

# Extensions of Vietoris's inequalities II <sup>1</sup>

Gavin Brown

University of Sydney, NSW 2006, Australia  
vice-chancellor@vcc.usyd.edu.au

Feng Dai

School of Mathematics and Statistics, University of Sydney, NSW 2006, Australia  
feng@maths.usyd.edu.au

and

Kunyang Wang

Department of Mathematics, Beijing Normal University, Beijing, 100875 China  
wangky@bnu.edu.cn

*Abstract.* For  $\alpha > -1$ , define  $\{c_k(\alpha)\}_{k=0}^{\infty}$  inductively by

$$c_0(\alpha) = c_1(\alpha) = 1, \quad c_{2k}(\alpha) = c_{2k+1}(\alpha) = \frac{2k-1+\alpha}{2k+\alpha} c_{2k-1}(\alpha), \quad k \geq 1,$$

and for  $n \geq 1$  and  $x \in (0, \pi)$ , define

$$T_n^\alpha(x) = \sum_{k=0}^n c_k(\alpha) \cos(kx).$$

Define the number  $\alpha_0 := 2.3308\dots$  to be the unique solution of the equation

$$\min_{x \in (0, \pi)} T_6^\alpha(x) = 0.$$

In this paper, it is shown that for  $-1 < \alpha \leq \alpha_0$ ,

$$T_n^\alpha(x) \geq 0, \quad x \in (0, \pi), \quad n \in \mathbb{N},$$

with an equality for some  $x \in (0, \pi)$  in the case when  $\alpha = \alpha_0$  and  $n = 6$ , that for any  $\alpha > \alpha_0$  there is a positive number  $N(\alpha) > 0$  depending only on  $\alpha$  such that

$$\min_{x \in (0, \pi)} T_6^\alpha(x) < 0,$$

and

$$T_n^\alpha(x) > 0, \quad x \in (0, \pi), \quad \text{whenever } n \geq N(\alpha), \quad (*)$$

and that the smallest positive integer  $N(\alpha)$  for which  $(*)$  holds tends to  $+\infty$  as  $\alpha \rightarrow \infty$ . In the case  $\alpha = 0$  the main result corresponds to the well known Vietoris's inequalities on cosine sums.

Similar results for the sine sums are also obtained.

---

<sup>1</sup>This work was supported by a grant from Australian Research Council. The third named author was also supported by NSF of China, grant #10071007

# 1 Statement of the main results

It was shown by Vietoris [1] in 1958 that for  $n \geq 1$  and  $x \in (0, \pi)$ ,

$$\sum_{k=0}^n c_k \cos(kx) > 0, \quad (1.1)$$

where  $c_0 = c_1 = 1$  and

$$c_{2k} = c_{2k+1} = \frac{2k-1}{2k} c_{2k-1} \quad k \in \mathbb{N}.$$

Here the coefficients can be expressed in terms of gamma functions:

$$c_{2k} = c_{2k+1} = \frac{\Gamma(k+0.5)}{\Gamma(0.5)\Gamma(k+1)}, \quad k = 0, 1, 2, \dots. \quad (1.2)$$

One way to extend Vietoris's inequality (1.1) is via taking the coefficients in (1.1) to be

$$d_{2k}(\beta) = d_{2k+1}(\beta) = \frac{\Gamma(k+1-\beta)}{\Gamma(1-\beta)\Gamma(k+1)}, \quad k = 0, 1, 2, \dots \quad (1.3)$$

with  $\beta < 1$ . In this direction, the current authors [4] recently have proved the following result:

*For  $0 < x < \pi$  and  $n \geq 1$ ,*

$$S_n^\beta(x) = \sum_{k=0}^n d_k(\beta) \cos kx > 0$$

*whenever  $\beta \geq \beta_0$ , where  $\beta_0 \in (0.308443, 0.308444)$  is the unique solution of the equation*

$$\int_0^{\frac{3\pi}{2}} \frac{\cos t}{t^\beta} dt = 0.$$

*Moreover, the number  $\beta_0$  is best possible in the sense that for any  $\beta < \beta_0$ ,*

$$\lim_{n \rightarrow \infty} \min_{x \in (0, \pi)} S_n^\beta(x) = -\infty.$$

In the present paper, we will extend Vietoris's inequality (1.1) in another way. More precisely, for  $\alpha > -1$  and  $k \geq 0$ , we let

$$c_{2k}(\alpha) = c_{2k+1}(\alpha) = \frac{\Gamma(\frac{2+\alpha}{2})\Gamma(k + \frac{1+\alpha}{2})}{\Gamma(\frac{1+\alpha}{2})\Gamma(k + \frac{2+\alpha}{2})} \quad (1.4)$$

and define for  $n \geq 1$ ,

$$T_n^\alpha(x) = \sum_{k=0}^n c_k(\alpha) \cos(kx). \quad (1.5)$$

Our main purpose is to find suitable conditions on  $\alpha$  under which

$$T_n^\alpha(x) > 0$$

holds true for  $n \in \mathbb{N}$  and  $x \in (0, \pi)$ .

It can be shown that the equation

$$\psi(\alpha) := \min_{x \in (0, \pi)} T_6^\alpha(x) = 0 \quad (1.6)$$

has a unique solution  $\alpha_0$  in  $(-1, \infty)$ . ( Its proof will be given at the end of this paper.) And mechanical computation shows  $\alpha_0 \in (2.3308, 2.3309)$ .

In this paper, we'll prove

**Theorem 1.** For  $-1 < \alpha \leq \alpha_0 = 2.3308\dots$ ,

$$T_n^\alpha(x) := \sum_{k=0}^n c_k(\alpha) \cos kx > 0, \quad x \in (0, \pi), \quad n \in \mathbb{N},$$

where the equality holds for some  $x \in (0, \pi)$  if and only if  $n = 6$  and  $\alpha = \alpha_0$ . Moreover, for  $\alpha > \alpha_0$ ,

$$\min_{x \in (0, \pi)} T_6^\alpha(x) < 0.$$

It is clear that Theorem 1 for  $\alpha = 0$  corresponds to Vietoris's inequality (1.1). We remark that Theorem 1 for  $\alpha = 1$  is due to Brown and Hewitt [2] while for  $\alpha = 2$  is due to a recent paper [3] by Brown and Yin.

As an immediate consequence of Theorem 1, we obtain, by Abel's transform,

**Corollary.** Let  $\alpha_0 = 2.3308\dots$  be as above and let  $\{a_k\}_{k=0}^\infty$  be a sequence of real numbers satisfying  $a_0 \geq a_1 \geq \dots \geq a_n > 0$  and  $(2k + \alpha_0)a_{2k} \leq (2k - 1 + \alpha_0)a_{2k-1}$ ,  $k \geq 1$ . Then we have

$$\sum_{k=0}^n a_k \cos kx > 0, \quad x \in (0, \pi), \quad n \in \mathbb{N},$$

where the equality holds for some  $x \in (0, \pi)$  if and only if  $n = 6$  and

$$a_0 = a_1 = 1, \quad a_2 = a_3 = \frac{1 + \alpha_0}{2 + \alpha_0},$$

$$a_4 = a_5 = \frac{(1 + \alpha_0)(3 + \alpha_0)}{(2 + \alpha_0)(4 + \alpha_0)},$$

$$a_6 = \frac{(1 + \alpha_0)(3 + \alpha_0)(5 + \alpha_0)}{(2 + \alpha_0)(4 + \alpha_0)(6 + \alpha_0)}.$$

We shall prove a second theorem as well. It can be stated as follows.

**Theorem 2.** For any  $\alpha > -1$  there's an  $M = M(\alpha)$  depending only on  $\alpha$  such that for  $n \geq M$  and  $x \in (0, \pi)$ ,

$$T_n^\alpha(x) := \sum_{k=0}^n c_k(\alpha) \cos kx > 0. \quad (1.7)$$

Furthermore, given an integer  $n \geq 2$  there is an  $\gamma = \gamma(n)$  depending only on  $n$  such that for  $\alpha > \gamma(n)$ ,

$$\min_{x \in (0, \pi)} T_n^\alpha(x) < 0.$$

For  $\alpha > -1$ , we denote by  $N(\alpha)$  the smallest positive integer for which (1.7) holds for all  $x \in (0, \pi)$  whenever  $n \geq N(\alpha)$ . Theorem 2 means that for each  $\alpha > -1$ ,  $N(\alpha)$  is finite and that

$$\lim_{\alpha \rightarrow \infty} N(\alpha) = \infty.$$

It would be interesting if one could find a better upper estimate of  $N(\alpha)$  for each specific  $\alpha$ . However, as will be demonstrated in our later proof of Theorem 1 where a better upper estimate for  $N(2.3309)$  is given, sometimes this seems to be very difficult.

We organize the paper as follows. Section 2 contains some fundamental lemmas that will play an important role in the later proofs of our main results. We shall first illustrate our method in the proof of Theorem 2 because it is technically easier to handle. This will be done in Section 3. After that, in Section 4 we shall prove Theorem 1 by more delicate computations. We conclude the paper in Section 5 with some remarks and further results on the sine sums.

