Measure-valued processes and related topics

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This is an introduction to some research results of the author and his collaborators by the year 2011. Most of the results are related to measure-valued branching processes, a class of infinite-dimensional Markov processes with beautiful mathematical structures and interesting applications. The reader may refer to Dawson [D1] for the backgrounds of the subject. The results summarized in paragraphs 1–4 have been retreated in the monograph [1] of the author. The publications of the author have been cited in papers by K.B. Athreya, D.A. Dawson, J.-F. Delmas, E.B. Dynkin, S.N. Ethier, P.J. Fitzsimmons, L.G. Gorostiza, R.C. Griffiths, K. Handa, T.G. Kurtz, A. Lambert, S. Méléard, K.V. Mitov, E.A. Perkins, M. Röckner, W. Schachermayer, B. Schmuland, A. Shied, A. Wakolbinger, L. Zambotti and many others.

1. Local and non-local branching measure-valued processes: A measure-valued branching process describes the evolution of a population evolving randomly in a region. The branching mechanism plays an important role in this modeling. In [2] the author gave a necessary and sufficient criterion for a class of integral represented functions. This result implies that the local and non-local branching mechanisms of the process, which arise as the limits of some sequences defined from probability generating functions, always take the usual forms. In particular, the local branching mechanism at any site is typically given by a function ϕ on $[0,\infty)$ with representation

$$\phi(z) = bz + cz^2 + \int_0^\infty (e^{-uz} - 1 - uz)m(du), \qquad z \geq 0,$$

where $c \geq 0$ and b are constants and $(u \wedge u^2)m(du)$ is a finite measure on $(0,\infty)$. In [7] the author constructed the corresponding measure-valued branching processes with local and non-local branching mechanism, extending the construction of Dynkin [D8]. From the non-local branching processes, the author derived several different models including multi-type superprocesses, mass-structured superprocesses, multi-level non-local branching superprocesses, age-reproduction-structured superprocesses and the superprocess-controlled immigration processes. This unified treatment simplifies considerably the proof of existence of those superprocesses; see, e.g., [A2, B1, B2, C3, F1, M1]. The results in [2] on the general representations of branching mechanisms give the universality of superprocesses as high density limits of branching particle systems; see e.g. Dynkin [D8, p.1250].

2. Skew convolution semigroups and immigration processes: A useful and realistic modification of the branching model is the addition of immigration into the population from an outside source. The processes allowing immigration are clearly of great appeal and importance from the point of view of applications; see, e.g., Athreya and Ney [A1, p.10 and p.262]. In the paper [4] the author gave an axiomatic formulation of the immigration processes as follows. Let $(Q_t)_{t\geq 0}$

be the transition semigroup of a measure-valued branching process X with state space $M(E)$, finite Borel measures on a metric space E. A family of probability measures $(N_t)_{t\geq 0}$ on $M(E)$ is naturally called a *skew convolution semigroup* associated with $(Q_t)_{t\geq 0}$ provided

$$N_{r+t} = (N_r Q_t) * N_t, \qquad r,t \geq 0, \tag{1}$$

where "$*$" denotes the convolution operation. The author observed that the above equation is satisfied if and only if

$$Q_t^N(\mu,\cdot) := Q_t(\mu,\cdot) * N_t, \qquad t \geq 0, \mu \in M(E), \tag{2}$$

defines a Markov semigroup $(Q_t^N)_{t\geq 0}$ on $M(E)$. A Markov process with transition semigroup $(Q_t^N)_{t\geq 0}$ is called an *immigration process* associated with X or $(Q_t)_{t\geq 0}$.

The intuitive meaning of the model is clear in view of (1) and (2). From (2) one can see that the population at any time $t \geq 0$ is made up of two parts; the native part generated by the mass $\mu \in M(E)$ has distribution $Q_t(\mu,\cdot)$ and the immigration in the time interval $(0,t]$ gives the distribution N_t. In a similar way, equation (1) decomposes the population immigrating to E during the time interval $(0, r+t]$ into two parts; the immigration in the interval $(r, r+t]$ gives the distribution N_t while the immigration in the interval $(0,r]$ generates the distribution N_r at time r and gives the distribution $N_r Q_t$ at time $r+t$. It is not hard to understand that (1) and (2) give a general formulation of the immigration independent of the current state of the population. In particular, the immigration models studied in [D1, D2, E1, K2] are essentially included by the above formulation. This formulation also contains some interesting new classes of immigration processes.

