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A CENTRAL LIMIT THEOREM FOR SUPER-BROWNIAN MOTION WITH SUPER-BROWNIAN IMMIGRATION[1]

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Abstract. We prove a central limit theorem for the super-Brownian motion with immigration governed by another super-Brownian. The limit theorem leads to Gaussian random fields in dimensions $d \geq 3$. For $d = 3$ the field is spatially uniform; for $d \geq 5$ its covariance is given by the potential operator of the underlying Brownian motion; and for $d = 4$ it involves a mixture of the two kinds of fluctuations.

Key words: super-Brownian motion, immigration, random medium, central limit theorem.

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Measure-valued branching processes with immigration (MBI-processes) have been considered by several authors; see e.g. [1, 3, 5, 6, 7]. A central limit theorem for super Brownian motion with immigration was given in Li and Shiga [9], where the immigration is governed by a deterministic measure; see also [8]. In the recurrent underlying motion case, the Gaussian field obtained in [9] is spatially uniform. In the transient case, its covariance kernel is given by the potential kernel of the underlying motion.

In this paper, we prove a central limit theorem for a super-Brownian motion with immigration in a random medium. We consider the situation where the immigration measure is given by the trajectory of another super-Brownian. We call the process under consideration a *super-Brownian motion with super-Brownian immigration* (SBMSBI). The investigation has been stimulated by the work of Dawson and Fleischmann [2], who studied the super-Brownian motion with random branching mechanism governed by another super-Brownian motion. We shall see that the randomization of the immigration measure causes some changes in the asymptotic behavior of the immigration process. Our limit theorem leads to Gaussian random fields for underlying dimension numbers $d \geq 3$. For $d = 3$ the field is spatially uniform; for $d \geq 5$ its covariance is given by the potential operator of the underlying Brownian motion; and for $d = 4$ it involves a mixture of the two kinds of fluctuations, which seems to be a new phenomenon in the asymptotic behavior of measure-valued processes.

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Let $C(I\!\!R^d)$ denote the space of continuous bounded functions on $I\!\!R^d$. We fix a constant $p > d$ and let $\phi_p(x) := (1+|x|^2)^{-p/2}$ for $x \in I\!\!R^d$ and $C_p(I\!\!R^d) := \{f \in C(I\!\!R^d) : |f(x)| \leq \text{const}\cdot\phi_p(x)\}$. Let $M_p(I\!\!R^d)$ be the space of Radon measures μ on $I\!\!R^d$ such that $\langle \mu, f\rangle := \int f(x)\mu(dx) < \infty$ for all $f \in C_p(I\!\!R^d)$. We endow $M_p(I\!\!R^d)$ with the p-vague topology, that is, $\mu_k \to \mu$ if and only if $\langle \mu_k, f\rangle \to \langle \mu, f\rangle$ for all $f \in C_p(I\!\!R^d)$. Throughout the paper, λ denotes the Lebesgue measure on $I\!\!R^d$.

Suppose that $W = (w_t, P_{r,a})$ is a standard Brownian motion in $I\!\!R^d$ with semigroup $(P_t)_{t\geq 0}$, where $P_{r,a}$ denote the conditional law of $\{w_t : t \geq r\}$ given $w_r = a$. A *super-Brownian motion* $X = (X_t, Q_{r,\mu})$ is an $M_p(I\!\!R^d)$-valued Markov process with transition probabilities given by

$$Q_{r,\mu} \exp\{-\langle X_t, f\rangle\} = \exp\{-\langle \mu, v(r,t,\cdot)\rangle\}, \quad f \in C_p^+(I\!\!R^d), \tag{1}$$

where $v(\cdot,\cdot,\cdot)$ is the unique solution of the evolution equation

$$v(r,t,a) = P_{r,a}f(w_t) - \int_r^t P_{r,a}v^2(s,t,w_s)ds, \quad t \geq r \geq 0. \tag{2}$$

Let $\{g_t(\cdot) : t \geq 0\}$ be a continuous $C_p^+(I\!\!R^d)$-valued path such that for each $a > 0$ there is a constant $C_a > 0$ such that $g_t \leq C_a\phi_p$ for all $t \in [0,a]$. The weighted occupation time of the super-Brownian motion may be determined by

$$Q_{r,\mu} \exp\Big\{ -\int_r^t \langle X_s, g_s\rangle ds\Big\} = \exp\{-\langle \mu, u(r,t,\cdot)\rangle\}, \quad f \in C_p^+(I\!\!R^d), \tag{3}$$

where $v(\cdot,\cdot,\cdot)$ is the unique solution of

$$u(r,t,a) = \int_r^t P_{r,a}g_s(w_s)ds - \int_r^t P_{r,a}u^2(s,t,w_s)ds, \quad t \geq r \geq 0. \tag{4}$$

See e.g. Dawson [1] or Iscoe [4].