## 2 Fundamental Lemmas

**Lemma 1.** For  $m \geq 0$  and  $x \in (0, \pi)$ ,

$$T_{2m+1}^\alpha(x) = 2 \cos\left(\frac{x}{2}\right) \sum_{k=0}^m c_{2k}(\alpha) \cos\left(\left(2k + \frac{1}{2}\right)x\right), \quad (2.1)$$

$$T_{2m+2}^\alpha(x) = T_{2m+1}^\alpha(x) + c_{2m+2}(\alpha) \cos((2m+2)x). \quad (2.2)$$

*Proof.* This is an immediate consequence of (1.4) and (1.5). □

**Lemma 2.** For  $x \in (0, \pi)$  and  $m \geq 100$ ,

$$T_{2m+1}^\alpha(x) = \frac{2 \cos \frac{x}{2}}{B\left(\frac{1+\alpha}{2}, \frac{1}{2}\right)} \left\{ I^\alpha(x) + J^\alpha(x) - K_m^\alpha(x) - r_m^\alpha(x) \right\} \quad (2.3)$$

where  $B(u, v)$  denotes the usual  $\beta$ -function defined by

$$B(u, v) = \int_0^1 t^{u-1} (1-t)^{v-1} dt = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} \quad (u, v > 0),$$

$$I^\alpha(x) = \cos \frac{x}{2} \int_0^1 \frac{t^{\frac{\alpha-1}{2}} \sqrt{1-t}}{(1-t)^2 + 4t \sin^2 x} dt, \quad (2.4)$$

$$J^\alpha(x) = 2 \sin \frac{x}{2} \sin x \int_0^1 \frac{t^{\frac{\alpha+1}{2}} dt}{\sqrt{1-t}((1-t)^2 + 4t \sin^2 x)}, \quad (2.5)$$

$$K_m^\alpha(x) = \sqrt{\pi} \sum_{k=m+1}^{\infty} \frac{\cos((2k + \frac{1}{2})x)}{(k + \frac{\alpha}{2} + \frac{1}{4})^{\frac{1}{2}}}$$

and  $r_m^\alpha(x)$  satisfies

$$|r_m^\alpha(x)| \leq \frac{0.0208}{(m + \frac{\alpha}{2} + \frac{1}{4})^{\frac{3}{2}}}. \quad (2.6)$$

*Proof.* Let

$$S_\infty^\alpha(x) = \sum_{k=0}^{\infty} c_{2k}(\alpha) \cos((2k + \frac{1}{2})x).$$

We first note that

$$\sum_{k=0}^{\infty} c_{2k}(\alpha) z^k = {}_2F_1(1, \frac{1+\alpha}{2}; \frac{2+\alpha}{2}; z),$$

where  ${}_2F_1(a, b; c; z)$  denotes Gauss' hypergeometric series which is defined by ( see [5], P. 56, formula (2))

$${}_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{\Gamma(k+a)\Gamma(k+b)\Gamma(c)}{\Gamma(a)\Gamma(b)\Gamma(k+c)\Gamma(k+1)} z^k.$$

We then use Euler's formula (see [5], P.59, formula (10))

$$F(a, b; c; z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1} (1-t)^{c-b-1} (1-tz)^{-1} dt$$

to obtain

$$S_\infty^\alpha(x) = \frac{1}{B(\frac{1+\alpha}{2}, \frac{1}{2})} \int_0^1 \frac{t^{\frac{\alpha-1}{2}}}{\sqrt{1-t}} \frac{\cos \frac{x}{2} - t \cos \frac{3x}{2}}{(1-t)^2 + 4t \sin^2 x} dt,$$

which, by elementary calculation, equals to

$$\begin{aligned} & \frac{\cos \frac{x}{2}}{B(\frac{1+\alpha}{2}, \frac{1}{2})} \int_0^1 \frac{t^{\frac{\alpha-1}{2}}}{\sqrt{1-t}} \frac{1-t+4t \sin^2 \frac{x}{2}}{(1-t)^2 + 4t \sin^2 x} dt \\ &= \frac{\cos \frac{x}{2}}{B(\frac{1+\alpha}{2}, \frac{1}{2})} \int_0^1 \frac{t^{\frac{\alpha-1}{2}} \sqrt{1-t}}{(1-t)^2 + 4t \sin^2 x} dt \\ & \quad + \frac{4 \sin^2 \frac{x}{2} \cos \frac{x}{2}}{B(\frac{1+\alpha}{2}, \frac{1}{2})} \int_0^1 \frac{t^{\frac{\alpha+1}{2}}}{\sqrt{1-t}((1-t)^2 + 4t \sin^2 x)} dt \\ &=: \frac{1}{B(\frac{1+\alpha}{2}, \frac{1}{2})} [I^\alpha(x) + J^\alpha(x)], \end{aligned} \quad (2.7)$$

where  $I^\alpha(x)$  and  $J^\alpha(x)$  are as defined in the lemma.

We write

$$R_m^\alpha(x) = \sum_{k=m+1}^{\infty} c_{2k}(\alpha) \cos\left(\left(2k + \frac{1}{2}\right)x\right). \quad (2.8)$$

Then by Lemma 1, we have

$$T_{2m+1}^\alpha(x) = 2 \cos \frac{x}{2} (S_\infty^\alpha(x) - R_m^\alpha(x)). \quad (2.9)$$

It will be shown that

$$R_m^\alpha(x) = \frac{1}{B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right)} \left[ K_m^\alpha(x) + r_m^\alpha(x) \right] \quad (2.10)$$

where  $K_m^\alpha(x)$  is as defined in the lemma and  $r_m^\alpha(x)$  satisfies (2.6). This, combined with (2.7) and (2.9), will give (2.3) and hence complete the proof.

To prove (2.10), we need a result from [4]. In fact, invoking Lemma 4.1 of [4] with  $\beta = \frac{1}{2}$  and  $x = k + \frac{\alpha}{2}$  yields that for  $k \geq 100$ ,

$$c_{2k}(\alpha) = \frac{\Gamma\left(\frac{2+\alpha}{2}\right)}{\Gamma\left(\frac{1+\alpha}{2}\right)} \left( \frac{1}{\left(k + \frac{\alpha}{2} + \frac{1}{4}\right)^{\frac{1}{2}}} - \frac{1}{64} \frac{1}{\left(k + \frac{\alpha}{2} + \frac{1}{4}\right)^{\frac{5}{2}}} + \frac{R(k+\alpha)}{\left(k + \frac{\alpha}{2} + \frac{1}{4}\right)^{\frac{7}{2}}} \right)$$

where, by formula (4.18) in [4],

$$|R(k+\alpha)| < 0.2.$$

This in turn implies that for  $k \geq 100$ ,

$$c_{2k}(\alpha) = \frac{\Gamma\left(\frac{2+\alpha}{2}\right)}{\Gamma\left(\frac{1+\alpha}{2}\right)} \left( \frac{1}{\left(k + \frac{\alpha}{2} + \frac{1}{4}\right)^{\frac{1}{2}}} + \frac{r(k+\alpha)}{\left(k + \frac{\alpha}{2} + \frac{1}{4}\right)^{\frac{5}{2}}} \right) \quad (2.11)$$

where

$$-0.0176 = -\frac{1}{64} - 0.2 < r(k+\alpha) < -\frac{1}{64} + 0.2 = -0.0136.$$

Now substitution (2.11) into (2.8) gives

$$\begin{aligned} R_m^\alpha(x) &= \frac{1}{B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right)} \left( \sqrt{\pi} \sum_{k=m+1}^{\infty} \frac{\cos\left(\left(2k + \frac{1}{2}\right)x\right)}{\left(k + \frac{\alpha}{2} + \frac{1}{4}\right)^{\frac{1}{2}}} + \sqrt{\pi} \sum_{k=m+1}^{\infty} \frac{r(k, \alpha) \cos\left(\left(2k + \frac{1}{2}\right)x\right)}{\left(k + \frac{\alpha}{2} + \frac{1}{4}\right)^{\frac{5}{2}}} \right) \\ &=: \frac{1}{B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right)} \left[ K_m^\alpha(x) + r_m^\alpha(x) \right] \end{aligned}$$

This will give (2.10) if we prove that  $r_m^\alpha(x)$  satisfies (2.6). In fact,

$$|r_m^\alpha(x)| \leq \sum_{k=m+1}^{\infty} \frac{0.0176\sqrt{\pi}}{\left(k + \frac{\alpha}{2} + \frac{1}{4}\right)^{\frac{5}{2}}} < \int_m^{\infty} \frac{0.0176\sqrt{\pi}}{\left(t + \frac{\alpha}{2} + \frac{1}{4}\right)^{\frac{5}{2}}} dt = \frac{0.028}{\left(m + \frac{\alpha}{2} + \frac{1}{4}\right)^{\frac{3}{2}}},$$

which is (2.6). □

As a consequence of Lemma 2, we have the following two lemmas, which will be used in our proofs in later sections.

