APPROXIMATION OF REAL SMOOTH FUNCTIONS ON THE UNIT SPHERE $\mathbb{S}^{d-1}$

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The undersigned hereby certify that they have read and recommend to the Faculty of Graduate Studies for acceptance a thesis entitled "**Approximation of real smooth functions on the unit sphere** $\mathbb{S}^{d-1}$" by **Dai Feng** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy**.

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To My Parents.

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Abstract

Let $\mathbb{S}^{d-1} := \Big\{(x_1, \cdots, x_d) \in \mathbb{R}^d \,:\, x_1^2 + \cdots + x_d^2 = 1\Big\}$ be the unit sphere of the d-dimensional Euclidean space $\mathbb{R}^d$ equipped with the usual rotation invariant measure $d\sigma(x)$. The thesis concerns mainly with the following three problems on the approximation of functions on $\mathbb{S}^{d-1}$: the asymptotic orders of the Kolmogorov and the linear widths of some function classes on $\mathbb{S}^{d-1}$; the realizations of the K-functionals on $\mathbb{S}^{d-1}$; the strong approximation by the Cesàro means of the Fourier-Laplace series. It is divided into four chapters.

Chapter 1 deals mainly with the orders of the Kolmogorov widths of some function classes. Two main results are obtained in this chapter. Their proofs are based on two important inequalities, which seem to us to be of independent interest.

Statement of the first main result. Given $r > 0$, let B_p^r denote the class of functions f on $\mathbb{S}^{d-1}$ representable in the form

$$f(x) = \int_{\mathbb{S}^{d-1}} \varphi(y) F_r(xy)\, d\sigma(y), \quad \|\varphi\|_p \le 1,$$

where

$$F_r(xy) = \sum_{k=1}^{\infty} (k(k+d-2))^{-\frac{r}{2}} \frac{\Gamma(\frac{d-2}{2})(k+\frac{d-2}{2})}{2\pi^{\frac{d}{2}}} C_k^{(\frac{d-2}{2})}(xy),$$

and the $C_k^{(\frac{d-2}{2})}$'s are the ultraspherical polynomials. For $1 \le p, q \le \infty$, let $d_N(B_p^r, L^q)$ denote the Kolmogorov N-width of B_p^r in L^q. The asymptotic orders of $d_N(B_p^r, L^q)$ for $1 \le p < q \le 2$ and for $1 \le p = q \le \infty$ were found by Kamzolov [Ka1]-[Ka3] in 1982–1989. Since then very few results in this direction had been obtained. Our result is

Theorem 0.0.1. *For $r \ge 2(d-1)^2$,*

$$d_N(B_p^r, L^q) \asymp \begin{cases} N^{-(\frac{r}{d-1} - \frac{1}{p} + \frac{1}{2})}, & \text{if } 1 \le p \le 2 < q \le \infty, \\ N^{-\frac{r}{d-1}}, & \text{if } 2 \le p < q \le \infty. \end{cases}$$

(The condition on r may be weakened somewhat, but at the cost of greatly increased computation.)

This result extends the well known Kashin theorem on the asymptotic orders of the Kolmogorov widths of the Sobolev classes in one variable.

Statement of the second main result. The second result is on the orders of the Kolmogorov widths of the function class $B_p^{\mathbf{r}}$ on the product spheres $\mathbb{S}^{d-1,n}$. Given a positive integer n, let $\mathbb{S}^{d-1,n} = \mathbb{S}^{d-1} \times \cdots \times \mathbb{S}^{d-1}$ (n times) denote the product space equipped with the usual Lebesgue measure $d\sigma(z) = d\sigma(z_1) \cdots d\sigma(z_n)$. For a vector $\mathbf{r} = (r_1, \cdots, r_n)$, let $B_p^{\mathbf{r}}$ $(1 \le p \le \infty)$ be the class of functions representable in the form

$$f(z_1, \cdots, z_n) = \int_{\mathbb{S}^{d-1,n}} \varphi(y_1, \cdots, y_n) F_{\mathbf{r}}(z_1 y_1, \cdots, z_n y_n)\, d\sigma(y_1), \cdots, d\sigma(y_n), \quad \|\varphi\|_p \le 1,$$

where

$$F_{\mathbf{r}}(z_1 y_1, \cdots, z_n y_n) = \sum_{k \in \mathbb{Z}^n_{>0}} \prod_{j=1}^{n} (k_j(k_j + 2\lambda))^{-\frac{r_j}{2}} \frac{\Gamma(\frac{d-2}{2})(k_j + \frac{d-2}{2})}{2\pi^{\frac{d}{2}}} C_{k_j}^{(\frac{d-2}{2})}(z_j y_j),$$

and $z_j, y_j \in \mathbb{S}^{d-1}$, $j = 1, \cdots, n$. We assume

$$0 < r = r_1 = \cdots = r_v < r_{v+1} \le \cdots \le r_n.$$

For $1 \le p, q \le \infty$, as before, denote by $d_M(B_p^{\mathbf{r}}, L^q)$ the Kolmogorov M-width of $B_p^{\mathbf{r}}$ in $L^q(\mathbb{S}^{d-1,n})$. With these notations, our next result can be stated as follows.

Theorem 0.0.2. *Let $1 < p \le 2 \le q < \infty$ and $r > (d-1)^2$. Then*

$$d_M(B_p^{\mathbf{r}}, L^q) \asymp M^{-\frac{r}{d-1} + \frac{1}{p} - \frac{1}{2}} \Big(\log M\Big)^{v-1}.$$

In the case $d = 2$, Theorem 0.0.2 was due to Temlyakov [Tem, 1986].

Statements of the two important inequalities. The proofs of Theorems 0.0.1 and 0.0.2 are based on the following two types of inequalities: the weighted Marcinkiewitz-Zygmund inequality for the spherical harmonics; the weighted Kashin type inequality for the Kolmogorov widths of the unit Euclidean ball. These inequalities are of independent interest.

For a non-negative integer k, let Π_k denote the set consisting of the restriction to $\mathbb{S}^{d-1}$ of all algebraic polynomials in d variables of total degree not exceeding k. The first type of inequality (M-Z inequality) can be stated as follows.

Theorem 0.0.3. ***(M-Z inequality.)*** *There exist $m_N = (4N+1)(2N)^{d-2}$ points $\xi_{N,1}, \xi_{N,2}, \cdots, \xi_{N,m_N}$ on $\mathbb{S}^{d-1}$ such that for any $f \in \Pi_{4N}$*

$$\int_{\mathbb{S}^{d-1}} f(x)\, d\sigma(x) = \frac{1}{m_N} \sum_{k=1}^{m_N} w_{N,k} f(\xi_{N,k}),$$

where the weights $\{w_{N,k}\}$ *satisfy the following two conditions:*

$$0 < w_{N,k} \le 1, \quad w_{N,k}^{-1} \le C_d N^{(d-2)^2};$$

$$\frac{1}{m_N} \sum_{k=1}^{m_N} \Big(w_{N,k} \Big)^t \le \begin{cases} C_d, & \text{if } t > -\frac{1}{d-2}, \\ C_d \log N, & \text{if } t = -\frac{1}{d-2}, \\ C_d N^{-(d-2)^2 t - 1}, & \text{if } t < -\frac{1}{d-2}. \end{cases}$$

Furthermore, if $f \in \Pi_N$ *and* $1 \le p \le \infty$*, then*

$$\Big(\int_{\mathbb{S}^{d-1}} |f(x)|^p \, d\sigma(x) \Big)^{\frac{1}{p}} \asymp \Big(\frac{1}{m_N} \sum_{k=1}^{m_N} w_{N,k} |f(\xi_{N,k})|^p \Big)^{\frac{1}{p}}.$$

An incorrect proof of the M-Z inequality was presented in [Ku, 2000].

To state the second type of inequality, we have to introduce some necessary notations. For a vector $w = (w_1, \cdots, w_m)$, denote by $\ell_{p,w}^m$ the m-dimensional space of vectors $x = (x_1, \cdots, x_m)$ with the norm

$$\|x\|_{p,w} = \Big(\sum_{i=1}^m |x_i|^p w_i \Big)^{\frac{1}{p}}, \ 1 \le p < \infty, \ \|x\|_{\infty,w} = \max_k |x_k|.$$

Let

$$b_{p,w}^m := \{x \in \ell_{p,w}^m : \ \|x\|_{p,w} \le 1\}$$

be the unit ball of $\ell_{p,w}^m$ and $d_n(b_{p,w}^m, \ell_{q,w}^m)$ the Kolmogorov n-widths of $b_{p,w}^m$ in the metric $\ell_{q,w}^m$. We say a vector $w = \{w_i\}_{i=1}^m$ is with the *Kashin type property* if it satisfies the following two conditions simultaneously:

$$0 < w_i \le 1, \quad w_i^{-1} \le C_1 m^a, \quad i = 1, \cdots, m; \tag{0.0.1}$$

$$\frac{1}{m} \sum_{k=1}^m (w_k)^t \le C_1, \quad \text{for any } t \ge -\alpha, \tag{0.0.2}$$

where a and α are some positive constants, $C_1 > 0$ is independent of m. Let us call the infimum over the constants C_1 on the right-hand side of (0.0.1) and (0.0.2) the *Kashin constant* of w.

Now the second type of inequality can be stated as follows.

Theorem 0.0.4. ***(Weighted Kashin type inequality.)*** *Suppose w is a vector with the Kashin type property. Then for $1 \le n \le m$,*

$$d_n(b_{2,w}^m, \ell_\infty^m) \le C_2 \Big(\frac{m}{n}\Big)^{\frac{\theta}{2(1-\theta)}} n^{-\frac{1}{2}} \Big(1 + \log \frac{m}{n}\Big)^{\frac{1}{2(1-\theta)}},$$

where $\theta = \frac{1}{1+\alpha} \in (0,1)$ and the constant $C_2 > 0$ depends only on the Kashin constant of w rather than on the vector w itself.

Obviously, the well known Kashin inequality corresponds to the limiting case $\alpha = \infty$ of Theorem 0.0.4.

Organization of Chapter 1. In Section 1.1, we give two counterexamples, which state that the Marcinkiewicz-Zygmund type inequality with equal weights for the multi-dimensional sphere is generally not true, which, in turn, shows that the proof in [Ku] is incorrect. The M-Z inequality and the weighted Kashin type inequality are established in Sections 1.2 and 1.3 respectively. Theorems 0.0.1 and 0.0.2 are then proved in Sections 1.4 and 1.5 respectively by applying the standard discretization technique.

Chapter 2 deals with the orders of the linear widths of B_p^r in L^q. It contains one main result, whose proof is based on an inequality for the ultraspherical polynomials, which seems to us to be of independent interest.

Statement of the main result. For a non-negative integer k, let $\mathcal{H}_k^d$ denote the space of spherical harmonics of degree k. Define the *linear spherical harmonic width* $\delta_m^{SH}(B_p^r, L^q)$ by

$$\delta_m^{SH}(B_p^r, L^q) := \inf_{T \in G_m} \sup_{f \in B_p^r} \|f - T(f)\|_q,$$

where

$$G_m = \Big\{T : T \text{ is continuous, linear on } L^q(\mathbb{S}^{d-1}) \text{ and there is} \\ \text{a vector space } L \in \mathcal{L}_m \text{ for which } T(L^q) \subset L\Big\},$$

and

$$\mathcal{L}_m := \Big\{\Big(\bigoplus_{k\in A} \mathcal{H}_k^d\Big) : \quad A \text{ is a finite set of non-negative} \\ \text{integers for which } \dim\Big(\bigoplus_{k\in A} \mathcal{H}_k^d\Big) \le m\Big\}.$$

As is easily seen, in the special case $d = 2$, $\delta_m^{SH}(B_p^r, L^q)$ returns to the well known trigonometric width of the Sobolev class on the unit circle, for which the asymptotic orders for all $1 \le p \le q \le \infty$ are known, due to the work of Makovoz [Mk, 1984], Maiorov [Mr2, 1986] and Belinskii [Be3, 1987]. However, for general $d \ge 3$, there are some essential difficulties in this problem. To the best of our knowledge, so far very few results in this direction had been obtained. Our main result partly answers the problem. We state it as follows.

Theorem 0.0.5.

$$\delta_m^{SH}(B_p^r, L^q) \asymp \begin{cases} m^{-\frac{r}{d-1}+\frac{1}{2}}, & \textit{for } p=1, \quad 2 \le q \le \infty, \quad r > \frac{d(d-1)}{2}, \\ m^{-\frac{r}{d-1}+\frac{1}{2}}, & \textit{for } 1 \le p \le 2, \quad q=\infty, \quad r > \frac{d(d-1)}{2}, \\ m^{-\frac{r}{d-1}+\frac{1}{p}-\frac{1}{2}}, & \textit{for } 1 \le p \le 2 \le q < \frac{2(d-1)p}{dp-2}, \quad r > d-1, \\ m^{-\frac{r}{d-1}+\frac{1}{2}-\frac{1}{q}}, & \textit{for } \frac{2(d-1)q}{dq-2} < p \le 2 \le q \le \infty, \quad r > d-1. \end{cases}$$

An inequality for the ultraspherical polynomials. For $\mu > 0$, let $\{C_k^\mu\}_{k=0}^\infty$ be the sequence of ultraspherical polynomials, whose precise definition can be found in [Sz]. The following inequality plays a crucial role in the proof of Theorem 0.0.5.

