

# Stochastic equations of non-negative processes with jumps

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*Abstract.* We study stochastic equations of non-negative processes with jumps. The existence and uniqueness of strong solutions are established under Lipschitz and non-Lipschitz conditions. Under suitable conditions, the comparison properties of solutions are proved. Those results are applied to construct continuous state branching processes with immigration as strong solutions of stochastic equations.

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**Abbreviated Title:** Stochastic equations of non-negative processes

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## 1 Introduction

Stochastic differential equations with jumps have been playing increasingly important roles in various applications. Under Lipschitz conditions, the existence and uniqueness of strong solutions of jump-type stochastic equations can be established by arguments based on Gronwall's inequality and the results on continuous-type equations; see e.g. Ikeda and Watanabe (1989). In view of the result of Yamada and Watanabe (1971), weaker conditions would be sufficient for the existence and uniqueness of strong solutions

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for one-dimensional equations. As an example of jump-type equations, let us consider the simple equation

$$dx(t) = \phi(x(t-))dz(t), \quad t \geq 0. \quad (1.1)$$

By a result of Bass (2003), if  $\{z(t)\}$  is a symmetric stable process of order  $1 < \alpha < 2$  and if  $x \mapsto \phi(x)$  is a bounded function with modulus of continuity  $x \mapsto \rho(x)$  satisfying

$$\int_{0+} \frac{1}{\rho(x)^\alpha} dx = \infty, \quad (1.2)$$

then (1.1) admits a strong solution and the solution is pathwise unique. This condition is exactly the analogue of the Yamada-Watanabe criterion for the diffusion coefficient. When the integral in (1.2) is finite, Bass (2003) constructed a continuous function  $x \mapsto \phi(x)$  having continuity modulus  $x \mapsto \rho(x)$  for which the pathwise uniqueness for (1.1) fails; see also Bass *et al.* (2004). In view of (1.2), if a power function  $\rho(x) = \text{const} \cdot x^\beta$  applies for all symmetric stable processes with parameters  $1 < \alpha < 2$ , we must have  $\beta = 1$ . In other words, a universal continuity modulus condition for jump-type stochastic differential equations would not be a great improvement of the Lipschitz condition.

For equations driven by non-symmetric noises, there is a new difficulty brought about by the compensators of the noises. For example, let us consider the equation (1.1) again with  $\{z(t)\}$  being a one-sided stable process of order  $1 < \alpha < 2$ . For any  $\varepsilon > 0$ , let

$$z_\varepsilon(t) = \sum_{0 < s \leq t} [z(s) - z(s-)] 1_{\{z(s) - z(s-) > \varepsilon\}}$$

and let  $c_\varepsilon = \mathbf{E}[z_\varepsilon(1)]$ . We can define another centered Lévy process  $\{w_\varepsilon(t)\}$  by

$$w_\varepsilon(t) = z(t) - z_\varepsilon(t) + c_\varepsilon t.$$

Between any two neighboring jumps of  $\{z_\varepsilon(t)\}$ , equation (1.1) reduces to

$$dx(t) = \phi(x(t-))dw_\varepsilon(t) - c_\varepsilon \phi(x(t-))dt. \quad (1.3)$$

Then one would expect, in order that the pathwise uniqueness holds for (1.1) or (1.3), the function  $x \mapsto \phi(x)$  should be as regular as the drift coefficient in the Yamada-Watanabe criterion. In other words, it should possess a continuity modulus  $x \mapsto r(x)$  satisfying

$$\int_{0+} \frac{1}{r(x)} dx = \infty, \quad (1.4)$$

which is much stronger than (1.2).

Continuous state branching processes with immigration (CBI-processes) constitute an important class of non-negative Markov processes with non-negative jumps. They were introduced in Kawazu and Watanabe (1971) as approximations of classical Galton-Watson branching processes with immigration. Many interesting applications of them have been found since then. In particular, CBI-processes are also known as Cox-Ingersoll-Ross models (CIR-models) and have been used widely in the study of mathematical

finance; see, e.g., Duffie *et al.* (2003) and Lamberton and Lapeyre (1996). Up to a minor moment assumption, a conservative CBI-process has generator  $A$  defined by

$$\begin{aligned} Af(x) = & axf''(x) + \int_0^\infty [f(x+z) - f(x) - zf'(x)]x\nu_0(dz) \\ & + (b + \beta x)f'(x) + \int_0^\infty [f(x+z) - f(x)]\nu_1(dz), \end{aligned} \quad (1.5)$$

where  $a \geq 0$ ,  $b \geq 0$  and  $\beta$  are constants, and  $\nu_0(dz)$  and  $\nu_1(dz)$  are  $\sigma$ -finite measures on  $(0, \infty)$  satisfying

$$\int_0^\infty (z \wedge z^2)\nu_0(dz) + \int_0^\infty (1 \wedge z)\nu_1(dz) < \infty. \quad (1.6)$$

Let  $\{B(t)\}$  be a standard Brownian motion and let  $\{N_0(ds, dz, du)\}$  and  $\{N_1(ds, dz)\}$  be Poisson random measures with intensities  $ds\nu_0(dz)du$  and  $ds\nu_1(dz)$ , respectively. Suppose that  $\{B(t)\}$ ,  $\{N_0(ds, dz, du)\}$  and  $\{N_1(ds, dz)\}$  are independent of each other. Let  $\tilde{N}_0(ds, dz, du)$  be the compensated measure of  $N_0(ds, dz, du)$ . Under a slightly stronger condition on  $\nu_1(dz)$  it was proved in Dawson and Li (2006) that the stochastic equation

$$\begin{aligned} x(t) = & x(0) + \int_0^t \sqrt{2ax(s)}dB(s) + \int_0^t \int_0^\infty \int_0^{x(s-)} z\tilde{N}_0(ds, dz, du) \\ & + \int_0^t (b + \beta x(s))ds + \int_0^t \int_0^\infty zN_1(ds, dz); \end{aligned} \quad (1.7)$$

has a unique non-negative strong solution. By Itô's formula one sees the solution  $\{x(t)\}$  of (1.7) is a CBI-process with generator given by (1.5).

The purpose of the present paper is to study stochastic equations of non-negative processes with jumps that generalizes the CBI-processes described above. By specifying to non-negative processes we can make the best use of the first moment analysis. This kind of processes arise naturally in various applications, but they are excluded by most of the existing results in the literature because their generators possess non-Lipschitz and degenerate coefficients. We provide here some criteria for the existence and uniqueness of strong solutions of those equations. The main idea of those criteria is to assume some monotonicity condition on the kernel associated with the compensated noise so that the continuity conditions can be released. It follows from our criterion that (1.1) has a unique non-negative strong solution if  $\{z(t)\}$  is a one-sided  $\alpha$ -stable process and if  $x \mapsto \phi(x)$  is a  $(1/2)$ -Hölder continuous non-decreasing function satisfying  $\phi(0) = 0$ .

To describe another consequence of our criteria, suppose that  $1 < \alpha < 2$  and  $(a, b, \beta, \nu_1)$  are given as above. Let  $\{B(t)\}$  be a standard Brownian motion,  $\{z_0(t)\}$  be a one-sided  $\alpha$ -stable process with characteristic measure  $z^{-1-\alpha}dz$  and  $\{z_1(t)\}$  be a non-decreasing pure jump Lévy process with characteristic measure  $\nu_1(dz)$ . Suppose  $\{B(t)\}$ ,  $\{z_0(t)\}$  and  $\{z_1(t)\}$  are independent of each other. We shall see for any  $\sigma \geq 0$  there is a unique non-negative strong solution to

$$dx(t) = \sqrt{2ax(t)}dB(t) + \sqrt[\alpha]{\sigma x(t-)}dz_0(t) + (\beta x(t) + b)dt + dz_1(t). \quad (1.8)$$

A particular case of this equation has been considered in Lambert (2007), where the uniqueness of the solution was left open. The solution of (1.8) is a CBI-process with generator given by (1.5) with  $\nu_0(dz) = \sigma z^{-1-\alpha} dz$ . Of course, for any  $\sigma > 0$  and  $1 < \alpha < 2$  the continuity modulus of the coefficient  $x \mapsto \sqrt[\alpha]{\sigma x}$  in (1.8) does not satisfy condition (1.4).

The theory of jump-type stochastic equations is not as well developed as that of continuous ones; see Bass (2004) and the references therein. We refer to Ikeda and Watanabe (1989) and Protter (2003) for the theory of stochastic analysis and to Sato (1999) for the theory of Lévy processes. Throughout this paper, we assume  $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})$  is a filtered probability space satisfying the usual hypotheses. Moreover, we make the conventions

$$\int_a^b = \int_{(a,b]} \quad \text{and} \quad \int_a^\infty = \int_{(a,\infty)}$$

for any real numbers  $a \leq b$ .

Some basic results for stochastic equations of non-negative processes are provided in Section 2. In particular, we give a Lipschitz condition for the existence and uniqueness of the strong solution. In Section 3 the pathwise uniqueness is studied. In Section 4 we prove a weak existence result by second moment arguments. The existence and uniqueness of strong solutions under non-Lipschitz conditions are established in Section 5, where only very weak moment conditions are required. We also prove some properties of the solutions, continuous dependence on the initial value and comparison properties. In Section 6, we illustrate some applications of the main results to stochastic differential equations driven by one-sided Lévy processes including (1.8).

## 2 Preliminaries

In this section, we prove some basic results on stochastic equations of non-negative processes with jumps. Suppose that  $U_0$  and  $U_1$  are complete separable metric spaces and that  $\mu_0(du)$  and  $\mu_1(du)$  are  $\sigma$ -finite Borel measures on  $U_0$  and  $U_1$ , respectively. Suppose that

- $x \mapsto \sigma(x)$  and  $x \mapsto b(x)$  are continuous functions on  $\mathbb{R}$  satisfying  $\sigma(x) = 0$  and  $b(x) \geq 0$  for  $x \leq 0$ ;
- $(x, u) \mapsto g_0(x, u)$  is a Borel function on  $\mathbb{R} \times U_0$  such that  $g_0(x, u) + x \geq 0$  for  $x > 0$ , and  $g_0(x, u) = 0$  for  $x \leq 0$ ;
- $(x, u) \mapsto g_1(x, u)$  is a Borel function on  $\mathbb{R} \times U_1$  such that  $g_1(x, u) + x \geq 0$  for all  $x \in \mathbb{R}$  and  $u \in U_1$ .

Let  $\{B(t)\}$  be a standard  $(\mathcal{F}_t)$ -Brownian motion and let  $\{p_0(t)\}$  and  $\{p_1(t)\}$  be  $(\mathcal{F}_t)$ -Poisson point processes on  $U_0$  and  $U_1$  with characteristic measures  $\mu_0(du)$  and  $\mu_1(du)$ , respectively. Suppose that  $\{B(t)\}$ ,  $\{p_0(t)\}$  and  $\{p_1(t)\}$  are independent of each other.