**Lemma 3.** For  $m \geq 100$  and  $x \in [0, \frac{\pi}{2}]$ ,

$$\frac{1}{2 \cos \frac{x}{2}} B\left(\frac{1+\alpha}{2}, \frac{1}{2}\right) T_{2m+1}^\alpha(x) = I^\alpha(x) + J^\alpha(x) - L_m^\alpha(x) - D_m^\alpha(x) \quad (2.12)$$

where  $I^\alpha(x)$ ,  $J^\alpha(x)$  are as defined in Lemma 2,

$$L_m^\alpha(x) = \sqrt{\frac{x}{\sin x}} \sqrt{\frac{\pi}{2 \sin x}} \int_{(2m+\frac{3}{2})x}^\infty \frac{\cos t}{\sqrt{t+\alpha x}} dt, \quad (2.13)$$

and  $D_m^\alpha(x)$  satisfies

$$|D_m^\alpha(x)| \leq \frac{0.0208}{(m + \frac{\alpha}{2} + \frac{1}{4})^{\frac{3}{2}}} + \frac{x}{\sin x} \frac{0.8863}{\sqrt{m + \frac{\alpha}{2} - \frac{1}{4}}}. \quad (2.14)$$

*Proof.* Let  $K_m^\alpha(x)$  be defined as in Lemma 2. Then we have

$$\begin{aligned} K_m^\alpha(x) &= \frac{\sqrt{\pi x}}{\sin x} \sum_{k=m+1}^\infty \int_{k-\frac{1}{4}}^{k+\frac{3}{4}} \frac{\cos(2xt)}{(k + \frac{\alpha}{2} + \frac{1}{4})^{\frac{1}{2}}} dt \\ &= \frac{\sqrt{\pi x}}{\sin x} \sum_{k=m+1}^\infty \int_{k-\frac{1}{4}}^{k+\frac{3}{4}} \left( \frac{1}{(k + \frac{\alpha}{2} + \frac{1}{4})^{\frac{1}{2}}} - \frac{1}{(t + \frac{\alpha}{2})^{\frac{1}{2}}} \right) \cos(2xt) dt \\ &\quad + \frac{\sqrt{\pi x}}{\sin x} \int_{m+\frac{3}{4}}^\infty \frac{\cos(2xu)}{(u + \frac{\alpha}{2})^{\frac{1}{2}}} du \\ &=: u_m^\alpha(x) + L_m^\alpha(x). \end{aligned}$$

For the second term  $L_m^\alpha(x)$ , a simple change of variable  $t = 2xu$  gives the identity (2.13), while for the first term  $u_m^\alpha(x)$ , elementary calculation gives

$$\begin{aligned} |u_m^\alpha(x)| &\leq \frac{\sqrt{\pi x}}{\sin x} \sum_{k=m+1}^\infty \frac{1}{4(k + \frac{\alpha}{2} - \frac{1}{4})^{\frac{3}{2}}} \\ &< \frac{\sqrt{\pi x}}{\sin x} \int_m^\infty \frac{dy}{4(y + \frac{\alpha}{2} - \frac{1}{4})^{\frac{3}{2}}} = \frac{\sqrt{\pi x}}{2 \sin x \sqrt{m + \frac{\alpha}{2} - \frac{1}{4}}}. \end{aligned} \quad (2.15)$$

Now invoking Lemma 2, we obtain Lemma 3 by letting

$$D_m^\alpha(x) = r_m^\alpha(x) + u_m^\alpha(x).$$

□

**Lemma 4.** For  $m \geq 100$  and  $y \in [0, \frac{\pi}{2}]$ ,

$$B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right) T_{2m+1}^\alpha(\pi - y) \geq 2 \sin \frac{y}{2} \left[ J^\alpha(\pi - y) - \tilde{L}_m^\alpha(y) - \tilde{D}_m^\alpha(y) \right],$$

where  $J^\alpha$  is as defined in Lemma 1,

$$\tilde{L}_m^\alpha(y) = \sqrt{\frac{\pi y}{2 \sin^2 y}} \int_{(2m+\frac{3}{2})y}^\infty \frac{\sin t}{\sqrt{t+\alpha y}} dt, \quad (2.16)$$

and  $\tilde{D}_m^\alpha(y)$  satisfies

$$|\tilde{D}_m^\alpha(y)| \leq \frac{0.0208}{(m + \frac{\alpha}{2} + \frac{1}{4})^{\frac{3}{2}}} + \frac{y}{\sin y} \frac{0.8863}{\sqrt{m + \frac{\alpha}{2} - \frac{1}{4}}}. \quad (2.17)$$

The proof of Lemma 4 is completely analogous to that of Lemma 3. So we omit its detail.

### 3 Proof of Theorem 2

Assume  $M \geq M(\alpha)$  with  $M(\alpha)$  a sufficiently large constant depending only on  $\alpha$ . We will keep this assumption throughout the proof. Some of our estimates below may not be true without this assumption. We will use the notation ‘ $O(\varepsilon_m)$ ’ to mean that there’s an inessential positive constant  $C$  (independent of  $m$  and  $x$ ) such that  $O(\varepsilon_m) \leq C\varepsilon_m$  as  $m \rightarrow \infty$ .

#### 3.1 Proof of Theorem 2 for $x \in [0, \frac{\pi}{2}]$

By Lemma 1, it’s obvious that for  $x \in [0, \frac{\pi}{4(m+1)}]$

$$T_{2m+1}^\alpha(x) > 0, \quad T_{2m+2}^\alpha(x) > 0.$$

It will be shown in this subsection that for  $x \in [\frac{\pi}{4(m+1)}, \frac{\pi}{2}]$ ,

$$T_{2m+1}^\alpha(x) \geq \frac{\varepsilon_\alpha}{\sqrt{x}} > 0 \quad (3.1)$$

with  $\varepsilon_\alpha$  independent of  $x$  and  $m$ , from which Theorem 2 for  $x \in [0, \frac{\pi}{2}]$  will follow. Here, as indicated above, we always assume  $m \geq M(\alpha)$ .

To prove (3.1), we take a  $\delta \in (0, 1)$  such that

$$(1 - \delta)^{\frac{\alpha-1}{2}} > \frac{99}{100}.$$

We then use (2.4) to obtain

$$\begin{aligned} I^\alpha(x) &\geq \min\{1, (1 - \delta)^{\frac{\alpha-1}{2}}\} \cos \frac{x}{2} \int_{1-\delta}^1 \frac{\sqrt{1-t}}{(1-t)^2 + 4\sin^2 x} dt \\ &> \frac{1}{\sqrt{2\sin x}} A^\alpha(x), \end{aligned}$$

where

$$A^\alpha(x) := \frac{99 \cos \frac{x}{2}}{100} \int_0^{\frac{\delta}{2\sin x}} \frac{\sqrt{t}}{1+t^2} dt.$$

Now invoking Lemma 3 we obtain

$$B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right) T_{2m+1}^\alpha(x) \geq \frac{2 \cos \frac{x}{2}}{\sqrt{2\sin x}} \left( A^\alpha(x) - B_m^\alpha(x) + O(m^{-\frac{1}{2}}) \right), \quad (3.2)$$

where

$$B_m^\alpha(x) = \sqrt{\frac{\pi x}{\sin x}} \int_{(2m+\frac{3}{2})x}^{\infty} \frac{\cos t}{\sqrt{t+\alpha x}} dt.$$

So, the proof of (3.1) is reduced to the proof of the following inequality:

$$A^\alpha(x) - B_m^\alpha(x) > \varepsilon'_\alpha > 0 \quad (3.3)$$

where  $x \in [\frac{\pi}{4(m+1)}, \frac{\pi}{2}]$ . To prove this inequality, we fix a number  $C$  which is sufficiently large and depends only on  $\alpha$ , and consider the following two cases:

*Case 1.*  $(m+1)x \geq C$ .

In this case,

$$A^\alpha(x) \geq \frac{99}{100} \cos \frac{\pi}{4} \int_0^{\frac{\delta}{2}} \frac{\sqrt{t}}{1+t^2} dt = 0.7 \int_0^{\frac{\delta}{2}} \frac{\sqrt{t}}{1+t^2} dt$$

while

$$|B_m^\alpha(x)| \leq \frac{\pi}{\sqrt{2}} \frac{4}{\sqrt{(2m+\frac{3}{2})x}} \leq \frac{2\sqrt{2}\pi}{\sqrt{C}}.$$

These last two inequalities imply

$$A^\alpha(x) - B_m^\alpha(x) \geq 0.7 \int_0^{\frac{\delta}{2}} \frac{\sqrt{t}}{1+t^2} dt - \frac{2\sqrt{2}\pi}{\sqrt{C}} > 0.6 \int_0^{\frac{\delta}{2}} \frac{\sqrt{t}}{1+t^2} dt$$

as long as

$$C > 800\pi^2 \left( \int_0^{\frac{\delta}{2}} \frac{\sqrt{t}}{1+t^2} dt \right)^{-2}.$$

*Case 2.*  $\frac{\pi}{4} \leq (m+1)x \leq C$ .

In this case,

$$\begin{aligned} A^\alpha(x) &\geq \frac{99}{100} \cos\left(\frac{C}{m+1}\right) \int_0^{\frac{\delta(m+1)}{2C}} \frac{\sqrt{t}}{1+t^2} dt \\ &= \frac{99}{100} \int_0^{\infty} \frac{\sqrt{t}}{1+t^2} dt + O(m^{-\frac{1}{2}}) > 2.1992 \end{aligned} \quad (3.4)$$

provided  $M(\alpha)$  is big enough.

Meanwhile

$$\begin{aligned} B_m^\alpha(x) &= \sqrt{\frac{\pi x}{\sin x}} \int_{(2m+\frac{3}{2})x}^{\infty} \frac{\cos t}{\sqrt{t}} dt + \sqrt{\frac{\pi x}{\sin x}} \int_{(2m+\frac{3}{2})x}^{\infty} \left( \frac{1}{\sqrt{t+\alpha x}} - \frac{1}{\sqrt{t}} \right) \cos t dt \\ &= \sqrt{\pi} \int_{(2m+\frac{3}{2})x}^{\infty} \frac{\cos t}{\sqrt{t}} dt + O(m^{-1}). \end{aligned} \quad (3.5)$$

For simplicity, let

$$G(z) = \sqrt{\pi} \int_z^\infty \frac{\cos t}{\sqrt{t}} dt.$$

Then we have

$$\begin{aligned} A^\alpha(x) - B_m^\alpha(x) &> 2.1992 - G\left(\left(2m + \frac{3}{2}\right)x\right) \geq 2.1992 - \max_{\frac{49\pi}{100} \leq z \leq 2C} G(z) \\ &= 2.1992 - \max\left\{G\left(\frac{49\pi}{100}\right), G\left(\frac{3}{2}\pi\right)\right\} = 2.1992 - 0.7950 = 1.4042. \end{aligned}$$

In summary, in both cases we have proved (3.3) and hence (3.1) from which Theorem 2 for all  $x \in [0, \frac{\pi}{2}]$  follows.