Theorem 0.0.6. *Suppose $0 < \mu < \infty$ and $T_N(t) = \sum_{k=0}^N c_k C_k^\mu(t)$ is an algebraic polynomial of degree not exceeding N. Then for $2 \le p \le \infty$ and $1 \le M \le N$, there exists a polynomial representable in the form*

$$T_{\theta_M}(t) = \sum_{k \in \theta_M} b_k C_k^\mu(t)$$

with $\theta_M \subset [0, 2N]$, $|\theta_M| = M$, such that

$$\|T_N(t) - T_{\theta_M}(t)\|_p \le C(p,\mu) \max\Big\{ \Big(\frac{N}{M}\Big)^{1/2}, \ \Big(\frac{N}{M}\Big)^{(1-\frac{2}{p})(\mu+1/2)} \Big(\log(1+\frac{N}{M})\Big)^{\mu(1-\frac{2}{p})} \Big\} \|T_N(t)\|_2,$$

here the norms are computed with respect to the measure $(1-t^2)^{\mu-1/2}\,dt$.

In the case of trigonometric polynomials (corresponding to the limiting case $\mu = 0$ here), Theorem 0.0.6 was due to Makovoz [Mk, 1984] for $2 \le p < \infty$ and Belinskii [Be1, 1998] for $p = \infty$. It should be pointed out that the probabilistic method used in [Mk,1984] and [Be1,1998] does not apply to the case of $\{C_k^\mu\}_{k=0}^\infty$ whenever $p \ge 2 + \frac{1}{\mu}$. Our approach is different.

Approximation of $F_{r,\mu}$. As an application of Theorem 0.0.6, we obtain sharp estimates of the quantities $e_m(F_{r,\mu}, L^p)$ for $2 \le p \le \infty$, which, in turn, are used to deduce our main result (Theorem 0.0.5). Before stating these sharp estimates, we describe the definitions of $F_{r,\mu}$ and $e_m(F_{r,\mu}, L^p)$. Let θ_m be a set consisting of m non-negative integers. Denote by T_{θ_m} an algebraic polynomial representable in the form

$$T_{\theta_m}(t) = \sum_{k\in\theta_m} a_k C_k^{\mu}(t).$$

For $f \in L^p$, define

$$e_m(f,\, L^p) = \inf_{\theta_m} \inf_{T_{\theta_m}} \|f - T_{\theta_m}\|_p,$$

where the norms are computed with respect to the measure $(1-t^2)^{\mu-\frac{1}{2}}\,dt$.

Let

$$F_{r,\mu}(t) = \sum_{k=1}^{\infty} \big(k(k+2\mu)\big)^{-\frac{r}{2}} \frac{\Gamma(\mu)(k+\mu)}{2\pi^{\mu+1}} C_k^{(\mu)}(t),\ \ \mu > 0.$$

Now the sharp estimates can be stated as follows.

Theorem 0.0.7. *Suppose $r > 2\mu + 1$ and $2 \le p \le \infty$. Then*

$$e_m(F_{r,\mu},\ L^p) \asymp m^{-r+\mu+1/2}.$$

Organizations of Chapter 2. We first establish Theorem 0.0.6 in Section 2.1 . As an application, we get Theorem 0.0.7 in Section 2.2. The main result Theorem 0.0.5 is then proved in Section 2.3 by invoking Theorem 0.0.7. In Section 2.4 we discuss some further extreme problems in a general setting.

Chapter 3 mainly deals with the realizations of the K-functionals. Strong equivalences between the K-functionals and some well known operators are established.

Organization of Chapter 3. In Section 3.1, we consider the Riesz means and the Cesàro means in a general setting, which was introduced by Ditzian [Di1]. Strong equivalence between each of these operators and the corresponding K-functional is established, which proves a conjecture raised by Ditzian [Di1, 1998]. In Section 3.2 we confine ourselves again to the case of the unit sphere $\mathbb{S}^{d-1}$. This section contains five subsections. Equivalences between the K-functionals and the average operators, the Steklov type means, are established in Subsections 3.2.2 and 3.2.4 respectively. A conjecture on the property of the norm of the derivatives of the average operator raised by Ditzian and Runovskii in [Di-Ru1, 2000] is solved in Subsection 3.2.3. In Subsection 3.2.5, we apply the technique developed in Subsections 3.2.2–3.2.3 to obtain a strong equivalence between the K-functional and the modulus of continuity on $\mathbb{S}^{d-1}$. As a consequence, the well known Jackson type inequality on $\mathbb{S}^{d-1}$ is deduced and its original proofs in [Ru2,1991] and [Ri-Wa,1995] are simplified.

In Section 3.3, we obtain some similar results as in Section 3.2 for the Jacobi expansions, which also improve some previously known results. A strong converse inequality for the average operator of high order and the K-functional related to the Laplacian on $\mathbb{R}^d$ is obtained in Section 3.4, which proves a conjecture raised by Ditzian and Runovskii [Di-Ru2, 1999].

Further statements of the main results in Section 3.2. Before stating the main results, we have to describe briefly some notations.

For $t \in (0, \pi)$ and $f \in L(\mathbb{S}^{d-1})$, *the average* $B_t(f)$ and *the Steklov type mean* $A_t(f)$ are defined by

$$B_t(f, x) := \frac{1}{\Phi(t)} \int_{\{y \in \mathbb{S}^{d-1}:\ xy \geq \cos t\}} f(y)\, d\sigma(y), \ x \in \mathbb{S}^{d-1}, \tag{0.0.3}$$

and

$$A_t(f)(x) = \frac{1}{\mu(t)} \int_0^t \frac{\Phi(\theta)}{(\sin\theta)^{d-2}} B_\theta(f, x)\, d\theta$$

respectively, where $\Phi(t)$ and $\mu(t)$ are chosen so that

$$B_t(\mathbb{1}, x) \equiv A_t(\mathbb{1}, x) \equiv \mathbb{1}(x).$$

Here $\mathbb{1}$ denotes the 1- valued constant function on $\mathbb{S}^{d-1}$.

Given $r > 0$, the *r-th order average* and the *r-th order Steklov type mean* are defined by

$$B_{t,r}(f)(x) = \sum_{j=1}^{\infty} \binom{\frac{r}{2}}{j} (-1)^{j+1} B_t^j(f)(x)$$

and

$$A_{t,r}(f)(x) = \sum_{j=1}^{\infty} \binom{\frac{r}{2}}{j} (-1)^{j+1} A_t^j(f)(x)$$

respectively.

Given $r > 0$, let D, D^r denote the Laplace-Beltrami operator, the $r-th$ order fractional differential operator respectively. Suppose that $\omega_r(f, t)_p$ $(1 \leq p \leq \infty)$ is the r-th order modulus of continuity of a function $f \in L^p(\mathbb{S}^{d-1})$, which was introduced by Rustamov [Rus1]. (See [Wa-L] for precise definitions.) For a non-negative integer N, let $E_N(f)_p$ denote the best approximation of a function $f \in L^p(\mathbb{S}^{d-1})$, $1 \leq p \leq \infty$, by spherical harmonics of degree $\leq N$:

$$E_N(f)_p := \inf_{g \in \bigoplus_{k=0}^{N} \mathcal{H}_k^d} \|f - g\|_p,$$

where, as before, $\mathcal{H}_k^d$ denotes the space of spherical harmonics of degree k on $\mathbb{S}^{d-1}$. For $r > 0$ and $f \in L^p(\mathbb{S}^{d-1})$ $(1 \leq p \leq \infty)$, define the r-th order K-functional $K_r(f,t)_p$ by

$$K_r(f,t)_p := \inf\Big\{\|f-g\|_p + t^r\|D^{\frac{r}{2}}g\|_P : \quad g \in L^p \quad \text{and} \quad D^{\frac{r}{2}}g \in L^p\Big\}.$$

Now we state the main results as follows.

Theorem 0.0.8. *Let $t \in (0,\pi)$, $r > 0$ and $1 \leq p \leq \infty$. Then for $f \in L^p(\mathbb{S}^{d-1})$,*

$$K_r(f,t)_p \approx \|f - B_{t,r}f\|_p.$$

Theorem 0.0.9. *Let $1 \leq p \leq \infty$, $\theta \in [0,\pi]$, D the Laplace -Beltrami operator and B_θ the average operator defined by (0.0.3). Then*

$$\lim_{m\to\infty} \sup_{\theta\in[0,\pi]} \|\theta^2 D B_\theta^m\|_{(p,p)} = 0.$$

Theorem 0.0.10. *Let $t \in (0,\pi)$, $r > 0$ and $1 \leq p \leq \infty$. Then for $f \in L^p(\mathbb{S}^{d-1})$,*

$$K_r(f,t)_p \approx \|f - A_{t,r}f\|_p.$$

Theorem 0.0.11. *Let $t \in (0,\pi)$, $r > 0$ and $1 \leq p \leq \infty$. Then for $f \in L^p(\mathbb{S}^{d-1})$,*

$$K_r(f,t)_p \approx \omega_r(f,t)_p.$$

As a consequence of Theorem 0.0.11, we have

Corollary 0.0.12. ***(Jackson type inequality.)*** *Let $r > 0$, $1 \leq p \leq \infty$ and N a non-negative integer. Then for $f \in L^p(\mathbb{S}^{d-1})$,*

$$E_N(f)_p \leq C_{r,p}\omega_r(f, \frac{1}{N})_p.$$

Theorems 0.0.8 and 0.0.10 are proved in Subsections 3.2.2 and 3.2.4 respectively. In the special case $r = 2$, both theorems improve results in [Di-Ru1, 2000].

Theorem 0.0.9 was conjectured by Ditzian and Runovskii [Di-Ru1,2000] and its proof is given in Subsection 3.2.3.

Theorem 0.0.11 and Corollary 0.0.12 are proved in Subsection 3.2.5. The Jackson type inequality (Corollary 0.0.12) is a basic inequality in the approximation theory on $\mathbb{S}^{d-1}$. Many mathematicians had contributed to the establishment of this inequality. It was Wang Kunyang and Riemenschneider [Ri-Wa, 1995] who finally gave an correct proof of this inequality for all $r>0$ and $1\le p\le\infty$. However, their proof was much complicated. Our proof is simpler.

Chapter 4 concerns mainly with the strong approximation by the Cesàro means of the Fourier–Laplace series. It contains two sections.

The first section deals with the strong approximation in the L^p $(1\le p\le\infty)$ metric. Some well known results in one variable are extended to the case of multi-dimensional spheres. Our approach is different from that of one dimensional case. It relies on the following equivalence theorem.

Theorem 0.0.13. *Suppose $\delta>0$, $1<p<\infty$ and f is a function defined on $\mathbb{S}^{d-1}$. Then there exists a constant $C(d,\delta)>0$ such that for any $x\in\mathbb{S}^{d-1}$,*

$$\frac{1}{C(d,\delta)}\sum_{k=0}^{\infty}|E_k^\delta(f)(x)-f(x)|^p\le\sum_{k=0}^{\infty}|\sigma_k^\delta(f)(x)-f(x)|^p\le C(d,\delta)\Big(\frac{p}{p-1}\Big)^p\sum_{k=0}^{\infty}|E_k^\delta(f)(x)-f(x)|^p,$$

where σ_k^δ denotes the Cesàro mean of order δ of the Fourier-Laplace series of f and

$$E_k^\delta(f)(x)=\frac{\Gamma(\delta+1)\Gamma(k+1)\Gamma(k+d-1)}{(4\pi)^{\frac{d-1}{2}}\Gamma(k+\delta+1)\Gamma(k+\frac{d-1}{2})}\int_{\mathbb{S}^{d-1}}f(y)P_k^{(\frac{d-1}{2}+\delta,\frac{d-3}{2})}(xy)\,d\sigma(y),\quad x\in\mathbb{S}^{d-1}$$

is the equiconvergent operator of order δ introduced by Wang K. Y. [Wa, 1991], $P_k^{(\frac{d-1}{2}+\delta,\frac{d-3}{2})}(t)$ denotes the Jacobi polynomial.

Since the kernels of the E_k^δ's are essentially the sequence of Jacobi polynomials, which are mutually orthogonal with respect to the measure $\sin^{d+2\delta}\frac{\theta}{2}\cos^{d-2}\frac{\theta}{2}\,d\theta$ on $[0,\pi]$, Theorem 0.0.13 provides convenience for us to investigate the strong approximation by the Cesàro means of Fourier –Laplace series.