Let  $N_0(ds, du)$  and  $N_1(ds, du)$  be the Poisson random measures associated with  $\{p_0(t)\}$  and  $\{p_1(t)\}$ , respectively. Let  $\tilde{N}_0(ds, du)$  be the compensated measure of  $N_0(ds, du)$ . By a *solution* of the stochastic equation

$$\begin{aligned} x(t) = & x(0) + \int_0^t \sigma(x(s-))dB(s) + \int_0^t \int_{U_0} g_0(x(s-), u)\tilde{N}_0(ds, du) \\ & + \int_0^t b(x(s-))ds + \int_0^t \int_{U_1} g_1(x(s-), u)N_1(ds, du) \end{aligned} \quad (2.1)$$

we mean a càdlàg and  $(\mathcal{F}_t)$ -adapted process  $\{x(t)\}$  that satisfies the equation almost surely for every  $t \geq 0$ . Since  $x(s-) \neq x(s)$  for at most countably many  $s \geq 0$ , we can also use  $x(s)$  instead of  $x(s-)$  in the integrals with respect to  $dB(s)$  and  $ds$  on the right hand side of (2.1).

**Proposition 2.1** *If  $\{x(t)\}$  satisfies (2.1) and  $\mathbf{P}\{x(0) \geq 0\} = 1$ , then  $\mathbf{P}\{x(t) \geq 0$  for all  $t \geq 0\} = 1$ .*

*Proof.* Suppose there exists a constant  $\varepsilon > 0$  so that  $\tau := \inf\{t \geq 0 : x(t) \leq -\varepsilon\} < \infty$  with strictly positive probability. The assumptions on  $g_0(x, u)$  and  $g_1(x, u)$  prevent  $\{x(t)\}$  from jumping into the interval  $[-\varepsilon, \infty)$ . Then on the event  $\{\tau < \infty\}$  we have  $x(\tau) = x(\tau-) = -\varepsilon$  and hence  $\tau > \tau_0 := \inf\{s < \tau : x(t) \leq 0 \text{ for all } s \leq t \leq \tau\}$ . Here  $\tau_0$  is a random time but not a stopping time. Then we choose a deterministic time  $r \geq 0$  so that  $\{\tau_0 \leq r < \tau\}$  occurs with strictly positive probability. On the event  $\{\tau_0 \leq r < \tau\}$  we have

$$x(t \wedge \tau) = x(r \wedge \tau) + \int_{r \wedge \tau}^{t \wedge \tau} b(x(s-))ds + \int_{r \wedge \tau}^{t \wedge \tau} \int_{U_1} g_1(x(s-), u)N_1(ds, du), \quad t \geq r,$$

so  $t \mapsto x(t \wedge \tau)$  is non-decreasing on  $[r, \infty)$ . Since  $x(r) > -\varepsilon$  on  $\{\tau_0 \leq r < \tau\}$ , we get a contradiction.  $\square$

In the sequel, we shall always assume the initial variable  $x(0)$  is non-negative, so Proposition 2.1 implies that any solution of (2.1) is non-negative. Then we can assume the ingredients are defined only for  $x \geq 0$ . For non-negative processes we can use the first moment estimates, which is essential for the CBI-process and solutions of stochastic equations driven by one-sided stable noises. Let  $U_2$  be a Borel subset of  $U_1$  satisfying  $\mu_1(U_1 \setminus U_2) < \infty$  and consider the stochastic equation

$$\begin{aligned} x(t) = & x(0) + \int_0^t \sigma(x(s-))dB(s) + \int_0^t \int_{U_0} g_0(x(s-), u)\tilde{N}_0(ds, du) \\ & + \int_0^t b(x(s-))ds + \int_0^t \int_{U_2} g_1(x(s-), u)N_1(ds, du). \end{aligned} \quad (2.2)$$

**Proposition 2.2** *If there is a strong solution to (2.2), there is also a strong solution to (2.1). If the pathwise uniqueness of solutions holds for (2.2), it also holds for (2.1).*

*Proof.* The results hold trivially if  $\mu_1(U_1 \setminus U_2) = 0$ , so we assume  $0 < \mu_1(U_1 \setminus U_2) < \infty$  in this proof. Suppose that (2.2) has a strong solution  $\{x_0(t)\}$ . Let  $\{S_k : k = 1, 2, \dots\}$  be the set of jump times of the Poisson process

$$t \mapsto \int_0^t \int_{U_1 \setminus U_2} N_1(ds, du).$$

We have clearly  $S_k \rightarrow \infty$  as  $k \rightarrow \infty$ . For  $0 \leq t < S_1$  set  $y(t) = x_0(t)$ . Suppose that  $y(t)$  has been defined for  $0 \leq t < S_k$  and let

$$\xi = y(S_k-) + \int_{\{S_k\}} \int_{U_1 \setminus U_2} g_1(y(S_k-), u) N_1(ds, du). \quad (2.3)$$

By the assumption there is also a strong solution  $\{x_k(t)\}$  to

$$\begin{aligned} x(t) &= \xi + \int_0^t \sigma(x(s-)) dB(S_k + s) + \int_0^t \int_{U_0} g_0(x(s-), u) \tilde{N}_0(S_k + ds, du) \\ &\quad + \int_0^t b(x(s-)) ds + \int_0^t \int_{U_2} g_1(x(s-), u) N_1(S_k + ds, du). \end{aligned} \quad (2.4)$$

For  $S_k \leq t < S_{k+1}$  we set  $y(t) = x_k(t - S_k)$ . By induction that defines a process  $\{y(t)\}$ , which is clearly a strong solution to (2.1). On the other hand, if  $\{y(t)\}$  is a solution to (2.1), it satisfies (2.2) for  $0 \leq t < S_1$  and the process  $\{y(S_k + t)\}$  satisfies (2.4) for  $0 \leq t < S_{k+1} - S_k$  with  $\xi$  given by (2.3). Then the pathwise uniqueness for (2.1) follows from that for (2.2) and (2.4).  $\square$

The results of Proposition 2.2 can be reformulated for weak solutions of (2.1) and (2.2). In fact, some similar results were given in Stroock (1975) in terms of martingale problems. Thanks to Proposition 2.2 we may focus on the existence and uniqueness of strong solutions of equation (2.2). For the convenience of the statements of the results, let us formulate the following conditions:

(2.a) For a constant  $K \geq 0$  and a Borel subset  $U_2 \subset U_1$  we have  $\mu_1(U_1 \setminus U_2) < \infty$  and

$$b(x) + \int_{U_2} |g_1(x, u)| \mu_1(du) \leq K(1 + x), \quad x \geq 0;$$

(2.b) There is a non-decreasing function  $x \mapsto L(x)$  on  $\mathbb{R}_+$  so that

$$\sigma(x)^2 + \int_{U_0} [|g_0(x, u)| \wedge g_0(x, u)^2] \mu_0(du) \leq L(x), \quad x \geq 0.$$

**Proposition 2.3** *Suppose that conditions (2.a,b) hold. Let  $\{x(t)\}$  be a non-negative solution of (2.2) and let  $\tau_m = \inf\{t \geq 0 : x(t) \geq m\}$  for  $m \geq 1$ . Then  $\tau_m \rightarrow \infty$  almost surely as  $m \rightarrow \infty$ . Moreover, we have*

$$\mathbf{E}[1 + x(t)] \leq \mathbf{E}[1 + x(0)] \exp\{Kt\} \quad (2.5)$$

and

$$(1 + m) \mathbf{P}\{\tau_m \leq t\} \leq \mathbf{E}[1 + x(0)] \exp\{Kt\}. \quad (2.6)$$

*Proof.* The first assertion is immediate since  $\{x(t)\}$  has càdlàg sample paths. It is easy to show that

$$\begin{aligned} & \mathbf{E} \left[ \left| \int_0^{t \wedge \tau_m} \int_{U_0} g_0(x(s-), u) 1_{\{|g_0(x(s-), u)| > 1\}} \tilde{N}_0(ds, du) \right| \right] \\ & \leq \mathbf{E} \left[ \int_0^{t \wedge \tau_m} ds \int_{U_0} |g_0(x(s-), u)| 1_{\{|g_0(x(s-), u)| > 1\}} \mu_0(du) \right]. \end{aligned}$$

Observe also that

$$\mathbf{E} \left[ \left| \int_0^{t \wedge \tau_m} \sigma(x(s-)) dB(s) \right|^2 \right] = \mathbf{E} \left[ \int_0^{t \wedge \tau_m} \sigma(x(s-))^2 ds \right]$$

and

$$\begin{aligned} & \mathbf{E} \left[ \left| \int_0^{t \wedge \tau_m} \int_{U_0} g_0(x(s-), u) 1_{\{|g_0(x(s-), u)| \leq 1\}} \tilde{N}_0(ds, du) \right|^2 \right] \\ & = \mathbf{E} \left[ \int_0^{t \wedge \tau_m} ds \int_{U_0} g_0(x(s-), u)^2 1_{\{|g_0(x(s-), u)| \leq 1\}} \mu_0(du) \right]. \end{aligned}$$

Since  $x(s-) \leq m$  for all  $0 < s \leq \tau_m$ , the above expectations are finite by (2.b), and so

$$t \mapsto \int_0^{t \wedge \tau_m} \sigma(x(s-)) dB(s) + \int_0^{t \wedge \tau_m} \int_{U_0} g_0(x(s-), u) \tilde{N}_0(ds, du)$$

is a martingale. From (2.2) and (2.a) we get

$$\begin{aligned} \mathbf{E}[1 + x(t \wedge \tau_m)] &= \mathbf{E}[1 + x(0)] + \mathbf{E} \left[ \int_0^{t \wedge \tau_m} b(x(s-)) ds \right] \\ &\quad + \mathbf{E} \left[ \int_0^{t \wedge \tau_m} ds \int_{U_2} g_1(x(s-), u) \mu_1(du) \right] \\ &\leq \mathbf{E}[1 + x(0)] + K \mathbf{E} \left[ \int_0^{t \wedge \tau_m} (1 + x(s-)) ds \right]. \end{aligned}$$

Thus  $t \mapsto \mathbf{E}[1 + x(t \wedge \tau_m)]$  is a locally bounded function. Moreover, since  $x(s-) \neq x(s)$  for at most countably many  $s \geq 0$ , it follows that

$$\begin{aligned} \mathbf{E}[1 + x(t \wedge \tau_m)] &\leq \mathbf{E}[1 + x(0)] + K \mathbf{E} \left[ \int_0^{t \wedge \tau_m} (1 + x(s)) ds \right] \\ &\leq \mathbf{E}[1 + x(0)] + K \int_0^t \mathbf{E}[1 + x(s \wedge \tau_m)] ds. \end{aligned}$$

By Gronwall's lemma,

$$\mathbf{E}[1 + x(t \wedge \tau_m)] \leq \mathbf{E}[1 + x(0)] \exp\{Kt\}, \quad t \geq 0.$$

By the right continuity of  $\{x(t)\}$  we have  $x(\tau_m) \geq m$ , so (2.6) holds, and (2.5) follows by an application of Fatou's lemma.  $\square$

**Proposition 2.4** Suppose that conditions (2.a,b) hold and for each  $m \geq 1$  there is a unique strong solution to

$$\begin{aligned} x(t) &= x(0) + \int_0^t \sigma(x(s-) \wedge m) dB(s) + \int_0^t b_m(x(s-) \wedge m) ds \\ &\quad + \int_0^t \int_{U_0} [g_0(x(s-) \wedge m, u) \wedge m] \tilde{N}_0(ds, du) \\ &\quad + \int_0^t \int_{U_2} [g_1(x(s-) \wedge m, u) \wedge m] N_1(ds, du), \end{aligned} \quad (2.7)$$

where

$$b_m(x) = b(x) - \int_{U_0} [g_0(x, u) - g_0(x, u) \wedge m] \mu_0(du).$$

Then there is a unique strong solution to (2.2).