### 3.2 Proof of Theorem 2 for $x \in [\frac{\pi}{2}, \pi]$

Let  $y = \pi - x$ . By Lemma 1, it's obvious that both  $T_{2m+1}^\alpha(\pi - y)$  and  $T_{2m+2}^\alpha(\pi - y)$  are positive when  $y \in (0, \frac{\pi}{4(m+1)}]$ . It will be shown that for  $y \in [\frac{\pi}{4(m+1)}, \frac{\pi}{2}]$ ,

$$T_{2m+1}^\alpha(\pi - y) \geq \varepsilon''_\alpha \sqrt{\sin y} \quad (3.6)$$

and

$$T_{2m+2}^\alpha(\pi - y) \geq \varepsilon'''_\alpha \sqrt{\frac{\pi}{m+1}} \quad (3.7)$$

with  $\varepsilon''_\alpha, \varepsilon'''_\alpha$  two positive constants depending only on  $\alpha$ , from which Theorem 2 for  $x \in [\frac{\pi}{2}, \pi]$  will follow.

To prove (3.6) and (3.7), we let  $\delta \in (0, 1)$  such that  $(1 - \delta)^{\frac{\alpha+1}{2}} > \frac{99}{100}$ . We then get from (2.5)

$$\begin{aligned} J^\alpha(\pi - y) &\geq 2(1 - \delta)^{\frac{\alpha+1}{2}} \cos \frac{y}{2} \sin y \int_{1-\delta}^1 \frac{dt}{\sqrt{1-t}((1-t)^2 + 4\sin^2 y)} \\ &> \frac{99}{100} \frac{\cos \frac{y}{2}}{\sqrt{2\sin y}} \int_0^{\frac{\delta}{2\sin y}} \frac{dt}{\sqrt{t}(t^2 + 1)}, \end{aligned}$$

which, by Lemma 4, implies

$$\begin{aligned} &B\left(\frac{1+\alpha}{2}, \frac{1}{2}\right) T_{2m+1}^\alpha(\pi - y) \\ &> \sqrt{2\sin y} \left\{ \frac{99}{200} \int_0^{\frac{\delta}{2\sin y}} \frac{dt}{\sqrt{t}(t^2 + 1)} - \frac{1}{2\cos \frac{y}{2}} \sqrt{\frac{\pi y}{\sin y}} \int_{(2m+\frac{3}{2})y}^\infty \frac{\sin t}{\sqrt{t + \alpha y}} dt \right\} \end{aligned} \quad (3.8)$$

provided  $M(\alpha)$  is large enough.

Meanwhile, by (2.2), we have

$$T_{2m+2}^\alpha(\pi - y) = T_{2m+1}^\alpha(\pi - y) + c_{2m+2}(\alpha) \cos(2m+2)y, \quad (3.9)$$

where

$$c_{2m+2}(\alpha) = \frac{1}{B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right)} \sqrt{\frac{\pi}{m+1}} + O(m^{-\frac{3}{2}}) \quad (3.10)$$

on account of (2.11).

Now we take a sufficiently large constant  $B$  depending only on  $\alpha$  and consider the following two cases:

*Case 1.*  $(m+1)y \geq B$ .

We use (3.8) and apply the second mean value theorem to its second integral to obtain

$$\begin{aligned}
B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right)T_{2m+1}^\alpha(\pi-y) &\geq \sqrt{2\sin y} \left\{ \frac{99}{200} \int_0^{\frac{\delta}{2}} \frac{dt}{\sqrt{t(1+t^2)}} - \frac{1}{2\cos\frac{\pi}{4}} \sqrt{\frac{\pi^2}{2}} \frac{4}{\sqrt{(m+1)y}} \right\} \\
&\geq \sqrt{2\sin y} \left\{ \frac{99}{200} \int_0^{\frac{\delta}{2}} \frac{dt}{\sqrt{t(1+t^2)}} - 2\pi B^{-\frac{1}{2}} \right\} \\
&\geq \sqrt{\sin y} \frac{99}{200} \int_0^{\frac{\delta}{2}} \frac{dt}{\sqrt{t(1+t^2)}}
\end{aligned} \tag{3.11}$$

provided  $B$  is large enough. This proves (3.6).

Meanwhile, by (3.9), (3.10) and (3.11), if  $B$  is sufficiently large, we have

$$\begin{aligned}
B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right)T_{2m+2}^\alpha(\pi-y) &\geq \sqrt{\sin y} \times \frac{99}{200} \int_0^{\frac{\delta}{2}} \frac{dt}{\sqrt{t(1+t^2)}} - \sqrt{\frac{\pi}{m+1}} \\
&\geq \left[ \frac{\sqrt{2B}}{\pi} \times \frac{99}{200} \int_0^{\frac{\delta}{2}} \frac{dt}{\sqrt{t(1+t^2)}} - 1 \right] \sqrt{\frac{\pi}{m+1}} \\
&\geq \frac{1}{2} \sqrt{\frac{\pi}{m+1}},
\end{aligned}$$

which proves (3.7) with  $\varepsilon''_\alpha = \frac{1}{2} \left( B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right) \right)^{-1}$  in this case.

*Case 2.*  $\frac{\pi}{4} \leq (m+1)y \leq B$ .

In this case, as  $y = O\left(\frac{1}{m+1}\right)$ , we get from (3.8)

$$B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right)T_{2m+1}^\alpha(\pi-y) > \sqrt{2y} \left[ \frac{99}{200} \int_0^\infty \frac{dt}{\sqrt{t(1+t^2)}} - \frac{\sqrt{\pi}}{2} \int_{(2m+\frac{3}{2})y}^\infty \frac{\sin t}{\sqrt{t}} dt \right]. \tag{3.12}$$

For convenience, let

$$F(z) := \frac{\sqrt{\pi}}{2} \int_z^\infty \frac{\sin t}{\sqrt{t}} dt.$$

Since

$$\frac{99}{200} \int_0^\infty \frac{dt}{\sqrt{t(1+t^2)}} = 1.0996,$$

(3.12) can be rewritten as

$$B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right)T_{2m+1}^\alpha(\pi-y) > \sqrt{2y} \left[ 1.0996 - F\left(\left(2m + \frac{3}{2}\right)y\right) \right]. \tag{3.13}$$

Clearly, for  $m$  sufficiently large,  $(2m + \frac{3}{2})y \geq \frac{49}{100}\pi$  and hence

$$F((2m + \frac{3}{2})y) \leq \max_{z \geq \frac{49}{100}\pi} F(z) = \left\{ F(2\pi), F(\frac{49}{100}\pi) \right\} = 0.3478.$$

This and (3.13) give (3.6).

Now it remains to prove (3.7) for  $\frac{\pi}{4} \leq (m+1)y \leq B$ . It follows from (3.9), (3.10) and (3.13) that

$$\begin{aligned} & B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right) T_{2m+2}^\alpha(\pi-y) \\ & \geq \left\{ \sqrt{(m+1)y} \sqrt{\frac{2}{\pi}} \left(1.0996 - F((2m + \frac{3}{2})y)\right) + \cos(2m+2)y \right\} \sqrt{\frac{\pi}{m+1}} \\ & = C_m(y) \sqrt{\frac{\pi}{m+1}}, \end{aligned} \quad (3.14)$$

where

$$C_m(y) := \sqrt{(m+1)y} \sqrt{\frac{2}{\pi}} \left[1.0996 - F((2m + \frac{3}{2})y)\right] + \cos(2m+2)y. \quad (3.15)$$

If  $\cos(2m+2)y \geq 0$  then it's clear that (3.9) and (3.6) imply (3.7). So we may assume  $\cos(2m+2)y < 0$ . By (3.14), it will suffice to show  $C_m(y) > \varepsilon_\alpha''' > 0$  for  $\frac{\pi}{4} \leq (m+1)y \leq B$ . To this end, we divide the interval  $[\frac{\pi}{4(m+1)}, \frac{\pi}{2}]$  into the following four parts and will consider each of them separately:

- I.  $\frac{\pi}{4} \leq (m+1)y \leq 0.36\pi$ ,      II.  $0.36\pi \leq (m+1)y \leq 0.5\pi$ ,  
 III.  $0.5\pi \leq (m+1)y \leq 0.75\pi$ ,      IV.  $1.25\pi \leq (m+1)y \leq B$ .

( We remind the reader that  $\cos 2(m+1)y \geq 0$  for  $0.75\pi \leq (m+1)y \leq 1.25\pi$ .)