The next section deals with the strong approximation by the Cesàro means with critical index in the Hardy spaces $H^p(\mathbb{S}^{d-1})$ ($0<p\le1$). For $0<p\le1$, denote by $H^p\equiv H^p(\mathbb{S}^{d-1})$ the real Hardy space on the unit sphere. Let σ_k^δ be the Cesàro mean of order δ of the Fourier–Laplace series. As is well known, the special value $\delta(p):=\frac{d-1}{p}-\frac{d}{2}$ (for a given $p\in(0,1]$) is the critical index for the uniform summability of the Cesàro means σ_k^δ in the metric H^p. Let $K_r(f,t)_{H^p}$, $E_j(f,H^p)$ denote the r-th order K-functional of a function $f\in H^p$, the best approximation of f by spherical harmonics of degree less than or equal to j, in the space H^p, respectively. The main result in this section can be stated as follows.

Theorem 0.0.14. *Let $0<p<1$ and $\delta=\delta(p):=\frac{d-1}{p}-\frac{d}{2}$. Then for $f\in H^p(\mathbb{S}^{d-1})$,*

$$\sum_{j=1}^{N}\frac{1}{j}\|\sigma_j^\delta(f)-f\|_{H^p}^p\approx\sum_{j=1}^{N}\frac{1}{j}E_j^p(f,H^p).$$

As a consequence, we have

Corollary 0.0.15. *Suppose* $0 < p < 1$ *and* $\delta = \delta(p) := \frac{d-1}{p} - \frac{d}{2}$*. Then for* $f \in H^p(\mathbb{S}^{d-1})$,

$$\frac{1}{\log N}\sum_{k=1}^{N}\frac{\|\sigma_k^\delta(f)-f\|_{H^p}^p}{k} \le CK_1^p\Big(f,\Big(\frac{1}{\log N}\Big)^{\frac{1}{p}}\Big)_{H^p}.$$

The proof of Theorem 0.0.14 is based on the following

Theorem 0.0.16. *Let* $0 < p < 1$, $\delta(p) := \frac{d-1}{p} - \frac{d}{2}$ *and* $f \in H^p$*. Then*

$$\frac{1}{\log N}\sum_{k=1}^{N}\frac{\|\sigma_k^\delta(f)\|_{H^p}^p}{k} \le C_p\|f\|_{H^p}^p. \tag{0.0.4}$$

In the case of the Riesz means for the periodic functions, Theorem 0.0.14, Corollary 0.0.15 and Theorem 0.0.16 are due to [Be2,1996], [C-J-Lu, 1989] and [J-Liu-Lu,1987] respectively. However, in the case of the multi-dimensional sphere $\mathbb{S}^{d-1}$, as far as we know, the best previously known result in this direction was due to Chen Guoliang [Chen, 1995], who proved the validity of Theorem 0.0.16, with the H^p norms in the left hand side of (0.0.4) replaced by the L^p norms.

The proof of Theorem 0.0.16 relies on a new characterization of the Hardy space $H^p(\mathbb{S}^{d-1})$ ($0 < p \le 1$), which is in terms of the maximal Cesàro operator and has an independent interest. Such a characterization is established in Section 4.2.2.

We point out it appears to be difficult to obtain Theorem 0.0.16 simply by following the technique developed in [J-Liu-Lu]. (This can be seen from the proof in [Chen, p154] and the proof of Theorem 4.10 of [CTW].)

General notations and basic facts

We shall use the symbol C to denote various large positive constants depending only on the inessential variables, and ε to denote various small positive constants. We shall write $x = O(y)$ or $x \lesssim y$ for the statement that $|x| \leq Cy$, and write $x \sim y$ if $x \lesssim y$ and $y \lesssim x$. We set $p' = \frac{p}{p-1}$ $(1 < p < \infty)$, $1' = \infty$ and $\infty' = 1$. An expression of the form $[x]$ denotes the greatest integer less than or equal to x when the brackets have no other functions, the letter d denotes the dimension of the Euclidean space $\mathbb{R}^d$, the symbol λ denotes the special value $\frac{d-2}{2}$, and P_k^d denotes the normalized ultraspherical polynomial $\frac{C_k^\lambda(t)}{C_k^\lambda(1)}$, whose definition will be given below.

First, we summarize briefly some definitions and facts about spherical harmonics. For further details of the material described below, the reader is referred to [Wa-L].

Let $\mathbb{S}^{d-1} = \{(x_1, \cdots, x_d) : x_1^2 + \cdots + x_d^2 = 1\}$ be the unit sphere of $\mathbb{R}^d$ and $d\sigma(x)$ denote the usual Lebesgue measure on $\mathbb{S}^{d-1}$. For an integer $k \geq 0$, the restriction to $\mathbb{S}^{d-1}$ of a homogeneous harmonic polynomial of degree k is called a spherical harmonic of degree k. The class of all spherical harmonics of degree k will be denoted by $\mathcal{H}_k^d$ and the class of all spherical harmonics of degree $k \leq N$ will be denoted by $\Pi_N^d \equiv \Pi_N^d(\mathbb{S}^{d-1})$. Of course, $\Pi_N^d = \bigoplus_{k=0}^{N} \mathcal{H}_k^d$ and it comprises the restriction to $\mathbb{S}^{d-1}$ of all algebraic polynomials in d variables of total degree not exceeding N. The dimension of $\mathcal{H}_k^d$ is given by

$$a_k^d := \dim \mathcal{H}_k^d = \begin{cases} \dfrac{(2k+d-2)\Gamma(k+d-1)}{(k+d-2)\Gamma(k+1)\Gamma(d-1)}, & \text{if } k \geq 1, \\ 1, & \text{if } k = 0 \end{cases}$$

$$\asymp k^{d-2}, \quad \text{as } k \to \infty$$

and that of Π_N^d is $\sum_{k=0}^{N} a_k^d \asymp N^{d-1}$. Throughout the rest of this thesis, the symbol c_k^d will always denote the special value $\frac{a_k^d}{|\mathbb{S}^{d-1}|}$.

The spaces of spherical harmonics are closely related to the Laplace -Beltrami operator $\triangle_{\mathbb{S}^{d-1}}$ on $\mathbb{S}^{d-1}$, which is defined as

$$\triangle_{\mathbb{S}^{d-1}} f(x) = \sum_{j=1}^{d} \frac{\partial^2 F}{\partial z_j^2}\Big|_{z=x},$$

where $x \in \mathbb{S}^{d-1}$, $f \in C^2(\mathbb{S}^{d-1})$ and $F(z) = f(\frac{z}{|z|})$. In fact, the operator $\triangle_{\mathbb{S}^{d-1}}$ is an elliptic, (unbounded), selfadjoint operator on $L^2(\mathbb{S}^{d-1})$ and its spectrum comprises distinct eigenvalues $\lambda_k = -k(k+d-2)$, $k = 0, 1, \cdots$, each having finite multiplicity $a_k^d := \dim H_k^d$. The space $\mathcal{H}_k^d$ can be characterized intrinsically as the eigenspace corresponding to λ_k.

Since the λ_k's are distinct, and the operator $\triangle_{\mathbb{S}^{d-1}}$ is selfadjoint, the spaces $\mathcal{H}_k^d$'s are mutually orthogonal relative to the inner product

$$\langle f, g \rangle_{\mathbb{S}^{d-1}} := \int_{\mathbb{S}^{d-1}} f(x)\overline{g(x)}\, d\sigma(x).$$

Also, $L^2(\mathbb{S}^{d-1}) = \text{closure}\Big\{\bigoplus_{k=0}^{\infty} \mathcal{H}_k^d\Big\}$. Hence, if we choose an orthonormal basis $\Big\{Y_{k,j} : j = 1, \cdots, a_k^d\Big\}$ for each $\mathcal{H}_k^d$, then the set $\Big\{Y_{k,j} : j = 1, \cdots, a_k^d,\ k = 0, 1, \cdots\Big\}$ is an orthonormal basis for $L^2(\mathbb{S}^{d-1})$. Furthermore, one has the well–known addition formula:

$$\sum_{j=1}^{a_k^d} Y_{k,j}(x)\overline{Y_{k,j}(y)} = c_k^d P_k^d(xy),\ x, y \in \mathbb{S}^{d-1},\ k = 0, 1, \cdots, \tag{0.0.5}$$

which allows us to express the orthogonal projection of $f \in L^2(\mathbb{S}^{d-1})$ onto $\mathcal{H}_k^d$ as a convolution

$$\begin{aligned} Y_k(f)(x) =& c_k^d f * P_k^d(x) \\ :=& c_k^d \int_{\mathbb{S}^{d-1}} f(y) P_k^d(xy)\, d\sigma(y). \end{aligned} \tag{0.0.6}$$

The identity (0.0.6) also gives the definition of $Y_k(f)$ for all $f \in L(\mathbb{S}^{d-1})$. For the remainder of this thesis, we regard Y_k as an operator defined on $L^1(\mathbb{S}^{d-1})$, valued in $L^\infty(\mathbb{S}^{d-1})$.

For $f \in L(\mathbb{S}^{d-1})$, let us write $\sigma(f) := \sum_{k=0}^{\infty} Y_k(f)$. We call $\sigma(f)$ *the Fourier-Laplace series* of f. In the special case $d = 2$, $\sigma(f)$ returns to the usual trigonometric series of periodic functions on the unit circle T, for which the theory is much more complete and wonderful. However, for $d \geq 3$, the theory is far from being perfect. Many problems in this direction remain open.

The Cesàro means of $\sigma(f)$ *of order* $\delta > -1$ are defined as

$$\sigma_N^\delta(f) := \sum_{k=0}^{N} \frac{A_{N-k}^\delta}{A_N^\delta} Y_k(f) \tag{0.0.7}$$

where $A_k^\delta = \frac{\Gamma(k+\delta+1)}{\Gamma(k+1)\Gamma(\delta+1)}$. The special value $\lambda := \frac{d-2}{2}$ of δ is known as the critical order of σ_N^δ. Indeed, it is well known that for $\delta > \lambda$, $1 \leq p \leq \infty$,

$$\sup_N \|\sigma_N^\delta\|_{(p,p)} < \infty, \tag{0.0.8}$$

while for $\delta \leq \lambda$, (0.0.8) for $p = 1, \infty$ is no longer valid. On the other hand, it follows from [CS] that (0.0.8) is valid whenever $\delta > \delta_p$ and $p \in J(d)$, where

$$\delta_p = \max\Big\{(d-1)\Big|\frac{1}{p} - \frac{1}{2}\Big| - \frac{1}{2}, 0\Big\} \tag{0.0.9}$$

and [1]

$$J(d) = \begin{cases} [1, \infty], & d = 3, \\ \Big\{p : \ 1 \leq p \leq \infty \ \text{ and } \ \Big|\frac{1}{2} - \frac{1}{p}\Big| > \frac{1}{d+1}\Big\}, & d \geq 4. \end{cases} \tag{0.0.10}$$

The translation operator S_θ on $\mathbb{S}^{d-1}$ with step $\theta \in [0, \pi]$ is defined by

$$S_\theta(f)(x) := \frac{1}{|\mathbb{S}^{d-2}| \sin^{d-2}\theta} \int_{\{y \in \mathbb{S}^{d-1}:\ xy = \cos\theta\}} f(y)\, d\ell_{x,\theta}(y),$$

where $f \in L(\mathbb{S}^{d-1})$, $x \in \mathbb{S}^{d-1}$ and $d\ell_{x,\theta}(y)$ denotes the Lebesgue measure element on the parallel $\{y \in \mathbb{S}^{d-1} : \ yx = \cos\theta\}$. It is well known that

$$Y_k(S_\theta(f))(x) = P_k^d(\cos\theta) Y_k(f)(x), \ x \in \mathbb{S}^{d-1}, \ k = 0, 1, \cdots \tag{0.0.11}$$

[1] Further development on this problem has been made recently. The restriction on the range of the value p here can be further relaxed in the case $d \geq 4$. We refer the reader to [Tao] for details.

and

$$\|S_\theta\|_{p,p} = 1 \tag{0.0.12}$$

for all $1 \le p \le \infty$ and $\theta \in [0, \pi]$. The translation operator S_θ on $\mathbb{S}^{d-1}$ was first introduced to investigate the uniqueness of Fourier Laplace series in the case $d = 3$ by W. Rudin [Rud]. It is a natural extension of the usual translation operator on the real line and the circle T.

Associated with the translation operator is the *rth order difference operator* Δ_t^r with step t (for a given $r > 0$), defined by

$$\Delta_t^r = (I - S_t)^{\frac{r}{2}} = \sum_{k=0}^{\infty} \frac{\psi^{(k)}(0)}{k!} (S_t)^k, \tag{0.0.13}$$

where I is the identity operator and $\psi(x) = (1-x)^{\frac{r}{2}}$. For $f \in L^p$ with $1 \le p \le \infty$, *the modulus of continuity of degree* r of f, which was introduced by Rustamov [Rus1], is defined by

$$\omega_r(f, t)_p = \sup_{0<\theta\le t} \|\Delta_t^r f\|_p.$$

As a generalization of the Laplace -Beltrami operator $\triangle_{\mathbb{S}^{d-1}}$, we define the *fractional Laplace -Beltrami operator* D^r (for a given $r > 0$), in the sense of distributions, by

$$D^r f \sim \sum_{k=1}^{\infty} (-k(k+d-2))^r Y_k(f). \tag{0.0.14}$$

It is well known that when $r = 2$, the operator D^r coincides with $\triangle_{\mathbb{S}^{d-1}}$. For simplicity, we write $f^{(r)} := D^{\frac{r}{2}} f$. Define the function class W_p^r by

$$W_p^r = \Big\{ f \in L^p(\mathbb{S}^{d-1}) : \int_{\mathbb{S}^{d-1}} f(y)\, d\sigma(y) = 0, \quad D^{\frac{r}{2}} f \in L^p(\mathbb{S}^{d-1}) \Big\}, \quad 1 \le p \le \infty.$$

In the case $d = 2$, as is well known, the space W_p^r coincides with the usual Sobolev space on the circle T.