*Proof.* For any  $m \geq 1$  let  $\{x_m(t)\}$  denote the unique strong solution to (2.7) and let  $\tau_m = \inf\{t \geq 0 : x_m(t) \geq m\}$ . Since  $0 \leq x_m(t) < m$  for  $0 \leq t < \tau_m$ , the trajectory  $t \mapsto x_m(t)$  has no jumps larger than  $m$  on the time interval  $[0, \tau_m)$ . Then we have

$$\begin{aligned} x_m(t) &= x(0) + \int_0^t \sigma(x_m(s-)) dB(s) + \int_0^t \int_{U_0} g_0(x_m(s-), u) \tilde{N}_0(ds, du) \\ &\quad + \int_0^t ds \int_{U_0} [g_0(x_m(s-), u) - g_0(x_m(s-), u) \wedge m] \mu_0(du) \\ &\quad + \int_0^t b_m(x_m(s-)) ds + \int_0^t \int_{U_2} g_1(x_m(s-), u) N_1(ds, du) \\ &= x(0) + \int_0^t \sigma(x_m(s-)) dB(s) + \int_0^t \int_{U_0} g_0(x_m(s-), u) \tilde{N}_0(ds, du) \\ &\quad + \int_0^t b(x_m(s-)) ds + \int_0^t \int_{U_2} g_1(x_m(s-), u) N_1(ds, du) \end{aligned}$$

for  $0 \leq t < \tau_m$ . In other words,  $\{x_m(t)\}$  satisfies (2.2) for  $0 \leq t < \tau_m$ . For  $n \geq m \geq 1$  let  $\{y(t) : t \geq 0\}$  be the unique solution to

$$\begin{aligned} y(t) &= \xi + \int_0^t \sigma(y(s-) \wedge n) dB(\tau_m + s) + \int_0^t b_n(y(s-) \wedge n) ds \\ &\quad + \int_0^t \int_{U_0} [g_0(y(s-) \wedge n, u) \wedge n] \tilde{N}_0(\tau_m + ds, du) \\ &\quad + \int_0^t \int_{U_2} [g_1(y(s-) \wedge n, u) \wedge n] N_1(\tau_m + ds, du), \end{aligned}$$

where

$$\begin{aligned} \xi &= x_m(\tau_m-) + \int_{\{\tau_m\}} \int_{U_0} [g_0(x_m(\tau_m-), u) \wedge n] N_0(ds, du) \\ &\quad + \int_{\{\tau_m\}} \int_{U_2} [g_1(x_m(\tau_m-), u) \wedge n] N_1(ds, du). \end{aligned}$$

We define  $x'_n(t) = x_m(t)$  if  $0 \leq t < \tau_m$  and  $x'_n(t) = y(t - \tau_m)$  if  $t \geq \tau_m$ . It is not hard to see that  $\{x'_n(t)\}$  is a solution to (2.7) with the  $m$  replaced by  $n$ . By the strong uniqueness we get  $x'_n(t) = x_n(t)$  for all  $t \geq 0$ . In particular, we infer  $x_n(t) = x_m(t) < m$  for  $0 \leq t < \tau_m$ . Consequently, the sequence  $\{\tau_m\}$  is non-decreasing. On the other hand, as in the proof of Proposition 2.3 we have

$$\mathbf{E}[1 + x_m(t \wedge \tau_m)] \leq \mathbf{E}[1 + x(0)] \exp\{Kt\}, \quad t \geq 0.$$

Then  $\tau_m \rightarrow \infty$  as  $m \rightarrow \infty$  first for a deterministic initial value and then for an arbitrary one. Let  $\{x(t)\}$  be the process such that  $x(t) = x_m(t)$  for all  $0 \leq t < \tau_m$  and  $m \geq 1$ . It is easily seen that  $\{x(t)\}$  is a strong solution of (2.2). The uniqueness of solution follows by a similar localization argument.  $\square$

Now we give a simple criterion for the existence and uniqueness of the strong solution to (2.1). We shall assume the following localized conditions:

(2.c)  $x \mapsto b(x)$  is locally Lipschitz on  $\mathbb{R}_+$  and for each integer  $m \geq 1$  there is a constant  $K_m > 0$  and a non-negative function  $u \mapsto h_m(u)$  so that

$$|g_1(x, u) - g_1(y, u)| \leq K_m |x - y| h_m(u), \quad 0 \leq x, y \leq m, u \in U_2,$$

and

$$\int_{U_2} h_m(u) \mu_1(du) < \infty;$$

(2.d)  $x \mapsto \sigma(x)$  is locally Lipschitz on  $\mathbb{R}_+$  and for each integer  $m \geq 1$  there is a constant  $K_m > 0$  and a non-negative function  $u \mapsto f_m(u)$  so that

$$|g_0(x, u) - g_0(y, u)| \leq K_m |x - y| f_m(u), \quad 0 \leq x, y \leq m, u \in U_0$$

and

$$\int_{U_0} [f_m(u) \wedge f_m(u)^2] \mu_0(du) < \infty.$$

**Theorem 2.5** *Suppose that conditions (2.a,c,d) are satisfied. Then there is a unique non-negative strong solution to (2.1).*

*Proof.* We first note that (2.b) follows from (2.d). By Propositions 2.2 and 2.4 we only need to show there is a unique strong solution to (2.7). Let  $V_0 = \{u \in U_0 : f_m(u) \leq 1/K_m\}$  and  $V_2 = \{u \in U_2 : h_m(u) \leq m\}$ . Then (2.c,d) imply  $\mu_0(U_0 \setminus V_0) + \mu_1(U_2 \setminus V_2) < \infty$ . We consider the equation

$$\begin{aligned} x(t) &= x(0) + \int_0^t \sigma(x(s-) \wedge m) dB(s) + \int_0^t b(x(s-) \wedge m) ds \\ &\quad + \int_0^t \int_{U_0} [g_0(x(s-) \wedge m, u) \wedge m] \tilde{N}_0(ds, du) \\ &\quad - \int_0^t ds \int_{U_0 \setminus V_0} [g_0(x(s-) \wedge m, u) - g_0(x(s-) \wedge m, u) \wedge m] \mu_0(du) \\ &\quad + \int_0^t \int_{U_2} [g_1(x(s-) \wedge m, u) \wedge m] N_1(ds, du), \end{aligned} \tag{2.8}$$

which can be rewritten as a jump-type equation with compensated Poisson integrals over  $V_0$  and  $V_2$ , non-compensated Poisson integrals over  $U_0 \setminus V_0$  and  $U_2 \setminus V_2$ , and drift coefficient

$$\begin{aligned} x \mapsto d_m(x) &:= b(x \wedge m) + \int_{V_2} [g_1(x \wedge m, u) \wedge m] \mu_1(du) \\ &\quad - \int_{U_0 \setminus V_0} g_0(x \wedge m, u) \mu_0(du). \end{aligned}$$

Clearly, we can choose a common sequence of constants  $\{K_m\}$  for conditions (2.c) and (2.d). Then there is a new constant  $C_m > 0$  so that

$$\begin{aligned} |d_m(x) - d_m(y)| &\leq |b(x \wedge m) - b(y \wedge m)| + K_m |x - y| \int_{V_2} h_m(u) \mu_1(du) \\ &\quad + K_m |x - y| \int_{U_0 \setminus V_0} f_m(u) \mu_0(du) \\ &\leq |b(x \wedge m) - b(y \wedge m)| + K_m |x - y| \int_{V_2} h_m(u) \mu_1(du) \\ &\quad + K_m |x - y| \int_{U_0 \setminus V_0} f_m(u) \wedge [K_m f_m(u)^2] \mu_0(du) \\ &\leq C_m |x - y| \end{aligned}$$

and

$$\begin{aligned} &\int_{V_0} l_0(x \wedge m, y \wedge m, u)^2 \mu_0(du) + \int_{V_2} l_1(x \wedge m, y \wedge m, u)^2 \mu_1(du) \\ &\leq K_m^2 |x - y|^2 \left( \int_{V_0} f_m(u)^2 \mu_0(du) + \int_{V_2} h_m(u)^2 \mu_2(du) \right) \\ &\leq K_m^2 |x - y|^2 \left( \int_{V_0} [K_m^{-1} f_m(u)] \wedge f_m(u)^2 \mu_0(du) + \int_{V_2} m h_m(u) \mu_2(du) \right) \\ &\leq C_m |x - y|^2, \end{aligned}$$

where  $l_i(x, y, u) = g_i(x, u) - g_i(y, u)$ . By Propositions 2.1 and the classical result it is simple to see that (2.8) has a unique strong solution; see, e.g., Ikeda and Watanabe (1989, pp.244-245). On the other hand, from (2.d) we have  $g_0(x, u) \leq K_m x f_m(u)$  for  $0 \leq x \leq m$ . Then  $f_m(u) \leq 1/K_m$  implies  $g_0(x \wedge m, u) \leq m$ , and hence

$$b_m(x \wedge m) = b(x \wedge m) - \int_{U_0 \setminus V_0} [g_0(x \wedge m, u) - g_0(x \wedge m, u) \wedge m] \mu_0(du).$$

Therefore (2.8) is equivalent to (2.7), so the later also has a unique non-negative strong solution.  $\square$

### 3 Pathwise uniqueness

In this section, we prove some results on the pathwise uniqueness of solutions to (2.1) under non-Lipschitz conditions. Suppose that  $(\sigma, b, g_0, g_1, \mu_0, \mu_1)$  are given as in the last

section. Given a function  $f$  defined on a subset of  $\mathbb{R}$ , we note

$$\Delta_z f(x) = f(x+z) - f(x) \quad \text{and} \quad D_z f(x) = \Delta_z f(x) - f'(x)z \quad (3.1)$$

if the right hand sides are meaningful. In the sequel, we assume the drift coefficient of (2.1) is written as  $b(x) = b_1(x) - b_2(x)$ , where  $x \mapsto b_1(x)$  is continuous and  $x \mapsto b_2(x)$  is continuous and non-decreasing. We consider the following condition:

(3.a) For each integer  $m \geq 1$  there is a non-decreasing and concave function  $z \mapsto r_m(z)$  on  $\mathbb{R}_+$  such that  $\int_{0+} r_m(z)^{-1} dz = \infty$  and

$$|b_1(x) - b_1(y)| + \int_{U_2} |g_1(x, u) - g_1(y, u)| \mu_1(du) \leq r_m(|x - y|)$$

for all  $0 \leq x, y \leq m$ .