For the first part,  $\frac{49}{100}\pi \leq (2m + \frac{3}{2})y \leq 0.72\pi$ . As

$$\max_{\frac{49}{100}\pi \leq z \leq 0.72\pi} F(z) = F(\frac{49}{100}\pi) = 0.1595,$$

we get from (3.15)

$$C_m(y) \geq \sqrt{\frac{2}{\pi}} \sqrt{\frac{\pi}{4}} (1.0996 - 0.1595) + \cos(0.72\pi) = 0.0273 > 0.$$

The other three parts can be handled similarly. In fact, we have, for the second part,

$$C_m(y) \geq \sqrt{\frac{2}{\pi}} \sqrt{0.36\pi} (1.0996 - F(0.71\pi)) - 1 = 0.1533 > 0,$$

for the third part,

$$C_m(y) \geq \sqrt{\frac{2}{\pi}} \sqrt{0.5\pi} (1.0996 - F(\frac{3}{2}\pi)) - 1 = 0.138 > 0,$$

and finally, for the last part,

$$C_m(y) \geq \sqrt{\frac{2}{\pi}} \sqrt{1.25\pi} (1.0996 - F(2\pi)) - 1 = 0.1886 > 0.$$

Putting these together, we obtain that if  $\cos 2(m+1)y \leq 0$  then  $C_m(y) \geq 0.027$ , which proves (3.7) with  $\varepsilon_\alpha'' = 0.027/B(\frac{\alpha+1}{2}, \frac{1}{2})$  in this case.

### 3.3 Concluding proof

In this subsection we shall prove the second conclusion of Theorem 2, which is quite easy. In fact, for a fixed integer  $k \geq 0$ ,

$$\lim_{\alpha \rightarrow \infty} c_{2k}(\alpha) = \lim_{\alpha \rightarrow \infty} c_{2k+1}(\alpha) = \lim_{\alpha \rightarrow \infty} \frac{(\alpha+1)(\alpha+3)\cdots(\alpha+2k-1)}{(\alpha+2)(\alpha+4)\cdots(\alpha+2k)} = 1.$$

This implies that for a given integer  $n \geq 2$ ,

$$\lim_{\alpha \rightarrow \infty} T_n^\alpha(x) = \sum_{k=0}^n \cos kx = \frac{\sin \frac{n+1}{2}x \cos \frac{n}{2}x}{\sin \frac{x}{2}},$$

which assumes a negative value at  $x = \frac{4\pi}{4n-1}$ . Therefore, there must be a  $\gamma(n)$  depending only on  $n$  such that for  $\alpha > \gamma(n)$ ,

$$\min_{x \in (0, \pi)} T_n^\alpha(x) \leq T_n^\alpha\left(\frac{4\pi}{4n-1}\right) < 0,$$

as desired.

## 4 Proof of Theorem 1

Theorem 1 is a consequence of the following lemma, whose proof will be the main part of this section.

**Lemma 5.** For  $n \neq 6$ ,  $\alpha_1 = 2.3309$  and  $x \in (0, \pi)$ ,

$$T_n^{\alpha_1}(x) > 0$$

holds true.

For the moment, we take Lemma 5 for granted and proceed with our proof. First, by mechanical computation, we know

$$\min_{x \in (0, \pi)} T_6^{2.3309}(x) < 0 < \min_{x \in (0, \pi)} T_6^{2.3308}(x).$$

So, there must be an  $\alpha_0 \in (2.3308, 2.3309)$  such that

$$\min_{x \in (0, \pi)} T_6^{\alpha_0}(x) = 0. \tag{4.1}$$

(Such an  $\alpha_0$  will be shown to be unique in Section 5.)

Let  $\mu_k = \frac{c_k(\alpha_0)}{c_k(\alpha_1)}$ . Then by (1.4) it is easy to verify that for even  $k$ ,  $\mu_k = \mu_{k+1}$  while for odd  $k$ ,

$$\mu_k = \frac{k+1+\alpha_0}{k+\alpha_0} \frac{k+\alpha_1}{k+\alpha_1+1} \mu_{k+1} > \mu_{k+1}.$$

Hence, by Abel's transform, for  $n \neq 6$ ,

$$T_n^{\alpha_0}(x) = \sum_{\substack{0 \leq k \leq n-1 \\ k \neq 6}} (\mu_k - \mu_{k+1}) T_k^{\alpha_1}(x) + \mu_n T_n^{\alpha_1}(x) > 0.$$

The last inequality follows from Lemma 5. Meanwhile, by (4.1),

$$T_6^{\alpha_0}(x) \geq 0.$$

For  $-1 < \alpha < \alpha_0$ , we let  $\mu_k(\alpha) = \frac{c_k(\alpha)}{c_k(\alpha_0)}$ . Repeating the above argument, we have, for  $n \geq 1$ ,

$$T_n^\alpha(x) = \sum_{0 \leq k \leq n-1} \left( \mu_k(\alpha) - \mu_{k+1}(\alpha) \right) T_k^{\alpha_0}(x) + \mu_n(\alpha) T_n^{\alpha_0}(x) > 0.$$

This proves Theorem 1.

Now it remains to prove Lemma 5. Mechanical computation (using maple)<sup>2</sup> shows that for  $1 \leq n \leq 200$ ,  $n \neq 6$  and  $x \in (0, \pi)$ ,

$$T_n^{2.3309}(x) > 0.$$

So, we need only to prove that for  $\alpha = 2.3309$ ,  $m \geq 100$  and  $x \in (0, \pi)$ ,

$$T_{2m+1}^\alpha(x) > 0, \quad T_{2m+2}^\alpha(x) > 0. \quad (4.2)$$

Throughout the rest of the proof in this section, we always assume that  $m \geq 100$  and  $\alpha = 2.3309$ . Some of our estimates below may not be true without these assumptions.

The proof is quite long, so we break it into several parts.

#### 4.1 Proof of Lemma 5 for $x \in (0, \frac{\pi}{2}]$

For  $x \in (0, \frac{\pi}{2}]$  and  $z \geq 0$ , we define

$$H(z, x) := \int_z^\infty \frac{\cos t}{\sqrt{t + \alpha x}} dt.$$

For simplicity, we denote  $H(z, 0)$  by  $H(z)$ . It is easy to show that for nonnegative integer  $k$ ,

$$\max_{z \geq 2k\pi + \frac{3\pi}{2}} H(z, x) = H(2k\pi + \frac{3\pi}{2}, x) \leq H(2k\pi + \frac{3\pi}{2}). \quad (4.3)$$

Below we shall use this fact frequently.

We consider the following three cases:

*Case I.*  $0 < x \leq \frac{\pi}{4m+4}$ .

In this case, by Lemma 1, (4.2) trivially holds.

*Case II.*  $\frac{\pi}{4m+4} < x < 0.15$ .

First, we claim that for  $0 < x < 0.15$  and  $m \geq 100$ ,

$$\begin{aligned} & \frac{1}{2 \cos \frac{x}{2}} B\left(\frac{1+\alpha}{2}, \frac{1}{2}\right) T_{2m+1}^\alpha(x) \\ & \geq \frac{1}{\sqrt{2 \sin x}} \left[ 0.6274 \int_0^{\frac{1}{4 \sin x}} \frac{\sqrt{t}}{t^2 + 1} dt - 1.7784 \max\left\{0, H\left(\left(2m + \frac{3}{2}\right)x, x\right)\right\} \right] - 0.0887 \quad (4.4) \end{aligned}$$

---

<sup>2</sup>For the details on the mechanical computation, the reader can contact the third author.

and for  $m \geq 100$ ,

$$c_{2m+2}(\alpha) \leq \frac{0.1751}{B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right)}. \quad (4.5)$$

In fact, (4.5) can be easily obtained from (1.4):

$$c_{2m+2}(\alpha) = \frac{\sqrt{\pi}}{B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right)} \frac{\Gamma(m+1 + \frac{\alpha+1}{2})}{\Gamma(m+1 + \frac{\alpha+2}{2})} \leq \frac{\sqrt{\pi}}{B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right)} \frac{\Gamma(101 + \frac{\alpha+1}{2})}{\Gamma(101 + \frac{\alpha+2}{2})} = \frac{0.1751}{B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right)},$$

while (4.4) is a consequence of Lemma 3 and the following elementary estimates:

$$\begin{aligned} |D_m^\alpha(x)| &\leq 0.0887, && \text{(by (2.14))} \\ I^\alpha(x) &> 2^{-0.6655} \cos \frac{x}{2} \int_{\frac{1}{2}}^1 \frac{\sqrt{1-t}}{(1-t)^2 + 4\sin^2 x} dt \geq \frac{0.6274}{\sqrt{2\sin x}} \int_0^{\frac{1}{4\sin x}} \frac{\sqrt{t}}{t^2+1} dt, \\ L_m^\alpha(x) &\leq \sqrt{\frac{0.15}{\sin(0.15)}} \frac{\sqrt{\pi}}{\sqrt{2\sin x}} \max\left\{0, H\left((2m + \frac{3}{2})x, x\right)\right\} \\ &= \frac{1.7784}{\sqrt{2\sin x}} \max\left\{0, H\left((2m + \frac{3}{2})x, x\right)\right\}. \end{aligned}$$

Next, we divide the interval  $[\frac{\pi}{4m+4}, 0.15]$  into three subintervals:  $[\frac{\pi}{4m+4}, \frac{5\pi}{4m+3}]$ ,  $[\frac{5\pi}{4m+3}, 0.1]$  and  $[0.1, 0.15]$ . We will consider each of them separately.