To each differential operator $D^{\frac{r}{2}}$, we associate a K-functional

$$K_r(f, t)_p := K(f, D^{\frac{r}{2}}, t^r)_p = \inf \Big\{ \|f - g\|_p + t^r \|D^{\frac{r}{2}} g\|_p : \ g \in W_p^r \Big\}, \tag{0.0.15}$$

where $f \in L^p$, $1 \le p \le \infty$ and $t > 0$.

Suppose $\{u_k\}_{k=0}^\infty$ is a sequence of real numbers. We define $\triangle u_k = u_k - u_{k+1}$ and $\triangle^{\ell+1} u_k = \triangle^\ell(\triangle u_k)$, $\ell \in \mathbb{N}$. The following lemma will be used repeatedly in the later sections.

Lemma 0.0.17. *Let $\{u_k\}_{k=0}^\infty$ be a sequence of complex numbers. Suppose the following two conditions are satisfied:*
(i)

$$\sup_{k\ge 0} |u_k| \le M < \infty;$$

(ii) For a positive integer $\ell > \lambda$,

$$\sum_{k=0}^\infty \Big|\triangle^{\ell+1} u_k\Big| A_k^\ell \le M.$$

For $1 \le p \le \infty$ and $f \in L^p(\mathbb{S}^{d-1})$, define

$$T(f)(x) := \sum_{k=0}^\infty \Big(\triangle^{\ell+1} u_k\Big) A_k^\ell \sigma_k^\ell(f)(x).$$

Then $T(f) \in L^p(\mathbb{S}^{d-1})$ and

$$Y_k(Tf) = (u_k - \overline{\lim_{k\to\infty}}\, u_k) Y_k(f), \quad k = 0, 1, \cdots . \tag{0.0.16}$$

Furthermore,

$$\|T(f)\|_p \lesssim M \|f\|_p. \tag{0.0.17}$$

Proof. Since the Cesàro operators σ_k^ℓ are uniformly bounded on L^p, we conclude that

$$\sum_{k=0}^\infty \Big(\triangle^{\ell+1} u_k\Big) A_k^\ell \sigma_k^\ell(f)(x)$$

converges absolutely in $L^p(\mathbb{S}^{d-1})$ and (0.0.17) holds.

Now it remains to prove (0.0.16). For a fixed non-negative integer k, since Y_k is continuous on L^p, we have

$$Y_k(T(f))(x) = \sum_{j=0}^\infty \Big(\triangle^{\ell+1} u_j\Big) A_j^\ell Y_k(\sigma_j^\ell(f))(x) = \Big(\sum_{j=k}^\infty \triangle^{\ell+1} u_j A_{j-k}^\ell\Big) Y_k(f)(x).$$

Therefore, it is sufficient to show

$$\sum_{j=k}^\infty \triangle^{\ell+1} u_j A_{j-k}^\ell = u_k - \overline{\lim_{k\to\infty}}\, u_k. \tag{0.0.18}$$

We start with the identity

$$\triangle^\ell u_k - \triangle^\ell u_N = \sum_{j=k}^{N-1} \triangle^{\ell+1} u_j. \tag{0.0.19}$$

Then we get from condition (ii) that

$$\lim_{k\to\infty} \triangle^\ell u_k$$

exists and is finite. Taking into account

$$\triangle^{\ell-1}u_k - \triangle^{\ell-1}u_{2k} = \sum_{j=k}^{2k} \triangle^\ell \mu_j,$$

from condition (i), we conclude $\lim_{k\to\infty} \triangle^\ell \mu_k = 0$. Hence, letting $N \to \infty$ in (0.0.19), we conclude

$$\triangle^\ell u_k = \sum_{j=k}^{\infty} \triangle^{\ell+1} u_j. \tag{0.0.20}$$

Substituting (0.0.20) into the sum

$$\sum_{k=0}^{\infty} |\triangle^\ell u_k| A_k^{\ell-1},$$

we get, by a straightforward computation,

$$\sum_{k=0}^{\infty} |\triangle^\ell u_k| A_k^{\ell-1} \le \sum_{k=0}^{\infty} |\triangle^{\ell+1} u_k| A_k^{\ell} \le M.$$

Repeating the above argument finite times yields

$$\sum_{k=0}^{\infty} \Big|\triangle^{i+1} u_k\Big| A_k^{i} < \infty, \quad 0 \le i \le \ell,$$

which also implies that $\lim_{k\to\infty} u_k$ exists and is finite. Without loss of generality, we may assume $\lim_{k\to\infty} u_k = 0$. (Otherwise, we may consider $u_k - \lim_{k\to\infty} u_k$.) Then we get that the identity (0.0.20) remains valid with ℓ replaced by $i = 0, 1, \cdots, \ell$. Now using this identity $\ell + 1$ times, we conclude that for $0 < r < 1$,

$$\sum_{j=0}^{\infty} u_j r^j = \sum_{k=0}^{\infty} \Big(\sum_{j=k}^{\infty} (\triangle^{\ell+1} u_j) A_{j-k}^{\ell} \Big) r^k.$$

Comparing the coefficients of r^k on both sides of the above identity, gives (0.0.18) and completes the proof. □

Next, we shall describe briefly some basic definitions and properties on Jacobi polynomials and ultraspherical polynomials. Most of the material we present below can be found in Szegö's treatise [Sz].

For α and β real and n a nonnegative integer

$$P_n^{(\alpha,\beta)}(x) = \sum_{m=0}^{n} \binom{n+\alpha}{m}\binom{n+\beta}{n-m}\left(\frac{x-1}{2}\right)^{n-m}\left(\frac{x+1}{2}\right)^{m}$$

is the usual Jacobi polynomial. Define

$$\phi_n^{(\alpha,\beta)}(x) = t_n^{(\alpha,\beta)} P_n^{(\alpha,\beta)}(\cos x)(\sin\frac{x}{2})^{\alpha+\frac{1}{2}}(\cos\frac{x}{2})^{\beta+\frac{1}{2}}, \tag{0.0.21}$$

where

$$t_n^{(\alpha,\beta)} = \left[\frac{2n+\alpha+\beta+1)n!\Gamma(n+\alpha+\beta+1)}{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}\right]^{\frac{1}{2}}.$$

For $\alpha > -1$ and $\beta > -1$, the functions $\phi_n^{(\alpha,\beta)}(x)$ are orthonormal on $[0,\pi]$ by (4.3.3) of [Sz].

From (4.1.3) of [Sz] we have

$$P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x). \tag{0.0.22}$$

From Theorem 7.32.2 of [Sz] and (0.0.22), it follows that if a is a fixed integer, α and β are fixed real numbers and $n \geq \max\{0,-a\}$, then

$$|P_{n+a}^{(\alpha,\beta)}(x)| \leq C E_n^{(\alpha,\beta)}(x), \tag{0.0.23}$$

where C is independent of n and x and

$$E_n^{(\alpha,\beta)}(x) = \begin{cases} (n+1)^{\alpha}, & 1-\frac{1}{(n+1)^2} \leq x \leq 1, \\ (n+1)^{-\frac{1}{2}}(1-x)^{-\frac{\alpha}{2}-\frac{1}{4}}, & 0 \leq x \leq 1-\frac{1}{(n+1)^2}, \\ (n+1)^{-\frac{1}{2}}(1+x)^{-\frac{\beta}{2}-\frac{1}{4}}, & -1+\frac{1}{(n+1)^2} \leq x \leq 0, \\ (n+1)^{\beta}, & -1 \leq x \leq -1+(n+1)^{-2}. \end{cases} \tag{0.0.24}$$

The *n-th Cesàro kernel of order θ* is defined by

$$K_n^{(\alpha,\beta),\theta}(s,t) = \frac{1}{A_n^{\theta}}\sum_{k=0}^{n} A_{n-k}^{\theta}\frac{P_k^{(\alpha,\beta)}(x)P_k^{(\alpha,\beta)}(y)}{\int_{-1}^{1} P_k^{(\alpha,\beta)}(t)^2(1-t)^{\alpha}(1+t)^{\beta}\,dt}.$$

The *n-th Cesàro mean* of f is then defined as

$$\sigma_n^{(\alpha,\beta),\theta}(f,x) = \int_{-1}^{1} f(y)K_n^{(\alpha,\beta),\theta}(x,y)(1-y)^{\alpha}(1+y)^{\beta}\,dy.$$

Let

$$p_1(\theta)=\frac{2\alpha+2}{\alpha+\theta+\frac{3}{2}},\ \ p_2(\theta)=\frac{2\alpha+2}{\alpha-\theta+\frac{1}{2}},$$

$$G(n,p,\theta)=\begin{cases}(n+1)^{\frac{2\alpha+2}{p}-(\alpha+\theta+\frac{3}{2})}, & 1\le p<p_1(\theta),\ \theta\le\alpha+\frac{1}{2},\\ \Big(\log(n+1)\Big)^{\frac{1}{p}}, & p=p_1(\theta),\ \theta\le\alpha+\frac{1}{2},\\ 1, & p_1(\theta)<p<p_2(\theta),\ 0\le\alpha+\frac{1}{2},\\ \Big[\log(n+1)\Big]^{\frac{1}{p'}}, & p=p_2(\theta),\ \theta\le\alpha+\frac{1}{2}\\ (n+1)^{(\alpha+\frac{1}{2}-\theta)-\frac{2\alpha+2}{p}}, & p_2(\theta)<p\le\infty,\ \theta\le\alpha+\frac{1}{2},\\ 1, & 1\le p\le\infty,\ \ \theta>\alpha+\frac{1}{2}.\end{cases}$$

It follows from ([Ch-Mu], P78, Corollary 18.11) that for $\alpha>-1,\ \beta>-1$ and $\theta>0$,

$$\|\sigma_n^{(\alpha,\beta),\theta}(f,x)\|_p\lesssim G(n,p,\theta)\|f\|_p,$$

where the norms $\|\cdot\|_p$ are computed with respect to the measure $(1-x)^\alpha(1+x)^\beta\,dx$ on $[-1,1]$.

The ultraspherical polynomials are usually defined via the generating function

$$(1-2tz+z^2)^{-\mu}=\sum_{k=0}^{\infty}C_k^{(\mu)}(t)z^k,$$

where $|z|<1,\ |t|<1$ and $\mu>0$. The coefficients $C_k^{(\mu)}(t)$ are algebraic polynomials of degree k and are termed the ultraspherical polynomials. As is well known,

$$C_k^{\mu}(t)=\frac{\Gamma(2\mu+k)\Gamma(\mu+\frac{1}{2})}{\Gamma(2\mu)\Gamma(\mu+\frac{1}{2}+k)}P_k^{(\mu-\frac{1}{2},\mu-\frac{1}{2})}(t).$$

The ultraspherical polynomials possess the following properties:

(A1) The set $\{C_k^{(\mu)}(\cos\theta)\}_{k=0}^{\infty}$ is orthogonal and complete over $(0,\ \pi)$ with respect to the measure $dm_\mu(\theta)=:(\sin\theta)^{2\mu}d\theta$. It leads the expansion $f(\theta)\sim\sum_{k=0}^{\infty}c_k(f)C_k^{(\mu)}(\cos\theta)$ for $f\in L\big([0,\ \pi],\ \sin^{2\mu}\theta\,d\theta\big)$, where

$$c_k(f):=\frac{1}{\|C_k^{(\mu)}(\cos\theta)\|^2_{L^2(dm_\mu)}}\int_0^{\pi}f(\theta)C_k^{(\mu)}(\cos\theta)\sin^{2\mu}\theta\,d\theta.$$

(A2)

$$\int_0^\pi |C_k^{(\mu)}(\cos\theta)|^2\, dm_\mu(\theta) = \frac{2^{1-2\mu}\pi\Gamma(k+2\mu)}{(k+\mu)(\Gamma(\mu))^2\Gamma(k+1)} \asymp k^{2\mu-2}. \tag{0.0.25}$$

(A3)

$$|C_k^{(\mu)}(\cos\theta)| \lesssim \begin{cases} k^{2\mu-1}, & 0 \le \theta \le \pi, \\ \\ \dfrac{k^{\mu-1}}{\theta^\mu(\pi-\theta)^\mu}, & 0 < \theta < \pi. \end{cases} \tag{0.0.26}$$

Chapter 1

Kolmogorov widths of the function classes B_p^r in the spaces L^q, $1 \le p \le q \le \infty$

For $r>0$, let $B_p^r := B_p^r(\mathbb{S}^{d-1})$ $(1 \le p \le \infty)$ denote the class of functions f on the sphere $\mathbb{S}^{d-1}$ representable in the form

$$f(x) = \varphi * F_r(x) := \int_{\mathbb{S}^{d-1}} \varphi(y) F_r(xy)\, d\sigma(y), \quad \|\varphi\|_p \le 1,$$

where

$$F_r(xy) = \sum_{k=1}^{\infty} \big(k(k+2\lambda)\big)^{-\frac{r}{2}} c_k^d P_k^d(xy).$$

By the definition of fractional derivatives, one may rewrite the class B_p^r as

$$B_p^r = \{f \in L^p : \int_{\mathbb{S}^{d-1}} f(y)\, d\sigma(y) = 0,\ \|f^{(r)}\|_p \le 1\}.$$

As is well known, when $d=2$, B_p^r coincides with the usual Sobolev class on the unit circle T.