**Theorem 3.1** *Suppose that conditions (2.a,b) and (3.a) hold. Then the pathwise uniqueness of solutions holds for (2.2) if for each integer  $m \geq 1$  there is a sequence of non-negative and twice continuously differentiable functions  $\{\phi_k\}$  with the following properties:*

- (i)  $\phi_k(z) \rightarrow |z|$  non-decreasingly as  $k \rightarrow \infty$ ;
- (ii)  $0 \leq \phi'_k(z) \leq 1$  for  $z \geq 0$  and  $-1 \leq \phi'_k(z) \leq 0$  for  $z \leq 0$ ;
- (iii)  $\phi''_k(z) \geq 0$  for  $z \in \mathbb{R}$  and as  $k \rightarrow \infty$ ,

$$\phi''_k(x - y)[\sigma(x) - \sigma(y)]^2 \rightarrow 0$$

uniformly in  $0 \leq x, y \leq m$ ;

- (iv) as  $k \rightarrow \infty$ ,

$$\int_{U_0} D_{l_0(x,y,u)} \phi_k(x - y) \mu_0(du) \rightarrow 0$$

uniformly in  $0 \leq x, y \leq m$ , where  $l_0(x, y, u) = g_0(x, u) - g_0(y, u)$ .

*Proof.* Suppose that  $\{x_1(t)\}$  and  $\{x_2(t)\}$  are two solutions of (2.2) with deterministic initial values. Proposition 2.3 implies that  $t \mapsto \mathbf{E}[x_1(t)] + \mathbf{E}[x_2(t)]$  is locally bounded. Let  $\zeta(t) = x_1(t) - x_2(t)$  for  $t \geq 0$ . From (2.2) we have

$$\begin{aligned} \zeta(t) &= \zeta(0) + \int_0^t [\sigma(x_1(s-)) - \sigma(x_2(s-))] dB(s) \\ &\quad + \int_0^t \int_{U_0} [g_0(x_1(s-), u) - g_0(x_2(s-), u)] \tilde{N}_0(ds, du) \\ &\quad + \int_0^t [b(x_1(s-)) - b(x_2(s-))] ds \\ &\quad + \int_0^t \int_{U_2} [g_1(x_1(s-), u) - g_1(x_2(s-), u)] N_1(ds, du). \end{aligned} \quad (3.2)$$

Let  $\tau_m = \inf\{t \geq 0 : x_1(t) \geq m \text{ or } x_2(t) \geq m\}$  for  $m \geq 1$ . By (3.2) and Itô's formula,

$$\begin{aligned}
\phi_k(\zeta(t \wedge \tau_m)) &= \phi_k(\zeta(0)) + \int_0^{t \wedge \tau_m} \phi'_k(\zeta(s-)) [b(x_1(s-)) - b(x_2(s-))] ds \\
&\quad + \frac{1}{2} \int_0^{t \wedge \tau_m} \phi''_k(\zeta(s-)) [\sigma(x_1(s-)) - \sigma(x_2(s-))]^2 ds \\
&\quad + \int_0^{t \wedge \tau_m} \int_{U_2} \Delta_{l_1(s-, u)} \phi_k(\zeta(s-)) N_1(ds, du) \\
&\quad + \int_0^{t \wedge \tau_m} \int_{U_0} D_{l_0(s-, u)} \phi_k(\zeta(s-)) N_0(ds, du) + \text{mart.} \\
&= \phi_k(\zeta(0)) + \int_0^{t \wedge \tau_m} \phi'_k(\zeta(s-)) [b(x_1(s-)) - b(x_2(s-))] ds \\
&\quad + \frac{1}{2} \int_0^{t \wedge \tau_m} \phi''_k(\zeta(s-)) [\sigma(x_1(s-)) - \sigma(x_2(s-))]^2 ds \\
&\quad + \int_0^{t \wedge \tau_m} ds \int_{U_2} \Delta_{l_1(s-, u)} \phi_k(\zeta(s-)) \mu_1(du) \\
&\quad + \int_0^{t \wedge \tau_m} ds \int_{U_0} D_{l_0(s-, u)} \phi_k(\zeta(s-)) \mu_0(du) + \text{mart.}, \tag{3.3}
\end{aligned}$$

where  $l_i(s, u) = g_i(x_1(s), u) - g_i(x_2(s), u)$ . Since  $b(x) = b_1(x) - b_2(x)$  and  $x \mapsto b_2(x)$  is non-decreasing, by property (ii) we have

$$\begin{aligned}
\phi'_k(\zeta(s-)) [b(x_1(s-)) - b(x_2(s-))] &\leq \phi'_k(\zeta(s-)) [b_1(x_1(s-)) - b_1(x_2(s-))] \\
&\leq |b_1(x_1(s-)) - b_1(x_2(s-))|.
\end{aligned}$$

Observe also that

$$\int_{U_2} \Delta_{l_1(s-, u)} \phi_k(\zeta(s-)) \mu_1(du) \leq \int_{U_2} |g_1(x_1(s-), u) - g_1(x_2(s-), u)| \mu_1(du).$$

By condition (3.a), for any  $s \leq \tau_m$  the summation of the right hand sides of the above two inequalities is no larger than  $r_m(|\zeta(s-)|)$ . By properties (iii) and (iv) we have

$$\phi''_k(\zeta(s-)) [\sigma(x_1(s-)) - \sigma(x_2(s-))]^2 \rightarrow 0$$

and

$$\int_{U_0} D_{l_0(s-, u)} \phi_k(\zeta(s-)) \mu_0(du) \rightarrow 0$$

uniformly on the event  $\{s \leq \tau_m\}$ . Then we can take the expectation in (3.3) and let  $k \rightarrow \infty$  to get

$$\mathbf{E}[|\zeta(t \wedge \tau_m)|] \leq \mathbf{E}[|\zeta(0)|] + \mathbf{E}\left[\int_0^{t \wedge \tau_m} r_m(|\zeta(s-)|) ds\right].$$

Since  $\zeta(s-) < m$  for  $0 < s \leq \tau_m$ , we infer that  $t \mapsto \mathbf{E}[|\zeta(t \wedge \tau_m)|]$  is locally bounded. Note also that  $\zeta(s-) \neq \zeta(s)$  for at most countably many  $s \geq 0$ . Then the concaveness of  $x \mapsto r_m(x)$  implies

$$\begin{aligned} \mathbf{E}[|\zeta(t \wedge \tau_m)|] &\leq \mathbf{E}[|\zeta(0)|] + \int_0^t \mathbf{E}[r_m(|\zeta(s \wedge \tau_m)|)] ds \\ &\leq \mathbf{E}[|\zeta(0)|] + \int_0^t r_m(\mathbf{E}[|\zeta(s \wedge \tau_m)|]) ds. \end{aligned}$$

If  $x_1(0) = x_2(0)$ , we can use a standard argument to show  $\mathbf{E}[|\zeta(t \wedge \tau_m)|] = 0$  for all  $t \geq 0$ ; see e.g. Ikeda and Watanabe (1989, p.184). Since  $\tau_m \rightarrow \infty$  as  $m \rightarrow \infty$  by Proposition 2.3, the right continuity of  $t \mapsto \zeta(t)$  implies  $\mathbf{P}\{\zeta(t) = 0 \text{ for all } t \geq 0\} = 1$ .  $\square$

We remark that the proof of Theorem 3.1 given above uses essentially the monotonicity of  $x \mapsto b_2(x)$ . A similar condition has been used in Rozovsky (1980) for continuous-type equations. The reader may refer Cherny and Engelbert (2005) for a thorough treatments of continuous-type stochastic differential equations with singular coefficients. For the statements of the next two theorems we introduce the following conditions:

(3.b) For every fixed  $u \in U_0$  the function  $x \mapsto g_0(x, u)$  is non-decreasing, and for each integer  $m \geq 1$  there is a non-negative and non-decreasing function  $z \mapsto \rho_m(z)$  on  $\mathbb{R}_+$  so that  $\int_{0+} \rho_m(z)^{-2} dz = \infty$  and

$$|\sigma(x) - \sigma(y)|^2 + \int_{U_0} [l_0(x, y, u) \wedge l_0(x, y, u)^2] \mu_0(du) \leq \rho_m(|x - y|)^2$$

for all  $0 \leq x, y \leq m$ , where  $l_0(x, y, u) = g_0(x, u) - g_0(y, u)$ ;

(3.c) For every fixed  $u \in U_0$  the function  $x \mapsto g_0(x, u)$  is non-decreasing, and for each integer  $m \geq 1$  there is a non-negative and non-decreasing function  $z \mapsto \rho_m(z)$  on  $\mathbb{R}_+$  so that  $\int_{0+} \rho_m(z)^{-2} dz = \infty$ ,

$$|\sigma(x) - \sigma(y)| \leq \rho_m(|x - y|) \quad \text{and} \quad |g_0(x, u) - g_0(y, u)| \leq \rho_m(|x - y|) f_m(u)$$

for all  $0 \leq x, y \leq m$  and  $u \in U_0$ , where  $u \mapsto f_m(u)$  is a non-negative function on  $U_0$  satisfying

$$\int_{U_0} [f_m(u) \wedge f_m(u)^2] \mu_0(du) < \infty.$$

**Theorem 3.2** *Suppose that conditions (2.a) and (3.a,b) are satisfied. Then the pathwise uniqueness of solutions holds for (2.1).*

*Proof.* By Proposition 2.2 it suffices to prove the pathwise uniqueness for (2.2). For each integer  $m \geq 1$  we shall construct a sequence of functions  $\{\phi_k\}$  that satisfies the properties required in Theorem 3.1. Let  $a_0 = 1$  and choose  $a_k \rightarrow 0$  decreasingly so that  $\int_{a_k}^{a_{k-1}} \rho_m(z)^{-2} dz = k$  for  $k \geq 1$ . Let  $x \mapsto \psi_k(x)$  be a non-negative continuous function

on  $\mathbb{R}$  which has support in  $(a_k, a_{k-1})$  and satisfies  $\int_{a_k}^{a_{k-1}} \psi_k(x) dx = 1$  and  $0 \leq \psi_k(x) \leq 2k^{-1} \rho_m(x)^{-2}$  for  $a_k < x < a_{k-1}$ . For each  $k \geq 1$  we define the non-negative and twice continuously differentiable function

$$\phi_k(z) = \int_0^{|z|} dy \int_0^y \psi_k(x) dx, \quad z \in \mathbb{R}. \quad (3.4)$$