Suppose that $\{\gamma_t, t \geq 0\}$ is an $M_p(I\!\!R^d)$-valued continuous path. A *super-Brownian motion with immigration* determined by $\{\gamma_t, t \geq 0\}$ is an $M_p(I\!\!R^d)$-valued Markov process $X^\gamma = (X_t^\gamma, Q_{r,\mu}^\gamma)$ with transition probabilities given by

$$Q_{r,\mu}^\gamma \exp\{-\langle X_t^\gamma, f\rangle\} = \exp\Big\{ -\langle \mu, v(r,t,\cdot)\rangle - \int_0^t \langle \gamma_s, v(s,t,\cdot)\rangle ds\Big\}, \quad f \in C_p^+(I\!\!R^d), \tag{5}$$

where $v(\cdot,\cdot,\cdot)$ is given by (2); see e.g. Dawson [1], Dynkin [3] and Li [6].

Based on (3) and (5) it is not difficult to construct a probability space $(\Omega, \mathcal{F}, Q)$ on which the processes $\{\varrho_t : t \geq 0\}$ and $\{Y_t : t \geq 0\}$ are defined, where $\{\varrho_t : t \geq 0\}$ is a super-Brownian motion with $\varrho_0 = \lambda$, and given $\{\varrho_t : t \geq 0\}$ the process $\{Y_t : t \geq 0\}$ is a super-Brownian motion with immigration determined by $\{\varrho_t : t \geq 0\}$ and $Y_0 = 0$. By (3) and (5) we have

$$Q \exp\{-\langle Y_t, f\rangle\} = \exp\{-\langle \lambda, u(0,t,\cdot)\rangle\} \tag{6}$$

where $u(\cdot,\cdot,\cdot)$ is the solution of the equation

$$u(r,t,a) = \int_r^t P_{r,a} v(s,t,w_s)ds - \int_r^t P_{r,a} u^2(s,t,w_s)ds, \quad t \geq r \geq 0. \tag{7}$$

The process $\{Y_t : t \geq 0\}$ is what we call an SBMSBI. Let $\mathcal{S}(I\!R^d)$ be the space of rapidly decreasing, infinitely differentiable functions on $I\!R^d$ whose all partial derivatives are also rapidly decreasing, and let $\mathcal{S}'(I\!R^d)$ be the dual space of $\mathcal{S}(I\!R^d)$. We define the $\mathcal{S}'(I\!R^d)$-valued process $\{Z_t : t > 0\}$ by

$$\langle Z_t, f\rangle := a_d(t)^{-1}[\langle Y_t, f\rangle - t\langle \lambda, f\rangle], \quad f \in \mathcal{S}(I\!R^d), \tag{8}$$

where $a_2(t) = t$, $a_3(t) = t^{3/4}$ and $a_d(t) = t^{1/2}$ for $d \geq 4$. Then we have

Theorem 1. *For $d \geq 3$, Z_t converges in distribution to a centered Gaussian random variable Z_∞ in $\mathcal{S}'(I\!R^d)$ with covariance*

$$\mathbf{Cov}(Z_\infty, f\rangle, \langle Z_\infty, g\rangle) = \begin{cases} \langle \lambda, f\rangle\langle \lambda, g\rangle / 6\pi^{3/2}, & d = 3, \\ \langle \lambda, f\rangle\langle \lambda, g\rangle / 8\pi^2 + \langle \lambda, fGg\rangle/2, & d = 4, \\ \langle \lambda, fGg\rangle/2, & d \geq 5, \end{cases} \tag{9}$$

where G denotes the potential operator of the Brownian motion.

An immediate consequence of Theorem 1 is the following

Corollary 2. *For $d \geq 3$ we have $t^{-1}Y_t \to \lambda$ in probability.*

Now we proceed to the proof of Theorem 1. We shall need the following two lemmas.

Lemma 3. *For $f \in \mathcal{S}(I\!R^d)^+$ let*

$$A_d(t,f) := a_d(t)^{-2} \int_0^t ds \int_s^t dr \int (P_{t-r}f(b))^2 db.$$

Then we have

$$\lim_{t\to\infty} A_d(t,f) = \begin{cases} 0, & d = 2,3, \\ \langle \lambda, fGf\rangle/2, & d \geq 4. \end{cases}$$