For $1 \le p,\ q \le \infty$, the Kolmogorov $N-$ width of B_p^r in $L^q(\mathbb{S}^{d-1})$ is given by

$$d_N(B_p^r, L^q) = \inf_{X_N} \sup_{f \in B_p^r} \inf_{g \in X_N} \|f-g\|_q,$$

with the left-most infimum being taken over all $N-$dimensional subspaces X_N of $L^q(\mathbb{S}^{d-1})$. We write $d_N(B_p^r, L^q) \asymp N^s$ to mean that there exist positive constants C and D, independent of N, for which

$$DN^s \le d_N(B_p^r, L^q) \le CN^s$$

for all N sufficiently large.

It is our aim in this chapter to investigate the asymptotic orders of the Kolmogorov widths $d_N(B_p^r, L^q)$ for some $1 \le p,\ q \le \infty$ as $N \to \infty$. This problem has been completely solved in the case $d = 2$. In fact, for $p = q = 2$ the exact values of the widths were obtained by Kolmogorov [Kol], and it is a result of Stechkin [Stec] for $p = 1$ and $q = 2$ and for $q = p = \infty$. (In the latter case Tikhomirov [Tik] obtained the exact values of the widths.) The orders of the widths were found by Babadzhanov and Tikhomirov [Ba–Tik] for $1 \le q = p < \infty$, by Ismagilov [Ism] for $1 \le p < q \le 2$, by Gluskin [Gl2] for $p = 1$ and $q > 2$ (for $r \ge 2$), and, finally, by Kashin [Kas] for all the remaining $1 \le p \le q \le \infty$ as a consequence of his determination of the widths of finite -dimensional sets.

In the case $d \ge 3$, the determination of the widths $d_N(B_p^r, L^q)$ began with the work of Kamzolov [Ka1]–[Ka3], where the following result was proved:

For $r > d - 1$,

$$d_N(B_p^r,\, L_q) \asymp \begin{cases} N^{-\frac{r}{d-1}+\frac{1}{p}-\frac{1}{q}}, & \text{if } 1 \le p < q \le 2, \\ N^{-\frac{r}{d-1}}, & \text{if } 1 \le p = q \le \infty. \end{cases} \tag{1.0.1}$$

Question. *If $1 \le p < q \le \infty$ and $q > 2$, what is the asymptotic order of $d_N(B_p^r,\, L_q)$?*

As a main result of this chapter, the following theorem gives an answer to this question.

Theorem 1.0.18. *For $r \ge 2(d-1)^2$,*

$$d_N(B_p^r, L^q) \asymp \begin{cases} N^{-(\frac{r}{d-1}-\frac{1}{p}+\frac{1}{2})}, & \textit{if } \ 1 \le p \le 2 < q \le \infty, \\ N^{-\frac{r}{d-1}}, & \textit{if } \ 2 \le p < q \le \infty, \end{cases}$$

(The condition on r may be weakened somewhat, but at the cost of greatly increased computation.)

Shortly after finishing the proof of the above theorem, we learnt that a result, which gives the orders of the widths $d_N(B_p^r, L^q)$ for all $1 \le p, q \le \infty$, had been presented in a recent paper [Ku]. However, as will be pointed out in Section 1.1 below, there were essential mistakes in the author's proof. Two counterexamples are given in Section 1.1.

The organization of this chapter is as follows. Section 1.1 contains two counterexamples, which state that the Marcinkiewicz-Zygmund type inequality with equal weights for the multi-dimensional sphere is generally not true and shows that the proof in [Ku] is incorrect. A quadrature formula and a weighted Marcinkiewicz-Zygmund type inequality for spherical harmonics are established in Section 1.2. In Section 1.3, we extend the well known Kashin type inequality for the Kolmogorov width of the unit Euclidean ball to the weighted case. Theorem 1.0.18 is then proved in Section 1.4 by the standard discretization technique. In Section 1.5, we obtain some similar results for the classes of functions with bounded mixed derivatives on the product space $\mathbb{S}^{d-1,n} = \mathbb{S}^{d-1} \times \cdots \times \mathbb{S}^{d-1}$ (n times).

1.1 Two counterexamples

Suppose $\xi_i \in \mathbb{S}^{d-1}$, $i = 1, \cdots, m$. We call $\{\xi_1, \cdots, \xi_m\}$ *an ε-net* of $\mathbb{S}^{d-1}$ if

$$\bigcup_{i=1}^{m} B(\xi_i, \varepsilon) \supset \mathbb{S}^{d-1},$$

where

$$B(\xi_i, \varepsilon) = \{y \in \mathbb{S}^{d-1} : y\xi_i \geq \cos\varepsilon\}.$$

The proof of Theorem 3.1 of [Ku] appears to yield the following stronger result.[1]

Let $\{\xi_1, \cdots, \xi_m\}$ be an ε-net on $\mathbb{S}^{d-1}$ with $m \asymp N^{d-1} \asymp \dim \Pi_N(\mathbb{S}^{d-1})$. If $\varepsilon < \frac{C}{N}$ with $C \in (0, \frac{1}{2})$, then the following Marcinkiewitz -Zygmund inequality

$$C(p)^{-1}\Big(\frac{1}{m}\sum_{k=1}^{m}|f(\xi_k)|^p\Big)^{\frac{1}{p}} \leq \|f\|_p \leq C(p)\Big(\frac{1}{m}\sum_{k=1}^{m}|f(\xi_k)|^p\Big)^{\frac{1}{p}} \tag{1.1.1}$$

holds for all $1 \leq p \leq \infty$ and $f \in \Pi_N(\mathbb{S}^{d-1})$, where $C(p)$ is independent of N and f.

The above result was shown for $p = \infty$ in [DHe]. However, for any $1 \leq p < \infty$, below we shall show that the result is no longer valid.

[1] There were some obvious misprints in both the description and the proof of Theorem 3.1 of [Ku].

For simplicity, we only consider the case $d=3$. Our argument below can be extended to the higher dimensional cases without difficulty. Before stating the results, we introduce some necessary notations. We introduce the following polar coordinates on $\mathbb{S}^2$:

$$x_1 := \cos\theta,$$

$$x_2 := \sin\theta\cos\varphi,$$

$$x_3 := \sin\theta\sin\varphi, \quad 0 \le \theta \le \pi, \quad 0 \le \varphi \le 2\pi.$$

We identify the point $(x_1, x_2, x_3) \in \mathbb{S}^2$ with its polar coordinates (θ, φ). For a finite subset A of $\mathbb{S}^2$, assume $|A|$ denotes its cardinality. let

$$A_\theta = \{\varphi \in [0, 2\pi] : \ (\theta, \varphi) \in A\}, \quad \theta \in [0, \pi]$$

and

$$\Theta_A = \{\theta \in [0, \pi] : \ A_\theta \neq \emptyset\}.$$

Theorem 1.1.1. *Let $A = \{\xi_1, \cdots, \xi_m\}$ be a subset of $\mathbb{S}^2$ with $m \asymp N^2$. Suppose the following condition is satisfied:*

$$|A_\theta| \ge cN, \quad \textit{for any } \theta \in \Theta_A, \tag{1.1.2}$$

with $c > 0$ independent of N and θ. Then for any $1 \le p < \infty$, the inequality (1.1.1), with the constant $C(p)$ independent of f and N, does not hold. More precisely, if for some $1 \le p < \infty$, the inequality

$$(C_N)^{-1}\Big(\frac{1}{|A|}\sum_{x\in A}|f(x)|^p\Big)^{\frac{1}{p}} \le \|f\|_p \le C_N\Big(\frac{1}{|A|}\sum_{x\in A}|f(x)|^p\Big)^{\frac{1}{p}}, \tag{1.1.3}$$

is satisfied for all $f \in \Pi_N(\mathbb{S}^2)$, with $C_N > 0$ independent of f, then

$$C_N \ge cN^{\frac{1}{2p}}, \tag{1.1.4}$$

with $c > 0$ independent of N.

Proof. First, we assume (1.1.3) holds simultaneously for $p = p_1$ and p_2 with $1 \le p_1 < p_2 \le \infty$. Let $h(t)$ be a nonzero algebraic polynomial on $[-1,1]$ of degree $< N$. Define

$$f(x) = h(x_1), \quad \text{for} \;\; x = (x_1, x_2, x_3) \in \mathbb{S}^2.$$

Then, from the assumption, one can easily obtain

$$\begin{aligned} C^{-1}C_N^{-1}&\Big(\frac{1}{N}\sum_{\theta\in\Theta_A}\frac{|A_\theta|}{N}|h(\cos\theta)|^{p_i}\Big)^{\frac{1}{p_i}} \\ &\le \Big(\int_{\mathbb{S}^2}|f(x)|^{p_i}\,d\sigma(x)\Big)^{\frac{1}{p_i}} = \Big(2\pi\int_{-1}^{1}|h(t)|^{p_i}\Big)^{\frac{1}{p_i}} \\ &\le CC_N\Big(\frac{1}{N}\sum_{\theta\in\Theta_A}\frac{|A_\theta|}{N}|h(\cos\theta)|^{p_i}\Big)^{\frac{1}{p_i}}, \quad i = 1, 2, \end{aligned}$$

which, together with the condition (1.1.2), implies

$$\frac{\|h\|_{p_2}}{\|h\|_{p_1}} \le CC_N^2 N^{\frac{1}{p_1}-\frac{1}{p_2}}, \tag{1.1.5}$$

here $C > 0$ is independent of N and h and the norms $\|\cdot\|_p$ are computed with respect to the usual Lebesgue measure on $[-1,1]$. Now taking $h(x) = P_{N-1}^{(4,0)}(x)$, in view of the fact (see [Sz, p 391, exercise 91]),

$$\|h\|_{p_i} \asymp N^{4-\frac{2}{p_i}}, \; i = 1, 2,$$

we get from (1.1.5)

$$C_N \ge CN^{\frac{1}{2}(\frac{1}{p_1}-\frac{1}{p_2})}, \tag{1.1.6}$$

with $C > 0$ independent of N.

Finally, notice that, as a consequence of a result in [DHe], (1.1.3) for $p = \infty$ holds automatically for all $f \in \Pi(\mathbb{S}^2)$ with the constant independent of N and f. We obtain (1.1.4) from (1.1.6) by taking $p_1 = p$, $p_2 = \infty$. This completes the proof. □

Theorem 1.1.2. *Let A be any finite subset of $\mathbb{S}^2$ of the form*

$$\Big\{(\cos\theta_i, \sin\theta_i\cos\varphi_j, \sin\theta_i\sin\varphi_j) : \; i = 1, \cdots, \ell, \; j = 1, \cdots, n\Big\},$$

where $\theta_i \in [0,\pi]$, $\varphi_j \in [0,2\pi]$ *and* $|A| = \ell n \asymp N^2$. *If for some* $p \in [1,\infty)$, *the inequality (1.1.3) is satisfied for all* $f \in \Pi_N(\mathbb{S}^2)$, *with the constant* $C_N > 0$ *independent of* f, *then (1.1.4) holds, with* $C > 0$ *independent of* N.

Proof. Let $m = [\frac{N-1}{2}]$. If $n \geq m$, then Theorem 1.1.2 follows directly from Theorem 1.1.1. Now we assume $n \leq m-1$. Let

$$a(\cos\theta) = \sum_{k=0}^{m} a_k \cos^{2k}\theta$$

be a nonzero trigonometric polynomial such that

$$a(\cos\varphi_j) = 0, \quad 1 \leq j \leq n.$$

(The existence of such a polynomial is obvious.)