For  $\frac{\pi}{4m+4} < x \leq \frac{5\pi}{4m+3} \leq \frac{5\pi}{403} = 0.0390$ , as  $(1 - \frac{1}{404})\frac{\pi}{2} < (2m + \frac{3}{2})x \leq \frac{5\pi}{2}$  and  $\frac{1}{4\sin x} > 6$ , we obtain from (4.4) and (4.3)

$$\begin{aligned} &\frac{1}{2\cos \frac{x}{2}} B\left(\frac{1+\alpha}{2}, \frac{1}{2}\right) T_{2m+1}^\alpha(x) \\ &> \frac{1}{\sqrt{2\sin x}} \left(0.6274 \int_0^6 \frac{\sqrt{t}}{t^2+1} dt - 1.7784 H\left(\frac{3}{2}\pi\right)\right) - 0.0887 \\ &\geq \frac{0.0869}{\sqrt{2\sin 0.0390}} - 0.0887 = 0.2225. \end{aligned}$$

Meanwhile, it follows from (2.2) and (4.5) that

$$T_{2m+2}^\alpha(x) > \frac{2\cos \frac{0.039}{2} \times 0.2225 - 0.1751}{B\left(\frac{1+\alpha}{2}, \frac{1}{2}\right)} = \frac{0.2676}{B\left(\frac{1+\alpha}{2}, \frac{1}{2}\right)}.$$

The other two subintervals can be handled similarly. In fact, for  $x \in [\frac{5\pi}{4m+3}, 0.1]$ , we have, by (4.4) and (4.3),

$$\begin{aligned} &\frac{1}{2\cos \frac{x}{2}} B\left(\frac{1+\alpha}{2}, \frac{1}{2}\right) T_{2m+1}^\alpha(x) \\ &> \frac{1}{\sqrt{2\sin x}} \left(0.6274 \int_0^{2.5} \frac{\sqrt{t}}{t^2+1} dt - 1.7784 H\left(\frac{7}{2}\pi\right)\right) - 0.0887 \\ &= \frac{0.0903}{\sqrt{2\sin x}} - 0.0887 > \frac{0.0903}{\sqrt{2\sin 0.1}} - 0.0887 = 0.1135, \end{aligned}$$

and, by (4.5) and (2.2),

$$T_{2m+2}^\alpha(x) > \frac{0.1135 \times 2 \cos \frac{0.1}{2} - 0.1751}{B\left(\frac{1+\alpha}{2}, \frac{1}{2}\right)} = \frac{0.0474}{B\left(\frac{1+\alpha}{2}, \frac{1}{2}\right)}.$$

For  $x \in [0.1, 0.15]$ , by (4.4) and (4.3),

$$\begin{aligned} & \frac{1}{2 \cos \frac{x}{2}} B\left(\frac{1+\alpha}{2}, \frac{1}{2}\right) T_{2m+1}^\alpha(x) \\ & > \frac{1}{\sqrt{2 \sin x}} \left( 0.6274 \int_0^{1.6729} \frac{\sqrt{t}}{t^2+1} dt - 1.7784 H\left(\frac{15}{2}\pi\right) \right) - 0.0887 \\ & \geq \frac{0.1160}{\sqrt{2 \sin 0.15}} - 0.0887 = 0.1235 \end{aligned}$$

and by (4.5) and (2.1),

$$T_{2m+2}^\alpha(x) \geq \frac{0.1235 \times 2 \cos \frac{0.15}{2} - 0.1751}{B\left(\frac{1+\alpha}{2}, \frac{1}{2}\right)} = \frac{0.0712}{B\left(\frac{1+\alpha}{2}, \frac{1}{2}\right)}.$$

In summary, we obtain, for  $x \in [\frac{\pi}{4(m+1)}, 0.15]$ ,

$$T_{2m+1}^\alpha(x) > 2 \cos \frac{x}{2} \frac{0.1135}{B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right)}, \quad T_{2m+1}^\alpha(x) > \frac{0.0474}{B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right)}.$$

This proves (4.2) in this case.

*Case III.*  $0.15 < x \leq \frac{\pi}{2}$ .

In this case, it follows from (2.14) that

$$|D_m^\alpha(x)| \leq 0.1382.$$

We divide the interval  $(0.15, \frac{\pi}{2}]$  into the following 46 subintervals:

$$\Delta_5 = \left(0.15, \frac{11.5}{201.5}\pi\right], \quad \Delta_\ell = \left(\frac{2\ell - 0.5}{201.5}\pi, \frac{2\ell + 1.5}{201.5}\pi\right], \quad \ell = 6, 7, \dots, 49; \quad \Delta_{50} = \left(\frac{99.5}{201.5}\pi, \frac{\pi}{2}\right].$$

We will consider each of them separately.

We let  $\theta_\ell = \frac{2\ell+1.5}{201.5}\pi$ . For  $x \in \Delta_\ell$ , by (2.4), (2.5) and (2.13) the following estimates are clear:

$$\begin{aligned} I^\alpha(x) & \geq \cos \frac{\theta_\ell}{2} \int_0^1 \frac{t\sqrt{1-t}}{(1-t)^2 + 4\sin^2\theta_\ell} dt =: a_\ell, \\ L_m^\alpha(x) & \leq \sqrt{\frac{\pi\theta_\ell}{2\sin^2\theta_\ell}} \int_{(2\ell-0.5)\pi}^\infty \frac{\cos t}{\sqrt{t}} dt =: b_\ell, \\ J^\alpha(x) & > 2 \sin \frac{\theta_\ell}{2} \sin \theta_\ell \int_0^1 \frac{t^2 dt}{\sqrt{1-t}((1-t)^2 + 4\sin^2\theta_\ell)}. \end{aligned}$$

So, it follows from Lemma 3 that for  $\ell = 5, 6, \dots, 50$  and  $x \in \Delta_\ell$ ,

$$\frac{1}{2 \cos \frac{x}{2}} B\left(\frac{1+\alpha}{2}, \frac{1}{2}\right) T_{2m+1}^\alpha(x) \geq a_\ell - b_\ell + c_\ell - 0.1382 =: d_\ell.$$

For each  $5 \leq \ell \leq 50$ , the value of  $d_\ell$  is computable. In fact, via using Maple in the computer, we can list the results as follows:

$\ell$	5	6	7	8	9	10	11	12	13	14
$d_\ell$	0.5508	0.4589	0.3889	0.3349	0.2925	0.2593	0.2326	0.2113	0.1900	0.1700
$\ell$	15	16	17	18	19	20	21	22	23	24
$d_\ell$	0.1685	0.1593	0.1517	0.1456	0.1406	0.1367	0.1336	0.1313	0.1295	0.1283
$\ell$	25	26	27	28	29	30	31	32	33	34
$d_\ell$	0.1276	0.1272	0.1273	0.1277	0.1283	0.1293	0.1305	0.1319	0.1336	0.1355
$\ell$	35	36	37	38	39	40	41	42	43	44
$d_\ell$	0.1376	0.1398	0.1423	0.1450	0.1478	0.1508	0.1541	0.1575	0.1611	0.1648
$\ell$	45	46	47	48	49	50				
$d_\ell$	0.1688	0.1730	0.1775	0.1821	0.1870	0.1904				

From the above table, it follows that for  $0.15 \leq x \leq \frac{\pi}{2}$ ,

$$\frac{1}{2 \cos \frac{x}{2}} B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right) T_{2m+1}^\alpha(x) \geq \min_{5 \leq \ell \leq 50} d_\ell = d_{26} = 0.1272.$$

Combing this with (4.5) and (2.2) yields

$$T_{2m+2}^\alpha(x) \geq \frac{1}{B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right)} \left(2 \cos \frac{x}{2} \times 0.1272 - 0.1751\right) \geq \frac{0.0048}{B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right)}.$$

This proves (4.2) in this case.

## 4.2 Proof of Lemma 5 for $x \in [\frac{\pi}{2}, \pi]$

Let  $y = \pi - x$ . By Lemma 1, it's plain that for  $y \in (0, \frac{\pi}{4m+4}]$  both  $T_{2m+1}^\alpha(\pi - y)$  and  $T_{2m+2}^\alpha(\pi - y)$  are positive. For the remainder we assume  $y \in [\frac{\pi}{4m+4}, \frac{\pi}{2}]$ . We will keep this assumption throughout the rest of the proof.

For simplicity, we define for  $z \geq 0$

$$F(z) := \int_z^\infty \frac{\sin t}{\sqrt{t}} dt \quad (4.6)$$

and for  $m \geq 100$ ,

$$F_m(y) := \int_{(2m+\frac{3}{2})y}^\infty \frac{\sin t}{\sqrt{t + \alpha y}} dt.$$

It is easy to verify that for  $\ell \in \mathbb{N}$ ,

$$\sup_{u \geq 0} \int_{2\ell\pi}^\infty \frac{\sin t}{\sqrt{t + u}} dt = F(2\ell\pi). \quad (4.7)$$

We start with the proof of the positivity of  $T_{2m+1}^\alpha(\pi - y)$ , which is relatively easier to handle. It follows from Lemma 4 that

$$B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right)T_{2m+1}^\alpha(\pi - y) \geq \sqrt{\tan \frac{y}{2}} \left[ v(y) - \sqrt{\frac{\pi y}{\sin y}} F\left(\left(2m + \frac{3}{2}\right)y, y\right) - \sqrt{2 \sin y} \tilde{D}_m^\alpha(y) \right], \quad (4.8)$$

where

$$v(y) = \cos \frac{y}{2} (2 \sin y)^{\frac{3}{2}} \int_0^1 \frac{t^2}{\sqrt{1-t}((1-t)^2 + 4 \sin^2 y)} dt \quad (4.9)$$

and by (2.17),

$$|\tilde{D}_m^\alpha(y)| \leq \frac{0.8864}{\sqrt{m + \frac{\alpha}{2} - \frac{1}{4}}} \frac{y}{\sin y} \leq 0.1382.$$

We first claim that  $v(y)$  is a strictly decreasing function on  $[0, \frac{\pi}{2}]$ . In fact, for  $0 \leq u \leq 1$ , let

$$h(u) = u^{\frac{3}{2}} \int_0^1 \frac{t^2 dt}{\sqrt{1-t}((1-t)^2 + u^2)}.$$

Then an easy computation shows

$$h'(u) = 2 \int_0^{\frac{1}{u}} \frac{(ut-1)\sqrt{t}}{t^2+1} dt < 0.$$

This, together with the obvious identity  $v(y) = \cos \frac{y}{2} h(2 \sin y)$ , implies that  $v(y)$  is strictly decreasing.