Clearly, the sequence  $\{\phi_k\}$  satisfies properties (i) and (ii) in Theorem 3.1. By condition (3.b) we have

$$\phi_k''(x-y)[\sigma(x) - \sigma(y)]^2 \leq \psi_k(|x-y|) \rho_m(|x-y|)^2 \leq 2/k$$

for  $0 \leq x, y \leq m$ . Thus  $\{\phi_k\}$  also satisfies property (iii). By Taylor's expansion,

$$D_h \phi_k(\zeta) = h^2 \int_0^1 \phi_k''(\zeta + th)(1-t) dt = h^2 \int_0^1 \psi_k(|\zeta + th|)(1-t) dt. \quad (3.5)$$

Consequently, the monotonicity of  $z \mapsto \rho_m(z)$  implies

$$D_h \phi_k(\zeta) \leq 2k^{-1} h^2 \int_0^1 \rho_m(|\zeta + th|)^{-2} (1-t) dt \leq k^{-1} h^2 \rho_m(|\zeta|)^{-2} \quad (3.6)$$

if  $\zeta h \geq 0$ . Observe also that

$$D_h \phi_k(\zeta) = \Delta_h \phi_k(\zeta) - \phi_k'(\zeta) h \leq \Delta_h \phi_k(\zeta) \leq |h| \quad (3.7)$$

if  $\zeta h \geq 0$ . Since  $x \mapsto g_0(x, u)$  is non-decreasing, for  $0 \leq x, y \leq m$  and  $n \geq 1$  we can use (3.6), (3.7) and (3.b) to get

$$\begin{aligned} \int_{U_0} D_{l_0(x,y,u)} \phi_k(x-y) \mu_0(du) &\leq \frac{1}{k \rho_m(|x-y|)^2} \int_{U_0} l_0(x, y, u)^2 \mathbf{1}_{\{|l_0(x,y,u)| \leq n\}} \mu_0(du) \\ &\quad + \int_{U_0} |l_0(x, y, u)| \mathbf{1}_{\{|l_0(x,y,u)| > n\}} \mu_0(du) \\ &\leq \frac{n}{k \rho_m(|x-y|)^2} \int_{U_0} |l_0(x, y, u)| \wedge l_0(x, y, u)^2 \mu_0(du) \\ &\quad + \int_{U_0} g_0(m, u) \mathbf{1}_{\{g_0(m,u) > n\}} \mu_0(du) \\ &\leq \frac{n}{k} + \int_{U_0} g_0(m, u) \mathbf{1}_{\{g_0(m,u) > n\}} \mu_0(du). \end{aligned}$$

Since (3.b) implies (2.b), we see that  $\{\phi_k\}$  satisfies property (iv) in Theorem 3.1. That proves the pathwise uniqueness for (2.2).  $\square$

**Theorem 3.3** *Suppose that conditions (2.a) and (3.a,c) are satisfied. Then the pathwise uniqueness for (2.1) holds.*

*Proof.* The first part of this proof is identical with that of Theorem 3.2. Under condition (3.c) we can use (3.6) and (3.7) to see

$$D_{l_0(x,y,u)}\phi_k(x-y) \leq k^{-1}l_0(x,y,u)^2\rho_m(|x-y|)^{-2} \leq k^{-1}f_m(u)^2$$

and

$$D_{l_0(x,y,u)}\phi_k(x-y) \leq |l_0(x,y,u)| \leq \rho_m(|x-y|)f_m(u) \leq \rho_m(m)f_m(u)$$

for all  $0 \leq x, y \leq m$  and  $u \in U_0$ . By (3.c) for any  $n \geq 1$  we have

$$\begin{aligned} \int_{U_0} D_{l_0(x,y,u)}\phi_k(x-y)\mu_0(du) &\leq \frac{1}{k} \int_{U_0} f_m(u)^2 1_{\{f_m(u) \leq n\}}\mu_0(du) \\ &\quad + \rho_m(m) \int_{U_0} f_m(u) 1_{\{f_m(u) > n\}}\mu_0(du) \\ &\leq \frac{n}{k} \int_{U_0} [f_m(u) \wedge f_m(u)^2]\mu_0(du) \\ &\quad + \rho_m(m) \int_{U_0} f_m(u) 1_{\{f_m(u) > n\}}\mu_0(du). \end{aligned}$$

By letting  $k \rightarrow \infty$  and  $n \rightarrow \infty$  we see that  $\{\phi_k\}$  satisfies property (iv) in Theorem 3.1. That proves the pathwise uniqueness first for (2.2) and then for (2.1) by an application of Proposition 2.2.  $\square$

## 4 Weak solutions

In this section, we prove a result on the existence of weak solutions of (2.1). The result will be used in the study of strong solutions in the next section. As for continuous-type equations, this is closely related with the corresponding martingale problems. Let us define the Lévy-type operator  $A$  from  $C^2(\mathbb{R}_+)$  to  $C(\mathbb{R}_+)$  by

$$\begin{aligned} Af(x) &= \frac{1}{2}\sigma(x)^2 f''(x) + b(x)f'(x) + \int_{U_0} D_{g_0(x,u)}f(x)\mu_0(du) \\ &\quad + \int_{U_1} \Delta_{g_1(x,u)}f(x)\mu_1(du), \end{aligned} \tag{4.1}$$

where  $(\sigma, b, g_0, g_1, \mu_0, \mu_1)$  are given as in Section 2. To simplify the statements we introduce the following conditions:

(4.a) There is a constant  $K \geq 0$  such that

$$\begin{aligned} \sup_{x \geq 0} [|b(x)| + \sigma(x)^2] + \sup_{x \geq 0} \int_{U_0} g_0(x,u)^2 \mu_0(du) \\ + \sup_{x \geq 0} \int_{U_1} [|g_1(x,u)| \vee g_1(x,u)^2] \mu_1(du) \leq K; \end{aligned}$$

(4.b)  $x \mapsto g_0(x, \cdot)$  is non-decreasing and continuous in  $L^2(\mu_0)$  and there is a non-decreasing sequence  $\{V_n\}$  of Borel subsets of  $U_0$  so that  $\cup_{n=1}^{\infty} V_n = U_0$  and

$$\mu_0(V_n) < \infty, \quad \int_{V_n} g_0(x, u) \mu_0(du) < \infty$$

for every  $n \geq 1$  and  $x \geq 0$ ;

(4.c)  $x \mapsto g_1(x, \cdot)$  is continuous in  $L^1(\mu_1)$ .

**Proposition 4.1** *If condition (4.a) holds, for any solution  $\{x(t)\}$  of (2.1) we have*

$$\mathbf{E} \left[ \sup_{0 \leq s \leq t} x(s)^2 \right] \leq 6\mathbf{E}[x(0)^2] + 24Kt + 6K^2t^2, \quad t \geq 0. \quad (4.2)$$

*Proof.* We first write (2.1) into

$$\begin{aligned} x(t) &= x(0) + \int_0^t \sigma(x(s-)) dB(s) + \int_0^t \int_{U_0} g_0(x(s-), u) \tilde{N}_0(ds, du) \\ &\quad + \int_0^t b(x(s-)) ds + \int_0^t \int_{U_1} g_1(x(s-), u) \tilde{N}_1(ds, du) \\ &\quad + \int_0^t ds \int_{U_1} g_1(x(s-), u) \mu_1(du). \end{aligned} \quad (4.3)$$

By applying Doob's inequality to the martingale terms in (4.3) we have

$$\begin{aligned} \mathbf{E} \left[ \sup_{0 \leq s \leq t} x(s)^2 \right] &\leq 6\mathbf{E}[x(0)^2] + 24\mathbf{E} \left[ \int_0^t \sigma(x(s-))^2 ds \right] + 6\mathbf{E} \left[ \left( \int_0^t |b(x(s-))| ds \right)^2 \right] \\ &\quad + 24\mathbf{E} \left[ \int_0^t ds \int_{U_0} g_0(x(s-), u)^2 \mu_0(du) \right] \\ &\quad + 24\mathbf{E} \left[ \int_0^t ds \int_{U_1} g_1(x(s-), u)^2 \mu_1(du) \right] \\ &\quad + 6\mathbf{E} \left[ \left( \int_0^t ds \int_{U_1} |g_1(x(s-), u)| \mu_1(du) \right)^2 \right] \\ &\leq 6\mathbf{E}[x(0)^2] + 24Kt + 6t\mathbf{E} \left[ \int_0^t b(x(s-))^2 ds \right] \\ &\quad + 6t\mathbf{E} \left[ \int_0^t \left( \int_{U_1} |g_1(x(s-), u)| \mu_1(du) \right)^2 ds \right] \\ &\leq 6\mathbf{E}[x(0)^2] + 24Kt + 6K^2t^2. \end{aligned}$$

That proves (4.2). □

**Proposition 4.2** *Suppose that condition (4.a) holds. Then a non-negative càdlàg process  $\{x(t)\}$  is a weak solution of (2.1) if and only if for every  $f \in C^2(\mathbb{R}_+)$ ,*

$$f(x(t)) - f(x(0)) - \int_0^t Af(x(s)) ds, \quad t \geq 0 \quad (4.4)$$

*is a martingale.*

*Proof.* Without loss of generality, we assume  $x(0)$  is deterministic. If  $\{x(t)\}$  is a solution of (2.1), by Itô's formula it is easy to see that (4.4) is a bounded martingale. Conversely, suppose that (4.4) is a martingale for every  $f \in C^2(\mathbb{R}_+)$ . By a standard stopping time argument, we have

$$x(t) = x(0) + \int_0^t b(x(s-))ds + \int_0^t ds \int_{U_1} g_1(x(s-), u)\mu_1(du) + M(t) \quad (4.5)$$

for a square-integrable martingale  $\{M(t)\}$ . Let  $N(ds, dz)$  be the optional random measure on  $[0, \infty) \times \mathbb{R}$  defined by

$$N(ds, dz) = \sum_{s>0} 1_{\{\Delta x(s) \neq 0\}} \delta_{(s, \Delta x(s))}(ds, dz),$$

where  $\Delta x(s) = x(s) - x(s-)$ . Let  $\hat{N}(ds, dz)$  be the predictable compensator of  $N(ds, dz)$  and let  $\tilde{N}(ds, dz)$  denote the compensated random measure. By (4.5) and Dellacherie and Meyer (1982, p.376) we have

$$x(t) = x(0) + \int_0^t b(x(s-))ds + \int_0^t ds \int_{U_1} g_1(x(s-), u)\mu_1(du) + M_c(t) + M_d(t), \quad (4.6)$$

where  $\{M_c(t)\}$  is a continuous martingale and

$$M_d(t) = \int_0^t \int_{\mathbb{R}} z \tilde{N}(ds, dz)$$

is a purely discontinuous martingale. Let  $\{C(t)\}$  denote the quadratic variation process of  $\{M_c(t)\}$ . By (4.6) and Itô's formula we have

$$\begin{aligned} f(x(t)) &= f(x(0)) + \int_0^t f'(x(s-))b(x(s-))ds + \frac{1}{2} \int_0^t f''(x(s-))dC(s) \\ &\quad + \int_0^t f'(x(s-))ds \int_{U_1} g_1(x(s-), u)\mu_1(du) \\ &\quad + \int_0^t \int_{\mathbb{R}} D_z f(x(s-))\hat{N}(ds, dz) + \text{mart.} \end{aligned} \quad (4.7)$$