Define

$$f(x,y,z) = \sum_{k=0}^{m} a_k (y^2+z^2)^{m-k} y^{2k}, \ (x,y,z) \in \mathbb{S}^2.$$

Then, obviously, $f \in \Pi_N(\mathbb{S}^2)$ and for all $1 \leq i \leq \ell$ and $1 \leq j \leq n$

$$\begin{aligned} f(\cos\theta_i, \sin\theta_i\cos\varphi_j, \sin\theta_i\sin\varphi_j) &= \sum_{k=0}^{m} a_k (\sin\theta_i)^{2m-2k} (\sin\theta_i\cos\varphi_j)^{2k} \\ &= (\sin\theta_i)^{2m} \sum_{k=0}^{m} a_k (\cos\varphi_j)^{2k} = 0. \end{aligned}$$

which contradicts (1.1.3) and completes the proof. □

Remark 1.1.1. The proof of Theorem 3.1 (M-Z inequality) of [Ku] is valid only for $p = \infty$, for which the result had been well known. (See [DHe].) For $1 \leq p < \infty$, the author of [Ku] used a wrong duality argument to derive his inequality (21) from his inequality (20), which yields the incorrect result described at the beginning of this section. We remark that such a mistake is essential. In fact, the author of [Ku] followed the well known discretization technique used by Kashin in one-dimensional case to derive the asymptotic orders of $d_N(B_p^r, L^q)$. But, as is well known, such a technique relies heavily on the M-Z inequality, whose proof in [Ku] is incorrect.

Remark 1.1.2. It should be pointed out that even if the proof of the inequality (1.1.1) presented in [Ku] were true, some gaps would remain in the proofs of the main results given there. In fact, the author claims to follow the method used in one dimensional case (see [Pin, P236-241]), to derive his estimate (28). In that case, instead of the well known Kashin inequality, one must use the following stronger estimates

$$d_n\Big(b_2^m \bigcap X, \ell_\infty^m \bigcap X\Big) \le Cn^{-\frac{1}{2}}\Big(1+\log\frac{m}{n}\Big)^{\frac{3}{2}}, \tag{1.1.7}$$

where b_2^m is the unit ball of ℓ_2^m and X is a subspace of $\mathbb{R}^m$ with $\dim X \asymp m$, since the number of points appearing in the inequality (1.1.1) is greater than the dimension of Π_N, which differs from the one dimensional case. However, the validity of the inequality (1.1.7) is not known. (Probably it does not generally hold.) We would meet some similar problems in dealing with the linear widths, which were also considered in [Ku].

Remark 1.1.3. To the best of our knowledge, the problem of establishing the Marcinkiewicz-Zygmund type inequality with equal weights for the multidimensional sphere remains open. (It is probable that this is not generally true. One can see this clearly in the next section.) But from the above two theorems, we know that the inequality of this type relies on the distribution of the points on the sphere, which may be a little complicated.

1.2 Quadrature formula and the Marcinkiewicz-Zygmund type inequality for spherical harmonics

Let $\{\tau_{N,k} = \cos\theta_{N,k}^{(\lambda)}\}_{k=1}^N$ be the zeros of the ultraspherical polynomial $C_N^{(\lambda)}(t)$ ordered so that $0 < \theta_{N,1}^{(\lambda)} < \theta_{N,2}^{(\lambda)} < \cdots < \theta_{N,N}^{(\lambda)} < \pi$.

Lemma 1.2.1 ([Sz], [KT]). *Suppose $-\frac{1}{2} < \lambda < \infty$. Then there exists a constant C depending only on λ so that*

$$\left|\theta_{N,k}^{(\lambda)} - \frac{k\pi}{N}\right| \le \frac{C}{N}.$$

Furthermore, there exists $C > 0$ *so that*

$$\left|\theta_{N,k}^{(\lambda)}\right| \geq \frac{Ck}{N}$$

if $k < \frac{N}{2}$.

Lemma 1.2.2 ([Sz]). *Suppose that* $-\frac{1}{2} < \lambda < \infty$. *Then for any algebraic polynomial* f *of degree not exceeding* $2N - 1$,

$$\int_0^\pi f(\cos\theta)(\sin\theta)^{2\lambda}\, d\theta = \frac{1}{N}\sum_{k=1}^{N} \alpha_{N,k}^{(\lambda)} f(\cos\theta_{N,k}^{\lambda}),$$

where

$$\alpha_{N,k}^{(\lambda)} = \frac{\pi}{\left(2^{\lambda}\Gamma(\lambda+1)\right)^2}\frac{\Gamma(N+2\lambda)}{\Gamma(N)}\frac{1}{\left|\sin(\theta_{N,k}^{\lambda})C_{N-1}^{\lambda+1}(\cos\theta_{N,k}^{\lambda})\right|^2} \asymp (\sin\theta_{N,k}^{\lambda})^{2\lambda}.$$

Lemma 1.2.3. *Suppose* f *is a trigonometric polynomial of degree* $\leq 4N + 1$. *Then*

$$\int_0^{2\pi} f(\theta)\, d\theta = \frac{\pi}{2N+1}\sum_{k=0}^{4N+1} f(\frac{k\pi}{2N+1})$$

Furthermore, if $1 \leq p \leq \infty$ *and* f *is a trigonometric polynomial of degree* $\leq N$, *then*

$$\Big(\int_0^{2\pi} |f(\theta)|^p\, d\theta\Big)^{\frac{1}{p}} \asymp \Big(\frac{\pi}{2N+1}\sum_{k=0}^{4N+1}\left|f(\frac{k\pi}{2N+1})\right|^p\Big)^{\frac{1}{p}}.$$

Proof. For $1 < p < \infty$, this result is contained in [Zy]. For $p = 1$ or $p = \infty$, it can be obtained by applying the Bernstein's inequality for trigonometric polynomials. We omit the detail here. □

Lemma 1.2.4. *Let* $a \in C_0^\infty(\mathbb{R})$ *with the properties that* $a(x) = 1$ *for* $|x| \leq 1$ *and* $a(x) = 0$ *for* $|x| \geq 2$. *Define*

$$V_N(f) = \sum_{k=0}^{\infty} a(\frac{k}{N})Y_k(f).$$

Then $V_N(f) = f$ *for any* $f \in \Pi_N(\mathbb{S}^{d-1})$. *Furthermore, for any* $g \in L^p(\mathbb{S}^{d-1})$ *and* $1 \leq p \leq \infty$,

$$\|V_N(g)\|_p \leq C_p\|g\|_p,$$

where C_p *is independent of* N.

Lemma 1.2.4 is a direct consequence of Lemma 0.0.17.

Theorem 1.2.5. *There exist $m_N = (4N+1)(2N)^{d-2}$ points $\xi_{N,1}, \xi_{N,2}, \cdots, \xi_{N,m_N}$ on $\mathbb{S}^{d-1}$ such that for any $f \in \Pi_{4N}$*

$$\int_{\mathbb{S}^{d-1}} f(x)\, d\sigma(x) = \frac{1}{m_N} \sum_{k=1}^{m_N} w_{N,k} f(\xi_{N,k}), \tag{1.2.1}$$

where the weights $\{w_{N,k}\}$ satisfy the following two conditions:

(i)

$$0 < w_{N,k} \le 1, \quad w_{N,k}^{-1} \le C_d N^{(d-2)^2};$$

(ii)

$$\frac{1}{m_N} \sum_{k=1}^{m_N} \Big(w_{N,k} \Big)^t \le \begin{cases} C_d, & \text{if } t > -\frac{1}{d-2}, \\ C_d \log N, & \text{if } t = -\frac{1}{d-2}, \\ C_d N^{-(d-2)^2 t - 1}, & \text{if } t < -\frac{1}{d-2}. \end{cases}$$

Furthermore, if $f \in \Pi_N$ and $1 \le p \le \infty$, then

$$\Big(\int_{\mathbb{S}^{d-1}} |f(x)|^p\, d\sigma(x) \Big)^{\frac{1}{p}} \asymp \Big(\frac{1}{m_N} \sum_{k=1}^{m_N} w_{N,k} |f(\xi_{N,k})|^p \Big)^{\frac{1}{p}}. \tag{1.2.2}$$

Proof. We apply induction on the dimension d. On account of Lemma 1.2.4, we assume Theorem 1.2.5 for $\mathbb{S}^{d-2}$ is valid. Now we turn to prove it for $\mathbb{S}^{d-1}$.

First step. We prove there are m_N points $\xi_{N,1}, \xi_{N,2}, \cdots, \xi_{N,m_N}$ on $\mathbb{S}^{d-1}$ such that (1.2.1) holds for all $f \in \Pi_{4N}(\mathbb{S}^{d-1})$, with the weights $\{w_{N,k}\}$ satisfying the required conditions.

It follows from (0.0.11) that for $f \in \Pi_{4N}$,

$$S_\theta(f)(\mathbb{1}) = \sum_{k=0}^{4N} P_k^d(\cos\theta) Y_k(f)(\mathbb{1})$$

is a polynomial of degree $\le 4N-1$ in $\cos\theta$, where $\mathbb{1} = (1, 0, \cdots, 0)$. So, according to Lemma 1.2.2,

$$\int_{\mathbb{S}^{d-1}} f(x)\, d\sigma(x) = \int_0^\pi S_\theta(f)(\mathbb{1}) \sin^{d-2}\theta\, d\theta = \frac{1}{2N} \sum_{k=1}^{2N} S_{\theta_{2N,k}^{(\lambda)}}(f)(\mathbb{1}) \alpha_{2N,k}^{(\lambda)}, \tag{1.2.3}$$

with $\alpha^{(\lambda)}_{2N,k} \asymp (\sin\theta^{(\lambda)}_{2N,k})^{d-2}$.

For a fixed $\theta^{(\lambda)}_{2N,k}$, obviously, $f(\cos\theta^{(\lambda)}_{2N,k}, y\sin\theta^{(\lambda)}_{2N,k})$ is a polynomial of degree $\le 4N-1$ in the variable $y \in \mathbb{S}^{d-2}$. Hence, from our assumption, it follows

$$\begin{aligned} S_{\theta^{(\lambda)}_{2N,k}}(f)(\mathbb{1}) &= \frac{1}{|\mathbb{S}^{d-2}|}\int_{\mathbb{S}^{d-2}} f(\cos\theta^{(\lambda)}_{2N,k}, y\sin\theta^{(\lambda)}_{2N,k})\, d\sigma(y) \\ &= \frac{1}{m_N^{(d-2)}}\sum_{j=1}^{m_N^{(d-2)}} w^{(d-2)}_{N,j} f(\cos\theta^{(\lambda)}_{2N,k}, \xi^{(d-2)}_{N,j}\sin\theta^{(\lambda)}_{2N,k}). \end{aligned} \tag{1.2.4}$$

Combining (1.2.3) with (1.2.4) yields (1.2.1). On the other hand, by Lemmas 1.2.1 and 1.2.2, one can easily verify that the corresponding weights satisfy the required conditions.

Second step. Prove that for $f \in \Pi_{2N}$ and $1 \le p \le \infty$,

$$\Big(\frac{1}{m_N}\sum_{k=1}^{m_N} w_{N,k}|f(\xi_{N,k})|^p\Big)^{\frac{1}{p}} \le C_{d,p}\Big(\int_{\mathbb{S}^{d-1}} |f(x)|^p\, d\sigma(x)\Big)^{\frac{1}{p}}. \tag{1.2.5}$$

It will suffice to prove (1.2.5) for $p = 1$. In fact, (1.2.5) for $p = \infty$ is trivial, and for $1 < p < \infty$ can be derived from the cases of $p = 1, \infty$ by applying Lemma 1.2.4 and the Riesz–Thö rin interpolation theorem to the operator

$$T_N(f)(k) = (V_{2N}f)(\xi^{(d-1)}_{N,k}),\ 1 \le k \le m_N^{(d-1)}.$$

For a fixed vector $\xi = (\xi_1, \cdots, \xi_{d-1}) \in \mathbb{S}^{d-2}$, let

$$h(\theta) := f(\cos\theta, \xi_1\sin\theta, \cdots, \xi_{d-1}\sin\theta)\sin^{d-2}\theta.$$

Then, clearly, $h(\theta)$ is a trigonometric polynomial of degree $\le d-2+4N$. Let us write

$$I_k = \Big\{\theta \in (0,\pi) : |\theta - \theta^{(\lambda)}_{2N,k}| < \frac{1}{2N}\Big\},\quad k = 1, \cdots, 2N.$$

Then Lemma 1.2.1 and Bernstein's inequality yield

$$\begin{aligned}\frac{1}{N}\sum_{k=1}^{2N}|h(\theta^{(\lambda)}_{2N,k})| &\le C\sum_{k=1}^{2N}\Big(\int_{I_k}|h(\theta^{(\lambda)}_{2N,k})-h(\theta)|\,d\theta+\int_{I_k}|h(\theta)|\,d\theta\Big)\\ &\le C\sum_{k=1}^{2N}\Big(\frac{1}{N}\int_{I_k}|h'(\theta)|\,d\theta+\int_{I_k}|h(\theta)|\,d\theta\Big)\\ &\le C\Big(\frac{1}{N}\int_{-\pi}^{\pi}|h'(\theta)|\,d\theta+\int_{-\pi}^{\pi}|h(\theta)|\,d\theta\Big)\le C\int_{-\pi}^{\pi}|h(\theta)|\,d\theta.\end{aligned}$$

