,(x_1,x_2),\widetilde{F}^c\big)=\frac{\delta\big(x_1,x_2;\widetilde{F}^c\big)}{\lambda+\tilde{q}(x_1,x_2)}=0,\qquad (x_1,x_2)\in\widetilde{F}.$$

Suppose

$$\widetilde{P}^{(n)}\big(\lambda,(x_1,x_2),\widetilde{F}^c\big)=0,\qquad (x_1,x_2)\in\widetilde{F},$$

then, by (29), we get

$$\begin{aligned}&\widetilde{P}^{(n+1)}\big(\lambda,(x_1,x_2),\widetilde{F}^c\big)\\&=\int\frac{\tilde{q}(x_1,x_2;\mathrm{d}y_1,\mathrm{d}y_2)}{\lambda+\tilde{q}(x_1,x_2)}\widetilde{P}^{(n)}\big(\lambda,(y_1,y_2),\widetilde{F}^c\big)+\widetilde{P}^{(0)}\big(\lambda,(x_1,x_2),\widetilde{F}^c\big),\\&=\int_{\widetilde{F}}\frac{\tilde{q}(x_1,x_2;\mathrm{d}y_1,\mathrm{d}y_2)}{\lambda+\tilde{q}(x_1,x_2)}\widetilde{P}^{(n)}\big(\lambda,(y_1,y_2),\widetilde{F}^c\big)=0,\qquad (x_1,x_2)\in\widetilde{F}.\end{aligned}$$

Hence, by induction, it follows that

$$\widetilde{P}^{(n)}\big(\lambda,(x_1,x_2),\widetilde{F}^c\big)=0,\qquad (x_1,x_2)\in\widetilde{F},\ n\geqslant 1$$

and so

$$\widetilde{P}\big(\lambda,(x_1,x_2),\widetilde{F}^c\big)=0,\qquad (x_1,x_2)\in\widetilde{F},\ \lambda>0.$$

This finishes the proof. □

## §5. Basic couplings

Let $\mu_1$ and $\mu_2$ be two finite measures on $(E,\mathscr{E})$. Denote by $(\mu_1-\mu_2)^+$ the Jordan-Hahn decomposition of $\mu_1-\mu_2$ and define

$$\mu_1\wedge\mu_2=\mu_1-(\mu_1-\mu_2)^+.$$

Clearly, $\mu_1\wedge\mu_2=\mu_2\wedge\mu_1$.

Let $(q_i(x_i),q_i(x_i,\cdot))$ be a given $q$-pair on $(E_i,\mathscr{E}_i)$, $i=1,2$. It often happens that

$$E_1\subset E_2\quad(\text{reap., } E_2\subset E_1).$$

and

$$E_1\in\mathscr{E}_2\quad(\text{reap., } E_2\in\mathscr{E}_1).$$

In this case, one can naturally extend the $q$-pair $(q_1(x_1),q_1(x_1,\cdot))$ to $(E_2,\mathscr{E}_2)$ simply by defining

$$q_1(x)=0,\qquad x\in E_2\setminus E_1.$$

Because of this reason, we may and will assume that

$$E_1=E_2=E,\qquad \mathscr{E}_1=\mathscr{E}_2=\mathscr{E}.$$

The simplest coupling is

**(31). Independent Coupling.**

$$\begin{aligned}\widetilde{\Omega}f(x_1,x_2)&=\int q_1(x_1,\mathrm{d}y_1)(f(y_1,x_2)-f(x_1,x_2))\\&=\int q_2(x_2,\mathrm{d}y_2)(f(x_1,y_2)-f(x_1,x_2))\\&=\big(\Omega_1 f(\cdot,x_2)\big)(x_1)+\big(\Omega_2 f(x_1,\cdot)\big)(x_2),\qquad (x_1,x_2)\in\widetilde{E},\ f\in{}_b\widetilde{E}.\end{aligned}$$

Perhaps the following coupling is the most useful one:

**(32). Basic Coupling.**

$$\begin{aligned}\widetilde{\Omega}f(x_1,x_2)=&\int\big(q_1(x_1,\cdot)-q_2(x_2,\cdot)\big)^+(\mathrm{d}y)[f(y,x_2)-f(x_1,x_2)]\\&+\int\big(q_2(x_2,\cdot)-q_1(x_1,\cdot)\big)^+(\mathrm{d}y)[f(x_1,y)-f(x_1,x_2)]\\&+\int\big(q_1(x_1,\cdot)\wedge q_2(x_2,\cdot)\big)(\mathrm{d}y)[f(y,y)-f(x_1,x_2)].\end{aligned}$$

For more examples of couplings, one can see [4] and [5].

It is not hard to check, for the basic coupling, that the order-preservation condition (29) becomes

(33). for each $(x_1,x_2)\in\widetilde{F}$,

$$\begin{aligned}(q_1(x_1,\cdot)-q_2(x_2,\cdot))^+\big(\{y\in E: y\not\leq x_2\}\big)=0,\\(q_2(x_2,\cdot)-q_1(x_1,\cdot))^+\big(\{y\in E: x_1\not\leq y\}\big)=0.\end{aligned}$$