In view of (4.4) and (4.7), the uniqueness of canonical decompositions of semi-martingales implies  $dC(s) = \sigma(x(s))^2 ds$  and

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} F(s, z)\hat{N}(ds, dz) &= \int_0^t ds \int_{U_0} F(s, g_0(x(s-), u))\mu_0(du) \\ &\quad + \int_0^t ds \int_{U_1} F(s, g_1(x(s-), u))\mu_1(du) \end{aligned}$$

for any non-negative Borel function  $F$  on  $\mathbb{R}_+ \times \mathbb{R}$ . Then we obtain the equation (2.1) on an extension of the probability space by applying martingale representation theorems to (4.6); see, e.g., Ikeda and Watanabe (1989, p.84 and p.93).  $\square$

For simplicity we assume the initial variable  $x(0)$  is deterministic in the sequel of this section. To prove the existence of a weak solution of (2.1) let  $\{\varepsilon_n\}$  be a sequence of strictly positive numbers decreasing to zero. If condition (4.b) holds, for every  $n \geq 0$ ,

$$x \mapsto \int_{V_n} g_0(x \wedge n, u) \mu_0(du)$$

is a bounded continuous non-decreasing function on  $\mathbb{R}_+$ . By the result on continuous-type stochastic equations, there is a weak solution to

$$\begin{aligned} x(t) = & x(0) + \int_0^t \sigma(x(s-)) dB(s) + \int_0^t b(x(s-)) ds \\ & - \int_0^t ds \int_{V_n} g_0(x(s-) \wedge n, u) \mu_0(du); \end{aligned} \quad (4.8)$$

see, e.g., Ikeda and Watanabe (1989, p.169). By Theorem 3.2 the pathwise uniqueness holds for (4.8), so the equation has a unique strong solution. Let  $\{W_n\}$  be a non-decreasing sequence of Borel subsets of  $U_1$  so that  $\cup_{n=1}^\infty W_n = U_1$  and  $\mu_1(W_n) < \infty$  for every  $n \geq 1$ . Following the proof of Proposition 2.2 one can see for every integer  $n \geq 1$  there is a weak solution  $\{x_n(t) : t \geq 0\}$  to

$$\begin{aligned} x(t) = & x(0) + \int_0^t \sigma(x(s-)) dB(s) + \int_0^t \int_{V_n} g_0(x(s-) \wedge n, u) \tilde{N}_0(ds, du) \\ & + \int_0^t b(x(s-)) ds + \int_0^t \int_{W_n} g_1(x(s-), u) N_1(ds, du). \end{aligned} \quad (4.9)$$

**Lemma 4.3** *Under conditions (4.a,b), the sequence  $\{x_n(t) : t \geq 0\}$  is tight in the Skorohod space  $D([0, \infty), \mathbb{R}_+)$ .*

*Proof.* Since  $x(0)$  is deterministic, by Proposition 4.1 it is easy to see that

$$t \mapsto C_t := \sup_{n \geq 1} \mathbf{E} \left[ \sup_{0 \leq s \leq t} x_n(s)^2 \right]$$

is locally bounded. Then for every fixed  $t \geq 0$  the sequence of random variables  $x_n(t)$  is tight. Moreover, in view of (4.3), if  $\{\tau_n\}$  is a sequence of stopping times bounded above by  $T \geq 0$ , we have

$$\begin{aligned} \mathbf{E}[|x_n(\tau_n + t) - x_n(\tau_n)|^2] & \leq 5\mathbf{E} \left[ \int_0^t \sigma(x_n(\tau_n + s))^2 ds + t \int_0^t b(x_n(\tau_n + s))^2 ds \right] \\ & \quad + 5\mathbf{E} \left[ \int_0^t ds \int_{U_0} g_0(x_n(\tau_n + s) \wedge n, u)^2 \mu_0(du) \right] \\ & \quad + 5\mathbf{E} \left[ \int_0^t ds \int_{U_1} g_1(x_n(\tau_n + s), u)^2 \mu_1(du) \right] \\ & \quad + 5t\mathbf{E} \left[ \int_0^t \left( \int_{U_1} g_1(x_n(\tau_n + s), u) \mu_1(du) \right)^2 ds \right] \\ & \leq 5Kt(1 + Kt), \end{aligned}$$

where the last inequality follows by (4.a). Consequently, as  $t \rightarrow 0$ ,

$$\sup_{n \geq 1} \mathbf{E}[|x_n(\tau_n + t) - x_n(\tau_n)|^2] \rightarrow 0.$$

Then  $\{x_n(t) : t \geq 0\}$  is tight in  $D([0, \infty), \mathbb{R}_+)$  by the criterion of Aldous (1978); see also Ethier and Kurtz (1986, pp.137-138).  $\square$

**Theorem 4.4** *Under the conditions (4.a,b,c), there exists a non-negative weak solution to (2.1).*

*Proof.* Let  $\{x_n(t) : t \geq 0\}$  be the unique non-negative strong solution of (4.9). By Proposition 4.2, for every  $f \in C^2(\mathbb{R}_+)$ ,

$$f(x_n(t)) - f(x_n(0)) - \int_0^t A_n f(x_n(s)) ds, \quad t \geq 0 \quad (4.10)$$

is a bounded martingale, where

$$\begin{aligned} A_n f(x) &= \frac{1}{2} \sigma(x)^2 f''(x) + \int_{V_n} D_{g_0(x \wedge n, u)} f(x) \mu_0(du) \\ &\quad + b(x) f'(x) + \int_{W_n} \Delta_{g_1(x, u)} f(x) \mu_1(du). \end{aligned}$$

By Lemma 4.3 there is a subsequence  $\{x_{n_k}(t) : t \geq 0\}$  of  $\{x_n(t) : t \geq 0\}$  that converges to some process  $\{x(t) : t \geq 0\}$  in distribution on  $D([0, \infty), \mathbb{R}_+)$ . By Skorokhod representation we may assume  $\{x_{n_k}(t) : t \geq 0\}$  converges to  $\{x(t) : t \geq 0\}$  almost surely in  $D([0, \infty), \mathbb{R}_+)$ . Let  $D(x) := \{t > 0 : \mathbf{P}\{x(t-) = x(t)\} = 1\}$ . Then the set  $[0, \infty) \setminus D(x)$  is at most countable; see, e.g., Ethier and Kurtz (1986, p.131). It follows that  $\lim_{k \rightarrow \infty} x_{n_k}(t) = x(t)$  almost surely for every  $t \in D(x)$ ; see, e.g., Ethier and Kurtz (1986, p.118). Using conditions (4.a,b,c) it is elementary to show that (4.4) is a bounded martingale. Then the theorem follows by another application of Proposition 4.2.  $\square$

As for continuous-type equations, one can derive some weak uniqueness results for (2.1) from the results on pathwise uniqueness given in the last section. The reader may refer to Stroock (1975) for a systematic treatment of existence and uniqueness of solutions to martingale problems associated with Lévy-type generators. However, it seems the results of Stroock (1975) do not yield immediately the results on non-negative solutions of (2.1).

## 5 Strong solutions

In this section, we give some criterion on the existence and uniqueness of the strong solution of equation (2.1). We also prove some properties of the solution, continuous dependence on the initial value and comparison properties. Recall that  $b(x) = b_1(x) - b_2(x)$ , where  $x \mapsto b_1(x)$  is continuous and  $x \mapsto b_2(x)$  is continuous and non-decreasing. Let us consider some further localizations of (3.a,b,c) as follows:

(5.a) For each integer  $m \geq 1$  there is a non-decreasing and concave function  $z \mapsto r_m(z)$  on  $\mathbb{R}_+$  such that  $\int_{0+} r_m(z)^{-1} dz = \infty$  and

$$|b_1(x) - b_1(y)| + \int_{U_2} |g_1(x, u) \wedge m - g_1(y, u) \wedge m| \mu_1(du) \leq r_m(|x - y|)$$

for all  $0 \leq x, y \leq m$ ;

(5.b) There is a non-decreasing sequence  $\{V_n\}$  of Borel subsets of  $U_0$  so that  $\cup_{n=1}^{\infty} V_n = U_0$  and

$$\mu_0(V_n) < \infty, \quad \int_{V_n} |g_0(x, u)| \mu_0(du) < \infty$$

for every  $n \geq 1$  and  $x \geq 0$ ;

(5.c) For each  $u \in U_0$  the function  $x \mapsto g_0(x, u)$  is non-decreasing, and for each integer  $m \geq 1$  there is a non-negative and non-decreasing function  $z \mapsto \rho_m(z)$  on  $\mathbb{R}_+$  so that  $\int_{0+} \rho_m(z)^{-2} dz = \infty$  and

$$|\sigma(x) - \sigma(y)|^2 + \int_{U_0} |g_0(x, u) \wedge m - g_0(y, u) \wedge m|^2 \mu_0(du) \leq \rho_m(|x - y|)^2$$

for all  $0 \leq x, y \leq m$ ;

(5.d) For each  $u \in U_0$  the function  $x \mapsto g_0(x, u)$  is non-decreasing, and for each integer  $m \geq 1$  there is a non-negative and non-decreasing function  $z \mapsto \rho_m(z)$  on  $\mathbb{R}_+$  so that  $\int_{0+} \rho_m(z)^{-2} dz = \infty$ ,  $|\sigma(x) - \sigma(y)| \leq \rho_m(|x - y|)$  and

$$|g_0(x, u) \wedge m - g_0(y, u) \wedge m| \leq \rho_m(|x - y|) f_m(u)$$

for all  $0 \leq x, y \leq m$  and  $u \in U_0$ , where  $u \mapsto f_m(u)$  is a non-negative function on  $U_0$  satisfying

$$\int_{U_0} [f_m(u) \wedge f_m(u)^2] \mu_0(du) < \infty.$$

**Theorem 5.1** *Suppose that conditions (2.a,b) and (5.a,b,c) are satisfied. Then there exists a unique non-negative strong solution to (2.1).*

*Proof.* By Propositions 2.2 and 2.4 we only need to show that (2.7) has a unique strong solution. Since  $x \mapsto g_0(x, u)$  is non-decreasing, so is

$$x \mapsto \beta_m(x) := \int_{U_0} [g_0(x, u) - g_0(x, u) \wedge m] \mu_0(du)$$

for every  $m \geq 1$ . One can use (2.b) and (5.c) to show that  $x \mapsto \beta_m(x)$  is continuous. By conditions (5.a,c) it is easy to see that  $x \mapsto g_0(x, \cdot) \wedge m$  is continuous in  $L^2(\mu_0)$  and  $x \mapsto g_1(x, \cdot) \wedge m$  is continuous in  $L^1(\mu_1)$ . Thus the ingredients of (2.7) satisfy conditions (2.a), (3.a,b) and (4.a,b,c). By Theorem 4.4 we conclude that (2.7) has a non-negative weak solution. The pathwise uniqueness of the solution follows from Theorems 3.2. Then (2.7) has a unique strong solution.  $\square$

**Corollary 5.2** *There exists a unique non-negative strong solution to (1.7).*

*Proof.* This follows by Theorem 5.1 applied with  $U_0 = (0, \infty)^2$ ,  $U_1 = (0, \infty)$  and  $U_2 = (0, 1]$ .  $\square$

**Theorem 5.3** *Suppose that conditions (2.a,b) and (5.a,d) are satisfied. Then there exists a unique non-negative strong solution to (2.1).*

*Proof.* Clearly, we may assume  $\{f_m\}$  is a non-decreasing sequence in (5.d). Let  $V_n := \{u \in U_0 : f_n(u) > 1/n\}$  and  $V_0 = \cup_{n=1}^{\infty} V_n$ . It is easy to see that  $g_0(x, u) = 0$  for every  $x \geq 0$  and  $u \in U_0 \setminus V_0$ . Then we can replace  $U_0$  by  $V_0$  in (2.1) so that condition (5.b) is satisfied. Based on the pathwise uniqueness stated in Theorem 3.3, the remaining arguments are similar to those in the proof of Theorem 5.1.  $\square$

It is easy to see that if (2.1) does have a unique strong solution, the solution is a strong Markov process with generator  $A$  defined by (4.1). In particular, the solution of (1.7) is a CBI-process with generator given by (1.5). In the special case where  $z\nu_1(dz)$  is a finite measure on  $(0, \infty)$ , the existence and uniqueness of the non-negative strong solution to (1.7) have been established in Dawson and Li (2006); see also Bertoin and Le Gall (2006).