This gives

$$\frac{1}{N}\sum_{k=1}^{2N}|f(\cos\theta^{(\lambda)}_{2N,k},\xi\sin\theta^{(\lambda)}_{2N,k})|(\sin\theta^{(\lambda)}_{2N,k})^{d-2}\le C\int_{-\pi}^{\pi}|f(\cos\theta,\xi\sin\theta)||\sin^{d-2}\theta|\,d\theta.$$

Noticing that

$$\begin{aligned}\int_{\mathbb{S}^{d-1}}|f(x)|^p\,d\sigma(x)&=\int_{\mathbb{S}^{d-2}}\int_0^{\pi}|f(\cos\theta,\xi\sin\theta)|\sin^{d-2}\theta\,d\theta d\sigma(\xi)\\ &=\int_{\mathbb{S}^{d-2}}\int_{-\pi}^{0}\Big|f(\cos\theta,-\xi\sin\theta)\Big|\Big|\sin^{d-2}\theta\Big|\,d\theta d\sigma(\xi)\\ &=\frac{1}{2}\int_{\mathbb{S}^{d-2}}\int_{-\pi}^{\pi}\Big|f(\cos\theta,\xi\sin\theta)\Big|\Big|\sin^{d-2}\theta\Big|\,d\theta d\sigma(\xi),\end{aligned}$$

we get,

$$\begin{aligned}\int_{\mathbb{S}^{d-1}}|f(x)|\,d\sigma(x)&=\frac{1}{2}\int_{\mathbb{S}^{d-2}}\Big(\int_{-\pi}^{\pi}\Big|f(\cos\theta,\xi\sin\theta)\sin^{d-2}\theta\Big|\,d\theta d\sigma(\xi)\\ &\ge C\frac{1}{N}\sum_{k=1}^{2N}\Big(\sin\theta^{(\lambda)}_{2N,k}\Big)^{d-2}\int_{\mathbb{S}^{d-2}}\Big|f(\cos\theta^{(\lambda)}_{2N,k},\xi\sin\theta^{(\lambda)}_{2N,k})\Big|\,d\sigma(\xi).\end{aligned}\tag{1.2.6}$$

But by the assumption for $\mathbb{S}^{d-2}$, we have

$$\begin{aligned}&\int_{\mathbb{S}^{d-2}}\Big|f(\cos\theta^{(\lambda)}_{2N,k},\xi\sin\theta^{(\lambda)}_{2N,k})\Big|\,d\sigma(\xi)\\ &\qquad\ge C\frac{1}{m_N^{(d-2)}}\sum_{j=1}^{m_N^{(d-2)}}w^{(d-2)}_{N,j}\Big|f(\cos\theta^{(\lambda)}_{2N,k},\xi^{(d-2)}_{N,j}\sin\theta^{(\lambda)}_{2N,k})\Big|.\end{aligned}\tag{1.2.7}$$

Now a combination of (1.2.6) and (1.2.7) yields the inequality (1.2.5).

Final step. Prove the converse inequality.

The converse inequality can be deduced from (1.2.5) by duality. In fact, for $f \in \Pi_N$ and $1 \le p \le \infty$, let $g \in L^{p'}(\mathbb{S}^{d-1})$ such that $\|g\|_{p'} \le 1$ and

$$\int_{\mathbb{S}^{d-1}} |f(x)|^p \, d\sigma(x) = \int_{\mathbb{S}^{d-1}} f(x)g(x)\, d\sigma(x).$$

Let $h(x) = V_N(g)(x)$ with V_N as in Lemma 1.2.4. Then (1.2.1),(1.2.5) and Hölder inequality imply

$$\begin{aligned}
&\int_{\mathbb{S}^{d-1}} f(x)g(x)\, d\sigma(x) = \int_{\mathbb{S}^{d-1}} f(x)h(x)\, d\sigma(x) = \frac{1}{m_N}\sum_{k=1}^{m_N} f(\xi_{N,k})h(\xi_{N,k})w_{N,k} \\
\le&\Big(\frac{1}{m_N}\sum_{k=1}^{m_N} |f(\xi_{N,k})|^p w_{N,k}\Big)^{\frac{1}{p}}\Big(\frac{1}{m_N}\sum_{k=1}^{m_N} |h(\xi_{N,k})|^{p'} w_{N,k}\Big)^{\frac{1}{p'}} \\
\le& C_{p,d}\Big(\frac{1}{m_N}\sum_{k=1}^{m_N} |f(\xi_{N,k})|^p w_{N,k}\Big)^{\frac{1}{p}}\Big(\int_{\mathbb{S}^{d-1}} |V_N(g)|^{p'}\, d\sigma(x)\Big)^{\frac{1}{p'}} \\
\le& C_{p,d}\Big(\frac{1}{m_N}\sum_{k=1}^{m_N} |f(\xi_{N,k})|^p w_{N,k}\Big)^{\frac{1}{p}},
\end{aligned}$$

which gives the converse inequality and completes the proof. □

Remark 1.2.1. The m_N points on $\mathbb{S}^{d-1}$ in Theorem 1.2.5 can be written explicitly. In fact, if we introduce the following polar coordinates on $\mathbb{S}^{d-1}$:

$$\begin{aligned}
x_1 &= \cos\varphi_1, \\
x_2 &= \sin\varphi_1 \cos\varphi_2, \\
&\ \vdots \\
x_{d-1} &= \sin\varphi_1 \cdots \sin\varphi_{d-2}\cos\varphi_{d-1}, \\
x_d &= \sin\varphi_1 \cdots \sin\varphi_{d-2}\sin\varphi_{d-1},
\end{aligned}$$

the set consisting of the m_N points can be written as

$$\Big\{(\varphi_1 \cdots,\ \varphi_{d-1}) :\ \varphi_i \in I_i,\ 1 \le i \le d-1\Big\},$$

where

$$\begin{aligned}
I_i &= \Big\{\theta_{2N,k}^{(\lambda_i)} :\ 1 \le k \le 2N\Big\},\quad \lambda_i = \frac{d-1-i}{2},\quad 1 \le i \le d-2, \\
I_{d-1} &= \Big\{\frac{k\pi}{2N+1} :\ 1 \le k \le 4N+1\Big\}.
\end{aligned}$$

Remark 1.2.2. For $d = 3$, with minimal change of the above proof, (1.2.1) and (1.2.2) can be written explicitly as

$$\int_{\mathbb{S}^2} f(x)\, d\sigma(x) = \frac{2\pi}{(2N+1)(N+2)} \sum_{j=1}^{2N+1} \sum_{k=0}^{4N+1} v_j f\Big(\frac{j\pi}{2N+2}, \frac{k\pi}{2N+1}\Big), \tag{1.2.8}$$

$$\Big(\int_{\mathbb{S}^2} |f(x)|^p\, d\sigma(x)\Big)^{\frac{1}{p}} \asymp \Big(\frac{1}{N^2} \sum_{j=1}^{2N+1} \sum_{k=0}^{4N+1} \sin\frac{j\pi}{2N+2} \Big| f\Big(\frac{j\pi}{2N+2}, \frac{k\pi}{2N+1}\Big)\Big|^p\Big)^{\frac{1}{p}}, \tag{1.2.9}$$

where

$$v_j = \sin\Big(\frac{j\pi}{2N+2}\Big) \sum_{\ell=1}^{N+1} \frac{1}{2\ell-1} \sin\frac{(2\ell-1)j\pi}{2N+2}, \quad j = 1, \cdots, 2N+1. \tag{1.2.10}$$

Formula (1.2.8) was due to [DHe].

Remark 1.2.3. For $d = 4$, (1.2.1) and (1.2.2) can be written as

$$\int_{\mathbb{S}^3} f(x)\, d\sigma(x) = \frac{2\pi^2}{(2N+1)^2(N+2)} \sum_{i=1}^{2N} \sum_{j=1}^{2N+1} \sum_{k=0}^{4N+1} v_j \sin^2\frac{i\pi}{2N+1} f\Big(\frac{i\pi}{2N+1}, \frac{j\pi}{2N+2}, \frac{k\pi}{2N+1}\Big), \tag{1.2.11}$$

$$\Big(\int_{\mathbb{S}^3} |f(x)|^p\, d\sigma(x)\Big)^{\frac{1}{p}} \asymp \Big(\frac{1}{N^3} \sum_{i=1}^{2N} \sum_{j=1}^{2N+1} \sum_{k=0}^{4N+1} \sin\frac{j\pi}{2N+2} \sin^2\frac{i\pi}{2N+1} \Big| f\Big(\frac{i\pi}{2N+1}, \frac{j\pi}{2N+2}, \frac{k\pi}{2N+1}\Big)\Big|^p\Big)^{\frac{1}{p}}, \tag{1.2.12}$$

with v_j as in (1.2.10).

Remark 1.2.4. After completing this paper, we learnt that the Marcinkiewicz-Zygmund type inequality for the spherical harmonics (also with weights) had been established in a recent paper [MNW, 2001] by a different method. Here our method is simpler. Furthermore, we can write the coefficients appearing in (1.2.1) and (1.2.2) explicitly, which is crucial for us to apply the weighted Kashin type inequality to prove Theorem 1.0.18 in Section 1.4 below.

1.3 The weighted Kashin type inequality

The following theorem was due to Kashin [Kas].

Theorem. (Kashin type inequality.) For $1 \le n \le m$,

$$d_n(b_2^m, \ell_\infty^m) \le 51 n^{-\frac{1}{2}} \sqrt{1 + \log \frac{m}{n}}. \tag{1.3.1}$$

The original upper estimate, which had $\log^{\frac{3}{2}}(1 + \frac{m}{n})$ instead of $\log^{\frac{1}{2}}(1 + \frac{m}{n})$, was also sufficient for Kashin's purposes. His proof [Kas, ,1977] relied on difficult probabilistic arguments. Afterwards, Gluskin [Gl, 1983] and Garnaev and Gluskin [Gar-Gl, 1984] simplified the proof by using an isoperimetric inequality in $\mathbb{R}^m$ instead. The final simplification was due to Makovoz [Mk1, 1988].

Since the Marcinkiewicz-Zygmund inequality established last section has positive weights, which are dependent on the dimension of $\Pi_N(\mathbb{S}^{d-1})$, Kashin's inequality (1.3.1) is not sufficient for our purposes. In order to use the well known discretization technique to derive our main result, we have to generalize Kashin type inequality (1.3.1) to the weighted case.

We first introduce some notations. For a vector $w = (w_1, \cdots, w_m)$, let $\ell_{p,w}^m$ be the m-dimensional space of vectors $x = (x_1, \cdots, x_m)$ with the norm

$$\|x\|_{p,w} = \Big(\sum_{i=1}^m |x_i|^p w_i\Big)^{\frac{1}{p}}, \ 1 \le p < \infty, \ \|x\|_{\infty,w} = \max_k |x_k|,$$

and the unit ball $b_{p,w}^m := \{x \in \ell_{p,w}^m : \ \|x\|_{p,w} \le 1\}$. Denote by $d_n(b_{p,w}^m, \ell_{q,w}^m)$ the Kolmogorov n-widths of $b_{p,w}^m$ in the metric $\ell_{q,w}^m$. For $x \in \mathbb{R}^m$, define

$$\widetilde{\|x\|}_{p,w} = \Big(\frac{1}{m}\sum_{i=1}^m |x_i|^p w_i\Big)^{\frac{1}{p}}.$$

If $w = (1, \cdots, 1)$, we simply write $\ell_{p,w}$ and $\|\cdot\|_{p,w}$ as ℓ_p and $\|\cdot\|_p$. For $b \in \mathbb{R}$ and $A \subset \mathbb{R}^m$, let $W^b A$ denote the following set of vectors in $\mathbb{R}^m$:

$$w^b A := \Big\{(w_1^b a_1, \cdots, w_m^b a_m) : \ a = (a_1, \cdots, a_m) \in A\Big\}.$$

We say a vector $w = \{w_i\}_{i=1}^m$ is with the *Kashin type property* if it satisfies the following two conditions simultaneously:

$$0 < w_i \le 1, \quad w_i^{-1} \le C_1 m^a, \ \ i = 1, \cdots, m; \tag{1.3.2}$$

$$\frac{1}{m}\sum_{k=1}^{m}(w_k)^t \le C_1, \quad \text{for any } t \ge -\alpha, \tag{1.3.3}$$

where $C_1 > 0$ is independent of m, a and α are some positive constants. Let us call the infimum over the constants C_1 on the right-hand side of (1.3.2) and (1.3.3) the Kashin constant of w. Clearly, the vector $\{w_{N,k}\}$ appearing in (1.2.2) has this property.

The main goal in this section is to prove the following weighted Kashin type inequality.

Theorem 1.3.1. *Suppose w is a vector with the Kashin type property. Then for $1 \le n \le m$,*

$$d_n(b_{2,w}^m, \ell_\infty^m) \le C_2\Big(\frac{m}{n}\Big)^{\frac{\theta}{2(1-\theta)}} n^{-\frac{1}{2}}\Big(1+\log\frac{m}{n}\Big)^{\frac{1}{2(1-\theta)}},$$

where $\theta = \frac{1}{1+\alpha} \in (0,1)$ and the constant $C_2 > 0$ depends only on the Kashin constant of w rather than on the vector w itself.