Next, for  $\frac{\pi}{4m+4} < y \leq \frac{\pi}{8}$ , it's plain that

$$F_m(y) \leq \max_{y \in [0, \frac{\pi}{8}]} \left\{ \int_{\frac{403}{404} \cdot \frac{\pi}{2}}^{\infty} \frac{\sin u}{\sqrt{u + \alpha y}} du, \int_{2\pi}^{\infty} \frac{\sin u}{\sqrt{u + \alpha y}} du \right\},$$

which, by mechanical calculation, equals  $F(2\pi)$ . Consequently, invoking (4.8), on account of the monotonicity of the function  $v$ , we obtain for  $\frac{\pi}{4m+4} < y \leq \frac{\pi}{8}$ ,

$$\begin{aligned} & B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right)T_{2m+1}^\alpha(\pi - y) \\ & \geq \sqrt{\tan \frac{y}{2}} \left[ v\left(\frac{\pi}{8}\right) - \sqrt{\frac{\pi^2}{8 \sin \frac{\pi}{8}}} F(2\pi) - 0.1382 \times \sqrt{2} \right] = 0.2239 \sqrt{\tan \frac{y}{2}}. \end{aligned}$$

Similarly, in view of (4.7), we have for  $\frac{\pi}{8} < y \leq \frac{\pi}{4}$ ,

$$\begin{aligned} & B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right)T_{2m+1}^\alpha(\pi - y) \\ & \geq \sqrt{\tan \frac{y}{2}} \left[ v\left(\frac{\pi}{4}\right) - \sqrt{\frac{\pi^2}{4 \sin \frac{\pi}{4}}} F(24\pi) - 0.1382 \times \sqrt{2} \right] = 0.4005 \sqrt{\tan \frac{y}{2}}, \end{aligned}$$

and for  $\frac{\pi}{4} < y \leq \frac{\pi}{2}$ ,

$$\begin{aligned} & B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right)T_{2m+1}^\alpha(\pi-y) \\ & \geq \sqrt{\tan \frac{y}{2}} \left[ v\left(\frac{\pi}{2}\right) - \sqrt{\frac{\pi^2}{2}}F(50\pi) - 0.1382 \times \sqrt{2} \right] = 0.1547 \sqrt{\tan \frac{y}{2}}. \end{aligned}$$

This proves Lemma 5 for  $T_{2m+1}^\alpha$ .

Now we turn to the proof of the positivity of  $T_{2m+2}^\alpha(x)$ . First, we note that if  $\cos(2m+2)y \geq 0$  then the positivity of  $T_{2m+2}^\alpha(x)$  follows from the positivity of  $T_{2m+1}^\alpha(x)$  and the identity (2.2). For the remainder, we assume  $\cos(2m+2)y < 0$ . We consider the following two cases:

*Case I.*  $0 < y \leq \frac{40\pi}{101} = 1.2442$ .

For simplicity, we define the following two functions:

$$\begin{aligned} A(y) &:= \sqrt{\frac{\pi y}{\sin y}}, \\ G(y) &:= v(y) - 0.1248 \frac{y}{\sqrt{\sin y}}, \end{aligned}$$

where  $v(y)$  is defined by (4.9). Since  $v(y)$  is a strictly decreasing function on  $[0, \frac{\pi}{2}]$ ,  $G(y)$  is also strictly decreasing on  $[0, \frac{\pi}{2}]$ . It follows from (2.11) that

$$B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right)c_{2m+2}(\alpha) \leq \sqrt{\frac{\pi}{m+1}}. \quad (4.10)$$

Now we use (4.8) and (2.2) to obtain

$$B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right)T_{2m+2}^\alpha(\pi-y) \geq \sqrt{\frac{\pi}{m+1}}C_m(y),$$

where

$$C_m(y) = \sqrt{\frac{(m+1)y}{2\pi}} \left( G(y) - A(y)F_m(y) \right) + \cos(2m+2)y. \quad (4.11)$$

So it's enough to show that  $C_m(y) > 0$  whenever  $\cos(2m+2)y < 0$  and  $y \in (0, 1.2442)$ . We will prove this by further considering the following cases:

- (1)  $\frac{\pi}{4m+4} \leq y \leq \frac{0.36\pi}{m+1}$ ,    (2)  $\frac{0.36\pi}{m+1} \leq y \leq \frac{0.5\pi}{m+1}$ ,    (3)  $\frac{0.5\pi}{m+1} \leq y \leq \frac{0.75\pi}{m+1}$ ,
- (4)  $\frac{1.25\pi}{m+1} \leq y \leq \frac{2\pi}{m+1}$ ,    (5)  $\frac{2\pi}{m+1} \leq y \leq \frac{4\pi}{m+1}$ ,    (6)  $\frac{4\pi}{m+1} \leq y \leq \frac{10\pi}{m+1}$ ,
- (7)  $\frac{10\pi}{m+1} \leq y \leq \frac{24\pi}{m+1}$ ,    (8)  $\frac{24\pi}{m+1} \leq y \leq 1.2442$ .

We start with the first case. In this case,  $\frac{\pi}{4m+4} \leq y \leq \frac{0.36\pi}{m+1} \leq 0.0112$  and

$$F_m(y) \leq \max_{0 \leq y \leq 0.02} \int_{\frac{403}{404} \frac{\pi}{2}}^{\infty} \frac{\sin t}{\sqrt{t + \alpha y}} dt \leq 0.1579.$$

So we obtain from (4.11),

$$\begin{aligned} C_m(y) &\geq \sqrt{\frac{(m+1)y}{2\pi}} \left( G(0.0112) - A(0.0112) \times 0.1579 \right) + \cos(0.72\pi) \\ &\geq \sqrt{\frac{1}{8}} (2.1254 - 0.2798) + \cos(0.72\pi) = 0.015. \end{aligned}$$

For the second case,

$$\frac{0.36\pi}{m+1} \leq y \leq \frac{0.5\pi}{m+1} \leq 0.0156$$

and

$$F_m(y) \leq \max_{0 \leq y \leq 0.02} \int_{\frac{403}{404} \cdot 0.72\pi}^{\infty} \frac{\sin t}{\sqrt{t + \alpha y}} dt \leq -0.3051.$$

So, in this case

$$\begin{aligned} C_m(y) &\geq \sqrt{\frac{(m+1)y}{2\pi}} \left( G(0.0156) + 0.3051\sqrt{\pi} \right) - 1 \\ &\geq \sqrt{0.18} (2.0944 + 0.5409) - 1 = 0.118. \end{aligned}$$

For the third case,

$$\frac{0.5\pi}{m+1} \leq y \leq \frac{0.75\pi}{m+1} \leq 0.0233$$

and

$$F_m(y) \leq \max_{0 \leq y \leq 0.02} \left\{ \int_{\frac{403}{404}\pi}^{\infty} \frac{\sin t}{\sqrt{t + \alpha y}} dt, \int_{\frac{3}{2}\pi}^{\infty} \frac{\sin t}{\sqrt{t + \alpha y}} dt \right\} \leq \max\{-0.5313, -0.0425\} = -0.0425.$$

Hence,

$$\begin{aligned} C_m(y) &\geq \sqrt{\frac{(m+1)y}{2\pi}} \left( G(0.0233) + 0.0425\sqrt{\pi} \right) - 1 \\ &\geq \sqrt{0.25} (2.044 + 0.0754) - 1 = 0.0597. \end{aligned}$$

The remaining five cases can be handled analogously but in a somewhat simpler way. In fact, we have, for the fourth case ( $\frac{1.25\pi}{m+1} \leq y \leq \frac{2\pi}{m+1} (\leq 0.0622)$ ),

$$C_m(y) \geq \sqrt{\frac{1.25}{2}} \left( G(0.0622) - A(0.0622)F(4\pi) \right) - 1 = 0.0583,$$

where  $F$  is defined by (4.6), for the fifth case ( $\frac{2\pi}{m+1} \leq y \leq \frac{4\pi}{m+1} (\leq 0.1244)$ ),

$$C_m(y) \geq G(0.1244) - A(0.1244)F(4\pi) - 1 = 0.097,$$

for the sixth case ( $\frac{4\pi}{m+1} \leq y \leq \frac{10\pi}{m+1} (\leq 0.311)$ ),

$$C_m(y) \geq \sqrt{2} \left( G(0.311) - A(0.311)F(8\pi) \right) - 1 = 0.1434,$$

for the seventh case ( $\frac{10\pi}{m+1} \leq y \leq \frac{24\pi}{m+1} (\leq 0.7465)$ ),

$$C_m(y) \geq \sqrt{5} \left( G(0.7465) - A(0.7465)F(20\pi) \right) - 1 = 0.085,$$

and for the final case ( $\frac{24\pi}{m+1} \leq y \leq \frac{40\pi}{101} (= 1.2442)$ ),

$$C_m(y) \geq \sqrt{12} \left( G(1.2442) - A(1.2442)F(48\pi) \right) - 1 = 0.0288.$$

Putting these together, noting that  $\cos(2m+2)y < 0$  for the case  $\frac{0.75\pi}{m+1} \leq y \leq \frac{1.25\pi}{m+1}$ , we prove  $T_{2m+2}^\alpha(\pi - y) > 0$  for  $0 < y \leq 1.2442$ .