The author proved that $(N_t)_{t\geq 0}$ is a skew convolution semigroup if and only if there is an infinitely divisible probability entrance law $(K_s)_{s>0}$ for $(Q_t)_{t\geq 0}$ such that

$$\log L_{N_t}(f) = \int_0^t [\log L_{K_s}(f)]ds, \quad t \geq 0, f \in \mathrm{bp}\mathscr{B}(E),$$

where L_{N_t} and L_{K_s} denote the Laplace functionals of the probability measures N_t and K_s, respectively. This result establishes a 1-1 correspondence between skew convolution semigroups and a class of infinitely divisible probability entrance laws. In particular, if K_0 is an infinitely divisible probability measure on $M(E)$, then $K_s := K_0 Q_s$ defines a *closeable* infinitely divisible probability entrance law $(K_s)_{s>0}$. A characterization of infinitely divisible probability entrance laws for a typical class of measure-valued branching processes, the Dawson-Watanabe superprocesses, was given in [4], developing the work of Dynkin [D7]. It was also shown in [4] that an immigration process has a Borel right realization if the corresponding entrance law is closable and the process may not even have a right continuous realization when the law is not closable.

3. Other applications of skew convolution semigroups: The concept of skew convolution semigroups can actually be introduced in an abstract setting. Roughly speaking, such a semigroup gives the law of evolution of a system with branching structure under the perturbation of random extra forces. The immigration process is only a special case of this formulation. Skew convolution semigroups were used in [10] to investigate the regularity of the affine Markov processes introduced by Duffie et al. [D6] in the study of mathematical finance. The techniques of [10] was developed in [K1] to settle the regularity problem.

There is another similar structure investigated by Bogachev et al. [B4], who formulated Ornstein-Uhlenbeck type processes on Hilbert spaces using generalized Mehler semigroups. In [9], a representation was given for generalized Mehler semigroups whose characteristic functions are not necessarily differentiable in time. A connection of the associated Ornstein-Uhlenbeck processes and catalytic branching processes with immigration was also established in [9] using fluctuation limits of the latter. See also [B5] and [S2] for some related developments.

4. Kuznetsov processes and excursion laws: In [6] the author proved that a general skew convolution semigroup is determined by an increasing continuous measure-valued path and an entrance rule for the relevant measure-valued branching process. Based on this result and the techniques from [D4, G1], the author gave a construction for the sample paths of the corresponding immigration process by Kuznetsov processes and studied the almost sure behaviors of the latter, providing useful insights into the trajectory structures of the immigration process. Some well-known results on excessive measures were formulated in [6] in terms of stationary immigration processes.

Let Q_μ denote the distribution on $C([0,\infty), M(E))$ of a continuous measure-valued branching diffusion process. A typical *excursion law* of the process is a *σ-finite Markovian measure* $Q^{(x)}$ on $C([0,\infty), M(E))$ which arises formally as the limit $Q^{(x)} = \lim_{\varepsilon\downarrow 0} \varepsilon^{-1} Q_{\varepsilon\delta_x}$. It was proved in [3] that $Q^{(x)}$ is carried by paths $w \in C([0,\infty), M(E))$ such that $w_0 = 0$ and $\lim_{t\to 0} w_t(E)^{-1} w_t = \delta_x$. This result was used by Dawson and Perkins [D3] to analyze the Poisson cluster structure of measure-valued branching diffusions. The sample paths of immigration branching diffusion processes were constructed in [3] by excursions. A class of measure-valued branching diffusions with dependent spatial motion and interactive immigration were constructed in [8] as the pathwise unique solution of a stochastic integral equation driven by Poisson processes of one-dimensional excursions carried by an isotropic stochastic flow.