**(34). Basic Coupling for $q$-Processes with Finite Product State Space.**

Let $S$ be a finite set. For each $u\in S$, let $(E_u,\mathscr{E}_u)$ be a measurable space as above. Suppose that $(q^\alpha(x),q^\alpha(x,\cdot))$ is a $q$-pair on $\big(\prod_{u\in S}E_u,\prod_{u\in S}\mathscr{E}_u\big)=:(E,\mathscr{E})$ satisfying that $q^\emptyset(x)=0$ for all $x\in E$ and the measure $q^\alpha(x,\cdot)$ is constrained on

$$\{y\in E: y_u\neq x_u,\ u\in\alpha;\ y_u=x_u,\ u\in S\setminus\alpha\}$$

for each $\alpha\subset S$. Now, set

$$q(x,\cdot)=\sum_{\alpha\in S}q^\alpha(x,\cdot),\qquad q(x)=q(x,E),\ x\in E.$$

Clearly, $(q(x),q(x,\cdot))$ is a $q$-pair on $(E,\mathscr{E})$. Corresponding to (32), we can define a coupling as follows:

$$\begin{aligned}\widetilde{\Omega}f(x_1,x_2)=&\sum_{\alpha\subset S}\big(q^\alpha(x_1,\cdot)-q^\alpha(x_2,\cdot)\big)^+(\mathrm{d}y_1)[f(y_1,x_2)-f(x_1,x_2)]\\&+\sum_{\alpha\subset S}\big(q^\alpha(x_2,\cdot)-q^\alpha(x_1,\cdot)\big)^+(\mathrm{d}y_2)[f(x_1,y_2)-f(x_1,x_2)]\\&+\sum_{\alpha\subset S}\big(q^\alpha(x_1,\cdot)\wedge q^\alpha(x_2,\cdot)\big)(\mathrm{d}y)[f(y,y)-f(x_1,x_2)],\\&\qquad(x_1,x_2)\in\widetilde{E},\ f\in{}_b\widetilde{\mathscr{E}}.\end{aligned}\tag{35}$$

The basic coupling will play an important role in the subsequent paper [6].

**In Addition**. After the present paper was written, J. L. Zheng and X. G. Zheng proved that the regularity of the marginal $q$-pairs implies the one of their coupled $q$-pair for Markov Chains under a slight assumption, by using martingale approach. Then the author and J. L. Zheng find a simple proof for general 'case. We present the proof in the following two theorems.

**(36). Theorem.** Given $q$-pair $(q(x), q(x,\cdot))$ and a sequence $\{E_n\} \subset \mathscr{E}$ such that

$$E_n \uparrow E, \qquad \sup_{x\in E_n} q_n(x) < \infty, \quad n \geqslant 1.$$

Define $(q_n(x), q_n(x,\cdot))$ by (20), Then $(q(x), q(x,\cdot))$ is regular iff

$$\lim_{n\to\infty} P_n\big(\lambda, x, E_n^c\big) = 0, \qquad \lambda > 0,\ x\in E.$$

*Proof.* The sufficiency follows from

$$P^{\min}\big(\lambda, x, E_n\big) \geqslant P_n\big(\lambda, x, E_n\big), \qquad \lambda>0,\ x\in E,\ n\geqslant 1$$

which we have seen in the proof of Theorem (16). To prove the necessity, note that by the backward Kolmogorov equation, Fatou lemma and the comparison theorem, we have

$$\varliminf_{n\to\infty} P_n\big(\lambda, x, E_n\big) \geqslant P^{\min}\big(\lambda, x, E\big), \qquad \lambda>0,\ x\in E.$$

Thus, if $(q(x), q(x,\cdot))$ is regular, then

$$\begin{aligned} 1 &\geqslant 1-\lambda \varlimsup_{n\to\infty} P_n\big(\lambda, x, E_n^c\big) = \lambda \varliminf_{n\to\infty} P_n\big(\lambda, x, E_n\big) \\ &\geqslant \lambda P(\lambda, x, E) = 1. \end{aligned}$$

and so the condition is necessary. □

**(37). Theorem.** If the marginal $q$-pairs $(q_i(x_i), q_i(x_i,\cdot))$ $(i=1,2)$ are regular, then so is each coupled $q$-pair satisfying (7).

*Proof.* Take

$$\begin{aligned} E_i^{(n)} &= \{x_i \in E_i : q_i(x_i) \leqslant n\}, \qquad i=1,\ 2,\ n\geqslant 1, \\ \widetilde{E}^{(n)} &= E_1^{(n)} \times E_2^{(n)}, \qquad n\geqslant 1. \end{aligned}$$

and define $\big(q_i^{(n)}(x_i), q_i^{(n)}(x_i,\cdot)\big)$, $i=1,2$ and $\big(\tilde{q}^{(n)}(x_i), \tilde{q}^{(n)}(x,\cdot)\big)$ by (20), respectively. Since

$$\sup_{\tilde{x}\in\widetilde{E}^{(n)}} \tilde{q}(x) \leqslant \sup_{x_1\in E_1^{(n)}} q_1(x_1) + \sup_{x_2\in E_2^{(n)}} q_2(x_1) < \infty$$

and Theorem (36), it suffices to show that

$$\begin{aligned} \widetilde{P}^{(n)}\big(\lambda;(x_1,x_2),(\widetilde{E}^{(n)})^c\big) &\leqslant P_1^{(n)}\big(\lambda, x_1, (E_1^{(n)})^c\big) + P_2^{(n)}\big(\lambda, x_2, (E_2^{(n)})^c\big) \\ &\qquad \lambda>0,\ (x_1,x_2)\in\widetilde{E},\ n\geqslant 1. \end{aligned}$$

where the $q$-processes are determined, respectively, by the above $q$-pairs. But this is an easy consequence of the condition (7) plus an application of the comparison theorem. □

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