Case II.  $1.2442 < y < \frac{\pi}{2}$ .

In this case, we use Lemma 4 directly. In fact, it's easy to derive the following estimates:

$$2 \sin \frac{y}{2} J^\alpha(\pi - y) \geq 2 \sin^2 1.2442 \int_0^1 \frac{t^2}{\sqrt{1-t}(1+t)^2} dt \geq 0.5717, \quad (\text{by (2.5)})$$

$$2 \sin \frac{y}{2} \tilde{L}_m^\alpha(y) \leq \sqrt{\pi y} F(80\pi) \leq 0.1401 \quad (\text{by (2.16)})$$

$$2 \sin \frac{y}{2} |D_m^\alpha(y)| \leq 0.1968. \quad (\text{by (2.17)})$$

Therefore, applying (2.2) and Lemma 4, in view of (4.10), we obtain

$$B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right) T_{2m+2}^\alpha(\pi - y) \geq 0.5717 - 0.1401 - 0.1969 - \sqrt{\frac{\pi}{101}} = 0.0584.$$

This concludes the proof.

## 5 Concluding remarks

**Remark 1.** As indicated in Section 1, it can be shown that the equation (1.6) has a unique solution  $\alpha_0$  in  $(-1, \infty)$ . In fact, the existence and uniqueness of  $\alpha_0$  in  $(-1, 2.3309]$  has already been shown in Section 4. It remains to prove that the equation (1.6) has no solution in  $(2.3309, \infty)$ .

Mechanical computation shows that

$$T_6^{2.3309}(x_0) = -2.3299 \times 10^{-6} < 0,$$

where  $x_0 = 0.7252$ . Since

$$c_k(\alpha) = \frac{\alpha+1}{\alpha+2} \cdot \frac{\alpha+3}{\alpha+4} \cdots \frac{\alpha+2[\frac{k}{2}]-1}{\alpha+2[\frac{k}{2}]}$$

increases with  $\alpha$  and  $\cos jx_0 < 0$  for  $1 \leq j \leq 6$ ,  $T_6^\alpha(x_0)$  decreases with  $\alpha$  and hence, for  $\alpha > 2.3309$ ,

$$\min_{x \in (0, \pi)} T_6^\alpha(x) \leq T_6^\alpha(x_0) \leq T_6^{2.3309}(x_0) < 0,$$

as desired.

**Remark 2.** It is clear that the proof of Theorem 1 actually yields the following result, which seems to us to be of independent interest.

**Theorem 1'** For  $\alpha_0 \leq \alpha \leq 2.3309$ ,  $x \in (0, \pi)$  and  $n \neq 6$ ,

$$T_n^\alpha(x) > 0$$

while for  $\alpha \geq \alpha_0$ ,

$$\min_{x \in (0, \pi)} T_6^\alpha(x) \leq 0$$

where the equality holds if and only if  $\alpha = \alpha_0$ .

**Remark 3.** Finally, we give some remarks concerning sine sums. It follows from [ Vietoris, 1] that for  $-1 < \alpha \leq 0$ ,  $n \geq 1$  and  $x \in (0, \pi)$ ,

$$\sigma_n^\alpha(x) := \sum_{k=1}^n c_k(\alpha) \sin kx > 0.$$

However, things are different for  $\alpha > 0$ . In fact, we have

**Theorem 3.** For the sine sums  $\sigma_n^\alpha(x)$  defined above, the following are true:

(i) Theorems 1, 1' and 2 remain valid with  $T_n^\alpha(x)$  replaced by  $\sigma_{2m+1}^\alpha(x)$ .

(ii) For  $\alpha > -1$  there exists an  $M = M(\alpha)$  such that for  $m \geq M(\alpha)$  and  $x \in (0, \pi - \frac{\pi}{2m+2})$ ,

$$\sigma_{2m+2}^\alpha(x) > 0. \tag{5.1}$$

(iii) For  $\alpha > 0$  and any  $m \geq 2$ ,

$$\min_{x \in (\pi - \frac{\pi}{2m+2}, \pi)} \sigma_{2m+2}^\alpha(x) < 0.$$

*Proof.* (i) By the definition and Lemma 1, we obtain

$$\sigma_{2m+1}^\alpha(x) = 2 \cos\left(\frac{x}{2}\right) \sum_{k=0}^m c_{2k}(\alpha) \sin\left(2k + \frac{1}{2}\right)x = \cot\left(\frac{x}{2}\right) T_{2m+1}^\alpha(\pi - x), \tag{5.2}$$

from which the desired conclusions for  $\sigma_{2m+1}^\alpha(x)$  follow.

(ii) The proof is similar to that for  $T_n^\alpha(x)$ . Different arguments are needed for each of the intervals  $0 < x \leq \frac{\pi}{2m+2}$ ,  $\frac{\pi}{2m+2} \leq x \leq \frac{\pi}{2}$ ,  $\frac{\pi}{2} \leq x \leq \pi - \frac{\pi}{2m+2}$ .

For  $0 < x \leq \frac{\pi}{2m+2}$ , all terms in the sum are nonnegative and the first is strictly positive.

For  $\frac{\pi}{2m+2} \leq x \leq \frac{\pi}{2}$ , we obtain from (5.2)

$$\sigma_{2m+2}^\alpha(x) = \cot\left(\frac{x}{2}\right)T_{2m+1}^\alpha(\pi - x) + c_{2m+2}(\alpha) \sin(2m+2)x.$$

This, together with (3.6) and (3.10), implies (5.1).

For  $\frac{\pi}{2} \leq x \leq \pi - \frac{\pi}{2m+2}$ , set  $y = \pi - x$  so that  $y \in [\frac{\pi}{2m+2}, \frac{\pi}{2}]$ . Then we obtain from (5.2)

$$\sigma_{2m+2}^\alpha(x) = \tan\left(\frac{y}{2}\right)T_{2m+1}^\alpha(y) - c_{2m+2}(\alpha) \sin(2m+2)y. \quad (5.3)$$

If  $y \in [\frac{\pi}{2m+2}, \frac{\pi}{m+1}]$  then  $\sin(2m+2)y \leq 0$  and (5.1) follows from (5.3) and Theorem 2. If  $(m+1)y \geq C$  with  $C$  sufficiently large then (5.1) is a simple consequence of (5.3), (3.1) and (3.10):

$$B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right)\sigma_{2m+2}^\alpha(x) \geq \left(\frac{\varepsilon_\alpha}{2}\sqrt{y(m+1)} - \sqrt{\pi}\right)\frac{1}{\sqrt{m+1}} \geq \left(\frac{\varepsilon_\alpha}{2}\sqrt{C} - \sqrt{\pi}\right)\frac{1}{\sqrt{m+1}} > 0.$$

Finally, for  $\pi \leq (m+1)y \leq C$ , we use (3.2), (3.4) (3.5) and (3.10) to obtain

$$\begin{aligned} B\left(\frac{\alpha+1}{2}, \frac{1}{2}\right)\sigma_{2m+2}^\alpha(x) &\geq \sqrt{\frac{\pi}{m+1}} \left\{ \sqrt{\frac{(m+1)y}{2\pi}} \left[ \frac{99}{100} \int_0^\infty \frac{\sqrt{t}}{1+t^2} dt - \sqrt{\pi} \int_{\frac{99}{50}\pi}^\infty \frac{\cos t}{\sqrt{t}} dt \right] - 1 \right\} \\ &\geq \sqrt{\frac{\pi}{m+1}} \left[ \frac{1}{\sqrt{2}}(2.1992 - 0.0967) - 1 \right] = 0.4866\sqrt{\frac{\pi}{m+1}}. \end{aligned}$$

Putting the above together, we prove (ii).

(iii) For  $x \in [\pi - \frac{\pi}{2m+2}, \pi)$ , set  $y = \pi - x$  so that  $y \in (0, \frac{\pi}{2m+2}]$ . Then we have

$$\begin{aligned} \sigma_{2m+2}^\alpha(x) &= \sum_{k=1}^{m+1} \left[ c_{2k-1}(\alpha) \sin(2k-1)y - c_{2k}(\alpha) \sin 2ky \right] \\ &= \sum_{k=1}^{m+1} (2k-1+\alpha)c_{2k-1}(\alpha) \left[ \frac{\sin(2k-1)y}{2k-1+\alpha} - \frac{\sin 2ky}{2k+\alpha} \right]. \end{aligned}$$

For  $y \in (0, \left(\frac{1}{4(m+1)}\right)^{\frac{3}{2}}\alpha^{\frac{1}{2}})$  the term in the square brackets is negative since the function  $\frac{\sin uy}{u+\alpha}$  increases strictly on  $[0, \frac{1}{2}\alpha^{\frac{1}{3}}y^{-\frac{2}{3}}]$ . This proves (iii).

## References

- [1] L. Vietoris, Über das Vorzeichen gewisser trigonometrischer Summen, *Sitzungsber., Öst. Acad. Wiss.Math.-Natur. Kl. S.-B. II*, **167** (1958), 125–135.
- [2] G.Brown and E.Hewitt, A class of positive trigonometric sums, *Math. Ann.* **268**(1984), 91–122.

- [3] G.Brown and Q.Yin, Positivity of a class of cosine sums, *Acta Sci. Math. (Szeged)* **67**(2001), 221–247.
- [4] G.Brown, F. Dai and K. Y. Wang, Extensions of Vietoris's inequalities II, preprint.
- [5] Erdélyi and et. Higher Transcendental Functions, Volume 1, McGraw-Hill Book Company, INC. 1953.