5. Reversibility of Fleming-Viot processes: A Fleming-Viot processes is a diffusion process taking values of probability measures on a metric space E. A typical Fleming-Viot process is determined by two parameters (A, σ), the mutation operator A and the selection density σ. It has been known for a long time that the Fleming-Viot process is reversible if its mutation operator is of the uniform-jump type, i.e., there is a finite measure ν on E such that

$$Af(x) = \int_E [f(y) - f(x)]\nu(dy), \qquad x \in E, f \in \mathscr{D}(A).$$

A problem which had remained open is whether a Fleming-Viot process with more general mutation operator can be reversible; see e.g. Ethier and Kurtz [E2, p.368]. This problem was settled in [5], where the author proved that a Fleming-Viot process with irreducible mutation operator A has a reversible stationary distribution if and only if A is of the uniform-jump type. There have been two alternate proofs of this result given respectively by Handa [H1] and Schmuland and Sun [S3]; see also [B3, H2, S3, T1] for some related developments.

6. Stochastic equations for branching processes: Continuous-state branching processes with immigration are one-dimensional prototypes of the measure-valued branching processes with immigration. The characterizations of both of those processes in terms of martingale problems are already well-known. In the continuous-state setting the processes can also be characterized using stochastic differential or partial differential equations. The pathwise uniqueness for the stochastic equation of a branching process is usually very difficult because of the non-Lipschitz coefficients. In [10] it was proved that a general continuous-state branching processes is the pathwise unique positive solution of the stochastic equation

$$\begin{aligned} y(t) &= y(0) + \int_0^t \sqrt{2cy(s)}dB(s) + \int_0^t \int_0^\infty \int_0^{y(s-)} z\tilde{N}_0(ds, dz, du) \\ &\quad + \int_0^t (\beta - by(s))ds + \int_0^t \int_0^\infty zN_1(ds, dz), \end{aligned}$$

where $\{B(t)\}$ is a Brownian motion, $\{N_1(ds, dz)\}$ is a Poisson random measure on $(0, \infty)^2$ and $\{\tilde{N}_0(ds, dz, du)\}$ is a compensated Poisson random measure on $(0, \infty)^3$. Some applications of the results in the paper have been given in [C1, C2, K1, L1].

In the recent work [11] the author showed that the distribution functions of some measure-valued branching processes can also be characterized as pathwise unique solutions of some new stochastic equations. The ideas introduced in the paper have stimulated a progress in the problem of characterizing measure-valued processes as pathwise unique solutions of stochastic differential equations; see, e.g., Xiong [X1]. They have also been used in the study of positive self-similar Markov processes; see, e.g., Döring and Barczy [D5].

Selected Publications

1. Li, Z.H. (2011): *Measure-Valued Branching Markov Processes*. Springer, Berlin.

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2. Li, Z.H. (1991): *Integral representations of continuous functions*. Chinese Science Bulletin (Chinese Edition) **36**, 81–84 / (English Edition) **36**, 12: 979–983.

3. Li, Z.H. and Shiga T. (1995): *Measure-valued branching diffusions: immigrations, excursions and limit theorems*. Journal of Mathematics of Kyoto University **35**, 2: 233–274.

4. Li, Z.H. (1996): *Immigration structures associated with Dawson-Watanabe superprocesses*. Stochastic Processes and their Applications **62**, 1: 73–86.

5. Li, Z.H., Shiga, T. and Yao, L.H. (1999): *A reversibility problem for Fleming-Viot processes*. Electronic Communications in Probability **4**, 71–82.

6. LI, Z.H. (2002): *Skew convolution semigroups and related immigration processes*. Theory of Probability and its Applications **46**, 2: 274–296.

7. Dawson, D.A., Gorostiza, L.G. and Li, Z.H. (2002): *Non-local branching superprocesses and some related models*. Acta Applicandae Mathematicae **74**, 1: 93–112.

8. Dawson, D.A. and Li, Z.H. (2003): *Construction of immigration superprocesses with dependent spatial motion from one-dimensional excursions*. Probability Theory and Related Fields **127**, 1: 37–61.

9. Dawson, D.A., Li, Z.H., Schmuland, B. and Sun, W. (2004): *Generalized Mehler semigroups and catalytic branching processes with immigration*. Potential Analysis **21**, 1: 75–97.

10. Dawson, D.A. and Li, Z.H. (2006): Skew convolution semigroups and affine Markov processes. *The Annals of Probability* **34**, 3: 1103-1142.

11. Dawson, D.A. and Li, Z.H. (2010+): Stochastic equations, flows and measure-valued processes. *The Annals of Probability*. To appear.

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