Proof. In this and the following proofs, C will denote a constant which may take different values in different lines. Then for any $f \in \mathcal{S}(I\!R^d)^+$ we have

$$||P_s f|| \leq C \cdot (1 \wedge s^{-d/2}). \tag{10}$$

It follows that

$$\begin{aligned} \lim_{t\to\infty} A_2(t,f) &= \lim_{t\to\infty} t^{-2} \int_0^t ds \int_0^s dr \int (P_r f(b))^2 db \\ &= \frac{1}{2} \lim_{t\to\infty} t^{-1} \int_0^t dr \int (P_r f(b))^2 db \\ &\leq C \cdot \lim_{t\to\infty} t^{-1} \langle \lambda, f\rangle \int_0^t (1 \wedge r^{-1}) dr \\ &= 0. \end{aligned}$$

Similarly, one may check that $\lim_{t\to\infty} A_3(t,f)=0$. When $d\geq 4$, we use l'Hospital's rule to get

$$\begin{aligned}
\lim_{t\to\infty} A_d(t,f) &= \lim_{t\to\infty} t^{-1}\int_0^t ds\int_0^s dr\int (P_rf(b))^2 db\\
&= \int_0^\infty dr\int (P_rf(b))^2 db\\
&= \langle\lambda, fGf\rangle/2,
\end{aligned}$$

as desired. □

Lemma 4. *For* $f\in\mathcal{S}(I\!\!R^d)^+$ *let*

$$B_d(t,f)=a_d(t)^{-2}\int_0^t\langle\lambda,(\int_s^t(P_{t-s}f)dr)^2\rangle ds.$$

Then we have

$$\lim_{t\to\infty} B_d(t,f)=\begin{cases}\langle\lambda,f\rangle^2/8\pi, & d=2,\\ \langle\lambda,f\rangle^2/12\pi^{3/2}, & d=3,\\ \langle\lambda,f\rangle^2/16\pi^2, & d=4,\\ 0, & d\geq 5.\end{cases}$$

Proof. We first observe that

$$\begin{aligned}
B_d(t,f) &= a_d(t)^{-2}\int_0^t s^2\langle\lambda,(P_sf)^2\rangle ds\\
&= a_d(t)^{-2}\int_0^t s^2 ds\int_{I\!\!R^d}\int_{I\!\!R^d} f(y)f(z)p(2s,y,z)dydz.
\end{aligned}$$

For $d=2$ we may set $s=t^{1-r}$ to get

$$\begin{aligned}
\lim_{t\to\infty} B_2(t,f) &= \lim_{t\to\infty} t^{-2}\int_1^t s^2\cdot(4\pi s)^{-1}ds\int_{R^4} e^{-\frac{(y-z)^2}{4s}}f(y)f(z)dydz\\
&= (4\pi)^{-1}\lim_{t\to\infty}\int_0^1 t^{-2r}\log t\; dr\int_{R^4} e^{-\frac{(y-z)^2}{4t^{1-r}}}f(y)f(z)dydz\\
&= (8\pi)^{-1}\langle\lambda,f\rangle^2.
\end{aligned}$$

Setting $s=tr$ for $d=3$ we get

$$\begin{aligned}
\lim_{t\to\infty} B_3(t,f) &= \lim_{t\to\infty} t^{-3/2}\int_0^t s^2\cdot(4\pi s)^{-3/2}ds\int_{R^6} e^{-\frac{(y-z)^2}{4s}}f(y)f(z)dydz\\
&= (4\pi)^{-3/2}\lim_{t\to\infty}\int_0^1 r^{1/2}dr\int_{R^6} e^{-\frac{(y-z)^2}{4tr}}f(y)f(z)dydz\\
&= \langle\lambda,f\rangle^2/12\pi^{3/2}.
\end{aligned}$$

Similarly, one can show $\lim_{t\to\infty} B_4(t,f) = \langle\lambda, f\rangle^2/16\pi^2$. When $d \geq 5$ we have

$$B_d(t,f) = t^{-1}\int_0^t s^2\langle\lambda,(P_sf)^2\rangle ds \leq C\cdot t^{-1}\langle\lambda,f\rangle\int_0^t s^2\cdot(1\wedge s^{-d/2})ds,$$

which goes to zero as $t\to\infty$. $\square$

Proof of Theorem 1. Let $f_t := a_d(t)^{-1}f$. By (2), (7) and (8) we get the Laplace functional

$$Q\exp\{-\langle Z_t,f\rangle\} = \exp\Big\{\int_0^t ds\int_s^t dr\int v(r,t,b)^2db + \int_0^t dr\int u(r,t,b)^2db\Big\} \tag{11}$$

where $v(\cdot,\cdot,\cdot)$ and $u(\cdot,\cdot,\cdot)$ is the solution of (2) and (8), respectively, with f being replaced by f_t. By (2) we get

$$\begin{aligned}(P_{t-r}f_t(b))^2 - (v(r,t,b))^2 &= 2P_{t-r}f_t(b)\int_r^t P_{s-r}(v(s,t,\cdot)^2)(b)ds\\ &\leq 2P_{t-r}f_t(b)\int_r^t P_{s-r}((P_{t-s}f_t)^2)(b)ds.\end{aligned}$$