We next prove some properties of the strong solutions of (2.1). In the following two theorems, we can replace conditions (5.b,c) by (5.d).

**Theorem 5.4** *Suppose that (2.a,b) and (5.a,b,c) hold with  $U_2 = U_1$  and the modulus functions  $r_m(z) \equiv r(z)$  and  $\rho_m(z) \equiv \rho(z)$  are independent of  $m \geq 1$ . For each integer  $n \geq 0$  let  $\{x_n(t)\}$  be a solution of (2.1) and assume  $\sup_{n \geq 0} \mathbf{E}[x_n(0)] < \infty$ . If  $\lim_{n \rightarrow \infty} \mathbf{E}[|x_n(0) - x_0(0)|] = 0$ , then*

$$\lim_{n \rightarrow \infty} \sup_{0 \leq s \leq t} \mathbf{E}[|x_n(s) - x_0(s)|] = 0, \quad t \geq 0. \quad (5.1)$$

*Proof.* Since (5.a,c) hold with the modulus functions independent of  $m \geq 1$ , by Fatou's lemma it is easy to show that (3.a,b) also hold with universal modulus functions. In particular, from (5.c) we get

$$|\sigma(x) - \sigma(y)|^2 + \int_{U_0} |g_0(x, u) - g_0(y, u)|^2 \mu_0(du) \leq \rho(|x - y|)^2. \quad (5.2)$$

Let  $\zeta_n(t) = x_n(t) - x_0(t)$  for  $t \geq 0$ . We then fix  $n \geq 1$  and let  $\tau_m = \inf\{t \geq 0 : x_0(t) \geq m \text{ or } x_n(t) \geq m\}$ . By Proposition 2.3 it is easy to see that  $t \mapsto \mathbf{E}[|\zeta_n(t)|]$  is uniformly bounded on each bounded interval. By the calculations in the proof of Theorem 3.1 we obtain

$$\begin{aligned} \mathbf{E}[|\zeta_n(t \wedge \tau_m)|] &\leq \mathbf{E}[|\zeta_n(0)|] + \mathbf{E}\left[\int_0^{t \wedge \tau_m} r(|\zeta_n(s-)|) ds\right] \\ &\leq \mathbf{E}[|\zeta_n(0)|] + \mathbf{E}\left[\int_0^t r(|\zeta_n(s-)|) ds\right] \\ &\leq \mathbf{E}[|\zeta_n(0)|] + \int_0^t r(\mathbf{E}[|\zeta_n(s)|]) ds. \end{aligned}$$

Since  $\tau_m \rightarrow \infty$  as  $m \rightarrow \infty$ , an application of Fatou's lemma gives

$$\mathbf{E}[|\zeta_n(t)|] \leq \mathbf{E}[|\zeta_n(0)|] + \int_0^t r(\mathbf{E}[|\zeta_n(s)|])ds.$$

For  $n \geq 1$  and  $t \geq 0$  let  $R_n(t) = \sup_{m \geq n} \sup_{0 \leq s \leq t} \mathbf{E}[|\zeta_m(s)|]$ . By Proposition 2.3 the sequence  $\{R_n(t)\}$  is uniformly bounded on each bounded interval. Then we use the monotonicity of  $z \mapsto r(z)$  to get

$$R_n(t) \leq R_n(0) + \int_0^t r(R_n(s))ds, \quad t \geq 0.$$

By letting  $n \rightarrow \infty$  we obtain

$$\lim_{n \rightarrow \infty} R_n(t) \leq \int_0^t r\left(\lim_{n \rightarrow \infty} R_n(s)\right)ds, \quad t \geq 0.$$

Thus  $\lim_{n \rightarrow \infty} R_n(t) = 0$  for every  $t \geq 0$ , as desired.  $\square$

**Theorem 5.5** *Suppose that (2.a,b) and (5.a,b,c) hold with  $U_2 = U_1$  and the modulus function  $r_m(x) \equiv r(x)$  is independent of  $m \geq 1$ . In addition, we assume  $x \mapsto x + g_1(x, u)$  is non-decreasing for every  $u \in U_1$ . If  $\{x_1(t)\}$  and  $\{x_2(t)\}$  are non-negative solutions of (2.1) satisfying  $\mathbf{P}\{x_1(0) \leq x_2(0)\} = 1$ , then we have  $\mathbf{P}\{x_1(t) \leq x_2(t) \text{ for all } t \geq 0\} = 1$ .*

*Proof.* The following arguments are similar to those in the proofs of Theorems 3.1 and 3.2. Let  $\zeta(t) = x_1(t) - x_2(t)$  and  $\tau_m = \inf\{t \geq 0 : x_1(t) \geq m \text{ or } x_2(t) \geq m\}$ . Instead of (3.4), for each  $k \geq 1$  we now define

$$\phi_k(z) = \int_0^z dy \int_0^y \psi_k(x)dx, \quad z \in \mathbb{R}. \quad (5.3)$$

Then  $\phi_k(z) \rightarrow z^+ := 0 \vee z$  non-decreasingly as  $k \rightarrow \infty$ . Recall that  $l_i(s, u) = g_i(x_1(s), u) - g_i(x_2(s), u)$  for  $i = 0, 1$ . Since  $\phi_k(z) = 0$  for  $z \leq 0$  and  $x \mapsto x + g_1(x, u)$  is non-decreasing, for  $\zeta(s-) \leq 0$  we have  $\zeta(s-) + l_1(s-, u) \leq 0$  and hence  $\Delta_{l_1(s-, u)}\phi_k(\zeta(s-)) = 0$ . By (3.2) and Itô's formula,

$$\begin{aligned} \phi_k(\zeta(t \wedge \tau_m)) &= \int_0^{t \wedge \tau_m} \phi'_k(\zeta(s-)) [b(x_1(s-)) - b(x_2(s-))] 1_{\{\zeta(s-) > 0\}} ds \\ &\quad + \frac{1}{2} \int_0^{t \wedge \tau_m} \phi''_k(\zeta(s-)) [\sigma(x_1(s-)) - \sigma(x_2(s-))]^2 ds \\ &\quad + \int_0^{t \wedge \tau_m} \int_{U_1} \Delta_{l_1(s-, u)} \phi_k(\zeta(s-)) 1_{\{\zeta(s-) > 0\}} N_1(ds, du) \\ &\quad + \int_0^{t \wedge \tau_m} \int_{U_0} D_{l_0(s-, u)} \phi_k(\zeta(s-)) N_0(ds, du) + \text{mart.} \\ &= \int_0^{t \wedge \tau_m} \phi'_k(\zeta(s-)) [b(x_1(s-)) - b(x_2(s-))] 1_{\{\zeta(s-) > 0\}} ds \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2} \int_0^{t \wedge \tau_m} \phi_k''(\zeta(s-)) [\sigma(x_1(s-)) - \sigma(x_2(s-))]^2 ds \\
& + \int_0^{t \wedge \tau_m} ds \int_{U_1} \Delta_{l_1(s-, u)} \phi_k(\zeta(s-)) 1_{\{\zeta(s-) > 0\}} \mu_1(du) \\
& + \int_0^{t \wedge \tau_m} ds \int_{U_0} D_{l_0(s-, u)} \phi_k(\zeta(s-)) \mu_0(du) + \text{mart.}
\end{aligned}$$

(If  $x \mapsto x + g_1(x, u)$  were allowed to decrease, we could not insert the indicator  $1_{\{\zeta(s-) > 0\}}$  into the integral with respect to  $N_1(ds, du)$ .) From the above equation and the estimates in the proof of Theorem 3.2 we obtain

$$\mathbf{E}[\zeta(t \wedge \tau_m)^+] \leq \int_0^t \mathbf{E}[r(\zeta(s \wedge \tau_m)^+)] ds \leq \int_0^t r(\mathbf{E}[\zeta(s \wedge \tau_m)^+]) ds.$$

Then  $\mathbf{E}[\zeta(t \wedge \tau_m)^+] = 0$  for all  $t \geq 0$ . Since  $\tau_m \rightarrow \infty$  as  $m \rightarrow \infty$ , that proves the desired comparison result.  $\square$

**Example.** The monotonicity of  $x \mapsto g_1(x, u)$  is necessary to assure the comparison property in Theorem 5.5. To see this, let  $g_1(x, u) = (1 - 2x)^+$  and let  $N_1(ds, du)$  be a Poisson random measure on  $(0, \infty) \times (0, 1]$  with intensity  $dsdu$ . We consider the stochastic equation

$$x(t) = x(0) + \int_0^t \int_0^1 g(x(s-), u) N_1(ds, du), \quad t \geq 0. \quad (5.4)$$

Let  $\tau = \inf\{t \geq 0 : N_1((0, t] \times (0, 1]) = 1\}$ . It is easy to see that

$$x_1(t) = 1_{\{t \geq \tau\}} \quad \text{and} \quad x_2(t) = 1/2, \quad t \geq 0,$$

defines two solutions of (5.4).

We next discuss the existence and uniqueness of the solution to a special type of stochastic equations of non-negative processes with non-negative jumps. In the sequel of this section, we suppose that

- $x \mapsto \sigma(x)$  and  $x \mapsto b(x)$  are continuous functions on  $\mathbb{R}_+$  satisfying  $\sigma(0) = 0$  and  $b(0) \geq 0$ ;
- $(x, u) \mapsto h_0(x, z)$  is a non-negative Borel function on  $\mathbb{R}_+ \times (0, \infty)$  so that  $h_0(0, z) = 0$ ;
- $(x, z) \mapsto h_1(x, z)$  is a non-negative Borel function on  $\mathbb{R}_+ \times (0, \infty)$ .