It follows that

$$\begin{aligned}&\int_0^t ds\int_s^t dr\int[(P_{t-r}f_t(b))^2-(v(r,t,b))^2]db\\ \leq\;& 2\int_0^t ds\int_s^t dr\int\Big[P_{t-r}f_t(b)\int_r^t P_{h-r}((P_{t-h}f_t)^2)(b)dh\Big]db\\ \leq\;& Ca_d(t)^{-2}\int_0^t ds\int_s^t dr\int(P_{t-r}f(b))^2db\cdot a_d(t)^{-1}\int_r^t(1\wedge(t-h)^{-d/2})dh\\ \leq\;& Ca_d(t)^{-3}\langle\lambda,f\rangle\int_0^t ds\int_s^t(1\wedge(t-r)^{-d/2})dr\cdot\int_r^t(1\wedge(t-h)^{-d/2})dh,\end{aligned}$$

where we have used (10) for two times. It is easy to check that the last value goes to zero as $t\to\infty$ if $d\geq 3$. Combining this with Lemma 3 we obtain

$$\lim_{t\to\infty}\int_0^t ds\int_s^t dr\int v(r,t,b)^2db = \begin{cases}0, & d=3\\ \langle\lambda, fGg\rangle/2, & d\geq 4\end{cases}$$

On the other hand, observe that

$$\begin{aligned}\Big(\int_s^t P_{t-s}f_tdr\Big)^2 - u(s,t)^2 &= \Big[\Big(\int_s^t P_{t-s}f_tdr\Big)^2 - \Big(\int_s^t P_{r-s}v(r,t)dr\Big)^2\Big]\\ &+\Big[\Big(\int_s^t P_{r-s}v(r,t)dr\Big)^2 - u(s,t)^2\Big].\end{aligned}$$

By a similar deduction as the above we get

$$\int_0^t\Big\langle\lambda,\Big(\int_s^t P_{r-s}P_{t-r}f_tdr\Big)^2 - \Big(\int_s^t P_{r-s}v(r,t)dr\Big)^2\Big\rangle ds$$

$$\begin{aligned}
&\le \int_0^t \Big\langle \lambda, 2\int_s^t P_{t-s} f_t dr \int_s^t P_{r-s}\Big[\int_r^t P_{h-r}(P_{t-h}f_t)^2 dh\Big] dr\Big\rangle ds \\
&\le Ca_d(t)^{-3}\int_0^t \Big\langle \lambda, \Big(\int_s^t P_{t-s} f dr\Big)^2\Big\rangle ds \cdot \int_0^t (1\wedge (t-h)^{-d/2}) dh \\
&\le Ca_d(t)^{-3}\int_0^t \langle \lambda, f\rangle (t-s)^2 (1\wedge (t-s)^{-d/2}) ds \\
&\le Ca_d(t)^{-3}\int_0^t \langle \lambda, f\rangle (1\wedge (t-s)^{2-d/2}) ds,
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t \Big\langle \lambda, \Big(\int_s^t P_{r-s} v(r,t) dr\Big)^2 - u(s,t)^2\Big\rangle ds \\
&\le \int_0^t \Big\langle \lambda, 2\int_s^t P_{r-s} v(r,t) dr \int_s^t P_{r-s}(u(r,t)^2) dr\Big\rangle ds \\
&\le 2\int_0^t \Big\langle \lambda, \int_s^t P_{t-s} f_t dr \int_s^t P_{r-s}\Big[\int_r^t P_{h-r}(P_{t-h}f_t) dh\Big]^2 dr\Big\rangle ds \\
&\le 2a_d(t)^{-3}\int_0^t \Big\langle \lambda, (t-s)P_{t-s} f \int_s^t P_{r-s}(P_{t-r}f)^2 (t-r) dr\Big\rangle ds \\
&\le Ca_d(t)^{-3}\int_0^t \Big\langle \lambda, (t-s)P_{t-s} f \int_s^t (1\wedge (t-r)^{1-d}) dr\Big\rangle ds \\
&\le Ca_d(t)^{-3}\int_0^t \langle \lambda, f\rangle (1\wedge (t-s)^{3-d}) ds.
\end{aligned}$$

Both values go to zero as $t \to \infty$ if $d \geq 3$. Then combining those with Lemma 4, we get

$$\lim_{t\to\infty}\int_0^t dr \int u(r,t,b)^2 db = \begin{cases} \langle \lambda, f\rangle^2/12\pi^{3/2}, & d=3, \\ \langle \lambda, f\rangle^2/16\pi^2, & d=4, \\ 0, & d>4. \end{cases} \qquad (12)$$

Returning to (11) we obtain the desired result. □

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