Suppose that  $\mu_0(dz)$  and  $\mu_1(dz)$  are  $\sigma$ -finite measures on  $(0, \infty)$  satisfying

$$\int_0^\infty h_0(x, z)(z \wedge z^2) \mu_0(dz) + \int_0^\infty h_1(x, z)(1 \wedge z) \mu_1(dz) < \infty, \quad x \geq 0.$$

In view of the characterization of Feller semigroups given by Courrège (1966), it is natural to consider the generator  $L$  defined by

$$\begin{aligned} Lf(x) &= \frac{1}{2}\sigma(x)^2 f''(x) + \int_0^\infty [f(x+z) - f(x) - zf'(x)]h_0(x,z)\mu_0(dz) \\ &\quad + b(x)f'(x) + \int_0^\infty [f(x+z) - f(x)]h_1(x,z)\mu_1(dz). \end{aligned} \quad (5.5)$$

Let  $\{B(t)\}$  be a standard  $(\mathcal{F}_t)$ -Brownian motion, and let  $\{p_0(t)\}$  and  $\{p_1(t)\}$  be  $(\mathcal{F}_t)$ -Poisson point processes on  $(0, \infty)^2$  with characteristic measures  $\mu_0(dz)du$  and  $\mu_1(dz)du$ , respectively. We assume  $\{B(t)\}$ ,  $\{p_0(t)\}$  and  $\{p_1(t)\}$  are independent of each other. Let  $N_0(ds, dz, du)$  and  $N_1(ds, dz, du)$  be the Poisson random measures associated with  $\{p_0(t)\}$  and  $\{p_1(t)\}$ , respectively. Let us consider the stochastic equation

$$\begin{aligned} x(t) &= x(0) + \int_0^t \sigma(x(s-))dB(s) + \int_0^t \int_0^\infty \int_0^\infty z \tilde{N}_0(ds, dz, du) \\ &\quad + \int_0^t b(x(s-))ds + \int_0^t \int_0^\infty \int_0^\infty z N_1(ds, dz, du). \end{aligned} \quad (5.6)$$

As before we assume  $b(x) = b_1(x) - b_2(x)$ , where  $x \mapsto b_1(x)$  is continuous and  $x \mapsto b_2(x)$  is continuous and non-decreasing. For the ingredients of the above equation we may rephrase (2.a,b) and (5.a,c) into the following conditions:

(5.e) There is a constant  $K \geq 0$  so that

$$b(x) + \int_0^\infty h_1(x,z)z\mu_1(dz) \leq K(1+x), \quad x \geq 0;$$

(5.f) For fixed  $z > 0$  the function  $x \mapsto h_0(x,z)$  is non-decreasing and there is a non-decreasing function  $x \mapsto L(x)$  on  $\mathbb{R}_+$  such that

$$\sigma(x)^2 + \int_0^\infty h_0(x,z)(z \wedge z^2)\mu_0(dz) \leq L(x), \quad x \geq 0;$$

(5.g) For each integer  $m \geq 1$  there is a non-decreasing concave function  $z \mapsto r_m(z)$  on  $\mathbb{R}_+$  such that  $\int_{0+} r_m(z)^{-1}dz = \infty$  and

$$|b_1(x) - b_1(y)| + \int_0^\infty |h_1(x,z) - h_1(y,z)|(z \wedge m)\mu_1(dz) \leq r_m(|x-y|)$$

for all  $0 \leq x, y \leq m$ ;

(5.h) For each integer  $m \geq 1$  there is a non-negative and non-decreasing function  $z \mapsto \rho_m(z)$  on  $\mathbb{R}_+$  such that  $\int_{0+} \rho_m(z)^{-2}dz = \infty$  and

$$|\sigma(x) - \sigma(y)|^2 + \int_0^\infty |h_0(x,z) - h_0(y,z)|(z \wedge m)^2\mu_0(dz) \leq \rho_m(|x-y|)^2$$

for all  $0 \leq x, y \leq m$ .

**Theorem 5.6** *If conditions (5.e,f,g,h) are satisfied, there exists a unique non-negative strong solution to (5.6) and the solution is a strong Markov process with generator given by (5.5).*

*Proof.* By applying Theorem 5.1 with  $U_i = (0, \infty)^2$  for  $i = 0, 1, 2$  it is simple to see that (5.6) has a unique non-negative strong solution  $\{x(t)\}$ . In particular, to verify condition (5.b) we can simply let  $V_n = [1/n, \infty) \times (0, \infty)$  for  $n \geq 1$ . The uniqueness implies the strong Markov property of  $\{x(t)\}$ . By Itô's formula it is easy to show that  $\{x(t)\}$  has generator given by (5.5).  $\square$

## 6 Applications

In this section, we illustrate some applications of the results established in the foregoing sections to stochastic differential equations driven by one-sided Lévy processes. Let  $\mu_0(dz)$  and  $\mu_1(dz)$  be  $\sigma$ -finite measures on  $(0, \infty)$  and let  $\nu_0(dz)$  and  $\nu_1(dz)$  be  $\sigma$ -finite measures on  $(0, 1]$ . We assume that

$$\int_0^\infty (z \wedge z^2) \mu_0(dz) + \int_0^\infty (1 \wedge z) \mu_1(dz) + \int_0^1 z^2 \nu_0(dz) + \int_0^1 z \nu_1(dz) < \infty.$$

Let  $\{B(t)\}$  be a standard  $(\mathcal{F}_t)$ -Brownian motion. Let  $\{z_0(t)\}$  and  $\{z_1(t)\}$  be  $(\mathcal{F}_t)$ -Lévy processes with exponents

$$u \mapsto \int_0^\infty (e^{iuz} - 1 - iuz) \mu_0(dz) \quad \text{and} \quad u \mapsto \int_0^\infty (e^{iuz} - 1) \mu_1(dz),$$

respectively. Let  $\{y_0(t)\}$  and  $\{y_1(t)\}$  be  $(\mathcal{F}_t)$ -Lévy processes with exponents

$$u \mapsto \int_0^1 (e^{iuz} - 1 - iuz) \nu_0(dz) \quad \text{and} \quad u \mapsto \int_0^1 (e^{iuz} - 1) \nu_1(dz),$$

respectively. Therefore  $\{z_0(t)\}$  and  $\{y_0(t)\}$  are centered and  $\{z_1(t)\}$  and  $\{y_1(t)\}$  are non-decreasing. Suppose that those processes are independent of each other. In addition, suppose that

- $x \mapsto \sigma(x)$  and  $x \mapsto b(x)$  are continuous functions on  $\mathbb{R}_+$  satisfying  $\sigma(0) = 0$  and  $b(0) \geq 0$ ;
- $x \mapsto \phi_0(x)$  and  $x \mapsto \phi_1(x)$  are continuous non-negative functions on  $\mathbb{R}_+$  so that  $\phi_0(0) = 0$ .

We consider the condition:

(6.a) There is a constant  $K \geq 0$  so that  $b(x) + \phi_1(x) \leq K(1 + x)$  for all  $x \geq 0$ .

**Theorem 6.1** Suppose that (6.a) holds. If, in addition, the functions  $\sigma$ ,  $b$ ,  $\phi_0$  and  $\phi_1$  are all locally Lipschitz, then there is a unique non-negative strong solution to

$$\begin{aligned} dx(t) &= \sigma(x(t))dB(t) + \phi_0(x(t-))dz_0(t) + b(x(t))dt \\ &\quad + \phi_1(x(t-))dz_1(t) - x(t-)dy_0(t) - x(t-)dy_1(t). \end{aligned} \quad (6.1)$$

*Proof.* To save the notation, we extend  $\mu_0$  and  $\mu_1$  to  $\sigma$ -finite measures on  $[-1, 0) \cup (0, \infty)$  by setting  $\mu_0([-x, 0)) = \nu_0((0, x])$  and  $\mu_1([-x, 0)) = \nu_1((0, x])$  for  $0 < x \leq 1$ . By the general result on Lévy-Itô decompositions, we have

$$z_0(t) = \int_0^t \int_0^\infty z \tilde{N}_0(ds, dz), \quad z_1(t) = \int_0^t \int_0^\infty z N_1(ds, dz)$$

and

$$y_0(t) = - \int_0^t \int_{[-1, 0)} z \tilde{N}_0(ds, dz), \quad y_1(t) = - \int_0^t \int_{[-1, 0)} z N_1(ds, dz),$$

where  $N_0(ds, dz)$  and  $N_1(ds, dz)$  are independent Poisson random measures with intensities  $ds\mu_0(dz)$  and  $ds\mu_1(dz)$ , respectively. By Theorem 2.5 applied with  $U_0 = U_1 = [-1, 0) \cup (0, \infty)$  and  $U_2 = [-1, 0) \cup (0, 1]$ , there is a unique strong solution to

$$\begin{aligned} x(t) &= x(0) + \int_0^t \sigma(x(s-))dB(s) + \int_0^t \int_0^\infty \phi_0(x(s-))z \tilde{N}_0(ds, dz) \\ &\quad + \int_0^t b(x(s-))ds + \int_0^t \int_0^\infty \phi_1(x(s-))z N_1(ds, dz) \\ &\quad + \int_0^t \int_{[-1, 0)} x(s-)z \tilde{N}_0(ds, dz) + \int_0^t \int_{[-1, 0)} x(s-)z N_1(ds, dz), \end{aligned}$$

which is just another form of (6.1).  $\square$

In particular, if  $\sigma \geq 0$  and  $b$  are real constants, by Theorem 6.1 there is a unique non-negative strong solution  $\{S(t)\}$  to the stochastic differential equation

$$dS(t) = \sigma S(t)dB(t) + bS(t)dt + S(t-)dz_0(t) - S(t-)dy_0(t).$$

The process  $\{S(t)\}$  is a generalization of the geometric Brownian motion and has been used widely in mathematical finance; see, e.g., Lamberton and Lapeyre (1996, p.144).

**Theorem 6.2** Suppose that (6.a) holds. If, in addition,  $x \mapsto \phi_0(x)$  is non-decreasing on  $\mathbb{R}_+$  and for each  $m \geq 1$  there is a constant  $K_m \geq 0$  so that

$$|\sigma(x) - \sigma(y)|^2 + |\phi_0(x) - \phi_0(y)|^2 + |b(x) - b(y)| + |\phi_1(x) - \phi_1(y)| \leq K_m|x - y|$$

for all  $0 \leq x, y \leq m$ , then there is a unique non-negative strong solution to

$$\begin{aligned} dx(t) &= \sigma(x(t))dB(t) + \phi_0(x(t-))dz_0(t) + b(x(t))dt \\ &\quad + \phi_1(x(t-))dz_1(t) - x(t-)dy_1(t). \end{aligned}$$

*Proof.* This follows from Theorem 5.3 by a simple modification of the proof of Theorem 6.1.  $\square$

**Corollary 6.3** *There exists a unique non-negative strong solution to (1.8).*

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