Obviously, Kashin inequality (1.3.1) corresponds to the limiting case $\alpha = \infty$ of Theorem 1.3.1.

Lemma 1.3.2. *Let $1 \le n \le m$, $p = 2(1+\alpha)$ and the numbers α, a as in (1.3.2)–(1.3.3). Then there exists an n-dimensional subspace Y_n of $\mathbb{R}^m$ such that*

$$\sup_{x\in b_{2,w}^m}\inf_{y\in Y_n}\|\widetilde{x-y}\|_{p,w} \le C_3 n^{-\frac{1}{2}}\sqrt{1+\log\frac{m}{n}},$$

$$\sup_{x\in b_{2,w}^m}\inf_{y\in Y_n}\|x-y\|_\infty \le C_3 m^{\frac{a}{2}} n^{-\frac{1}{2}}\sqrt{1+\log\frac{m}{n}},$$

where the constant C_3 depends only on the Kashin constant of w.

Lemma 1.3.2 is a simple consequence of (1.3.1)–(1.3.3) and the fact $b_{p,w}^m = w^{-\frac{1}{p}} b_p^m$. We omit the detail.

Proof of Theorem 1.3.1. Let Y_n be the n-dimensional subspace of $\mathbb{R}^m$ defined as in Lemma 1.3.2. Define three sequences $\{a_k\}$, $\{b_k\}$, $\{d_k\}$ of real numbers as follows:

$$\begin{aligned} a_0 &= a > 0, \ \ a_{k+1} = (a_k + \tfrac{1}{2})\theta; \\ b_0 &= -\frac{1}{2}, \ \ b_{k+1} = b_k\theta - \frac{1}{2}; \qquad k = 0,1,2,\cdots \\ d_0 &= \frac{1}{2}, \ \ d_{k+1} = (d_k + \delta)\theta + \frac{1}{2}; \end{aligned}$$

where

$$\theta = \frac{2}{p}, \quad \delta = \ell\Big(\log(1+\log\frac{m}{n})\Big)^{-1},$$

and ℓ is a specified positive constant. (Its precise definition will be given below.)

The key part of the proof of the theorem is to show that the following statement is valid:

For any positive integer k, any vector $x \in \mathbb{R}^m$, there exists an element $z \in Y_n$ such that

$$\|x - z\|_\infty \le C_4 m^{a_k} n^{b_k} (1 + \log\frac{m}{n})^{d_k} \|x\|_{2,w}, \tag{1.3.4}$$

where $C_4 > 0$ is independent of k.

Observing that

$$\lim_{k\to\infty} a_k = \frac{\theta}{2(1-\theta)}, \quad \lim_{k\to\infty} b_k = -\frac{1}{2}\frac{1}{1-\theta}, \quad \lim_{k\to\infty} d_k = \frac{\delta\theta + \frac{1}{2}}{1-\theta},$$

we deduce the theorem from the above statement by taking k sufficiently large.

It remains to prove the statement. We apply induction on k. Assume the statement is true for k. We prove it for $k+1$.

It follows from Lemma 1.3.2 that there exists an element $y \in Y_n$, such that

$$\|\widetilde{x - y}\|_{p,w} \le C_1 n^{-\frac{1}{2}} (1 + \log\frac{m}{n})^{\frac{1}{2}} \|x\|_{2,w},$$

with $p = 2(1+\alpha)$. For any $\eta > 0$, we decompose $x - y = (x_1 - y_1, \cdots, x_m - y_m)$ as

$$x - y = u + v := (u_1 + v_1, \cdots, u_m + v_m),$$

where

$$u_i = \begin{cases} x_i - y_i, & \text{if } \ |x_i - y_i| \le \eta^{\frac{1}{p}} \|\widetilde{x - y}\|_p, \\ \eta^{\frac{1}{p}} sgn(x_i - y_i) \|\widetilde{x - y}\|_p, & \text{otherwise.} \end{cases}$$

An easy calculation shows

$$\|u\|_\infty \le C_5 \eta^{\frac{1}{p}} n^{-\frac{1}{2}} (1 + \log\frac{m}{n})^{\frac{1}{2}} \|x\|_{2,w},$$

$$\widetilde{\|v\|_{2,w}} \le C_5 \eta^{\frac{1}{p}-\frac{1}{2}} n^{-\frac{1}{2}} (1+\log \frac{m}{n})^{\frac{1}{2}} \|x\|_{2,w}.$$

Invoking the assumption, we know there exists an element $f \in Y_n$ such that

$$\|v-f\|_\infty \le C_4 m^{a_k} n^{b_k} (1+\log \frac{m}{n})^{d_k} m^{\frac{1}{2}} \widetilde{\|v\|_{2,w}} \le C_4 C_5 m^{a_k+\frac{1}{2}} n^{b_k-\frac{1}{2}} (1+\log \frac{m}{n})^{d_k+\frac{1}{2}} \eta^{\frac{1}{p}-\frac{1}{2}} \|x\|_{2,w}.$$

Let $z = y + f$. Then $z \in Y_n$ and

$$\begin{aligned}\|x-z\|_\infty &\le \|u\|_\infty + \|v-f\|_\infty \\ &\le C_4 \|x\|_{2,w} \Big(\frac{C_5}{C_4} \eta^{\frac{1}{p}} n^{-\frac{1}{2}} (1+\log \frac{m}{n})^{\frac{1}{2}} + C_5 m^{a_k+\frac{1}{2}} n^{b_k-\frac{1}{2}} (1+\log \frac{m}{n})^{d_k+\frac{1}{2}} \eta^{\frac{1}{p}-\frac{1}{2}} \Big).\end{aligned}$$

Letting $\eta^{\frac{1}{2}} = m^{a_k+\frac{1}{2}} n^{b^k} (1+\log \frac{m}{n})^{d_k+\delta}$, taking into account $\delta = \ell \Big(\log(1+\log \frac{m}{n}) \Big)^{-1}$, we conclude

$$\|x-z\|_\infty \le C_4 m^{a_{k+1}} n^{b_{k+1}} (1+\log \frac{m}{n})^{d_{k+1}} \Big(\frac{C_5}{C_4} + C_5 e^{-\ell} \Big) \|x\|_{2,w},$$

with $\theta = \frac{2}{p} \in (0,1)$. Taking $C_4 = 3C_5$, $\ell = \log(2C_5)$, we prove the statement for $k+1$. This completes the proof. □

1.4 The widths of classes B_p^r

The main goal in this section is to prove Theorem 1.0.18. To this end, we need two lemmas.

Lemma 1.4.1 ([Ka1]). *Suppose $1 \le p \le 2$. Then for any $f \in \Pi_N$,*

$$\|f\|_2 \le C(d,p) N^{(d-1)(\frac{1}{p}-\frac{1}{2})} \|f\|_p.$$

Recall that

$$W_p^r = \Big\{ f \in L^p(\mathbb{S}^{d-1}) : \; f^{(r)} \in L^p(\mathbb{S}^{d-1}) \Big\}.$$

Lemma 1.4.2. *Suppose $1 \le p \le \infty$ and V_N is an operator defined as in Lemma 1.2.4. Then for every $f \in W_p^r$,*

$$\|f - V_N(f)\|_p \le C_{p,d} N^{-r} \|f^{(r)}\|_p.$$

Lemma 1.4.2 can be easily proved by using the Abelian transform finite times. We omit the detail.

Next, we introduce the concept of "extension property".

Definition 1.4.1 ([N]). Let X be a given normed space. X is said to have the extension property if, for any normed space Y, for any vector subspace Z of Y, and for every continuous linear transformation $f:\ Z \longrightarrow X$, there is at least one linear , continuous transformation $F:\ Y \longrightarrow X$ such that

$$F(z) = f(z), \quad z \in Z$$

and

$$\|F\|_{(Y,X)} = \|f\|_{(Z,X)}.$$

Lemma 1.4.3 ([N]). *Let Ω be a measure space endowed with a non-negative completely additive measure such that Ω can be expressed as a union of countably many measurable sets with finite measure. Then the space $L^\infty(\Omega)$ has the extension property.*

We thank Professor Allan Pinkus for providing us with some references on the concept of extension property.

Proof of Theorem 1.0.18. We only sketch the proof here. For details of the proof, we refer the reader to ([Pin], P236–241).

It is sufficient to show the first estimate. The second estimate follows from the first one.

The lower bound follows from Kamzolov's result (1.0.1). It remains to prove the upper estimate. Since the norm $\|\cdot\|_{L^q(\mathbb{S}^{d-1})}$ is increasing with q, it is sufficient to consider the case $q = \infty$.

First, for a non-negative integer k, define the operator

$$U_k:\ \Pi_{2^{k+1}}(\mathbb{S}^{d-1}) \longrightarrow \mathbb{R}^{m_{2^{k+1}}}$$

by

$$U_k(f) = (f(\xi_{2^{k+1},1}), \cdots, f(\xi_{2^{k+1},m_{2^{k+1}}})),$$

with $\xi_{2^{k+1},j}$, $j = 1, \cdots, m_{2^{k+1}}$, $m_{2^{k+1}} \asymp 2^{(d-1)k}$ the same as in Theorem 1.2.5. For simplicity, we write $b_k = m_{2^{k+1}}$. According to Theorem 1.2.5, the mapping U_k is one to one (not onto). By Lemma 1.4.3, there is a linear operator $\widetilde{U}_k^{-1} : \ell_\infty^{b_k} \longrightarrow L_\infty$, such that

$$\widetilde{U}_k^{-1} U_k = I_k,$$

$$\|\widetilde{U}_k^{-1}\|_{(\ell_\infty^{b_k}, L_\infty)} \|U_k\|_{(\Pi_{2^{k+1}} \cap L_\infty, \ell_\infty)} = 1,$$

where I_k denotes the identity operator on $\Pi_{2^{k+1}}(\mathbb{S}^{d-1})$.

Next, we define a sequence of linear operators $\{T_k\}_{k=0}^\infty$ as follows.

$$T_0 f = V_1(f),$$

$$T_k f = V_{2^k} f - V_{2^{k-1}} f, \ k = 1, 2, \cdots,$$

with the operator V_j as in Lemma 1.4.2. Then, obviously, $f = \sum_{k=0}^{\infty} T_k f$.

Finally, we use Theorem 1.2.5, Lemmas 1.4.1– 1.4.3 to get

$$d_N(B_p^r, L^\infty) \le C \sum_{k=0}^{\infty} 2^{-(d-1)k(\frac{r}{d-1} - \frac{1}{p})} d_{n_k}(b_{2,w_k}, \ell_\infty^{b_k}), \tag{1.4.1}$$

where the n_k's are non-negative integers for which $\sum_{k=0}^{\infty} n_k \le N$.

Now set $a = 2^{d-1}$, suppose $N \sim a^m$, and let

$$n_k = \begin{cases} b_k \asymp a^k, & 0 \le k \le m, \\ a^{2m-k}, & m < k < 2m, \\ 0, & k > 2m. \end{cases}$$

Then applying Theorem 1.3.1, and by an easy computation, we know the right-hand side of (1.4.1) is dominated by

$$C_d a^{-m(\frac{r}{d-1} - \frac{1}{p} + \frac{1}{2})},$$

which gives the required upper estimate and completes the proof. □

Remark 1.4.1. We remark that the method used in this chapter may be useful for some other problems. As an example, let us consider the classes $W^{\mathbf{r}}_{p,\alpha}$ of functions $f \in L(\mathrm{T}^d)$ with bounded mixed derivative. More precisely, let $W^{\mathbf{r}}_{p,\alpha}$ be the class of functions representable in the form

$$f(x) =: \varphi * F_{\mathbf{r}}(x,\alpha) = (2\pi)^d \int_{\mathrm{T}^d} \varphi(y) F_{\mathbf{r}}(x-y,\alpha)\, dy,$$

where $\varphi \in L^p(\mathrm{T}^d)$, $\|\varphi\|_p \le 1$,

$$\mathbf{r} = (r_1, \cdots, r_d), \quad \alpha = (\alpha_1, \cdots, \alpha_d),$$

$$0 < r = r_1 = \cdots r_v < r_{v+1} \le \cdots \le r_d,$$

and

$$F_{\mathbf{r}}(x,\alpha) = 2^d \sum_{k \in \mathbb{Z}^d_{>0}} \prod_{j=1}^{d} k_j^{-r_j} \cos(k_j x_j - \frac{\alpha_j \pi}{2}).$$

Note that the problem of finding the orders of the widths $d_M(W_{p,\alpha}, L^\infty)$ for $1 \le p \le 2$, $v \ge 2$ and $d \ge 3$ remains open.

Using the Marcinkiewicz-Zygmund inequality for $q = \infty$ and slightly modifying the proof in [Tem, Chapter III], one can obtain the following results.

For $1 < p \le 2$,

$$C_1 M^{-r+\frac{1}{p}-\frac{1}{2}} \Big(\log M\Big)^{(v-1)(r-\frac{1}{p}+\frac{1}{2})} \le d_M(W^{\mathbf{r}}_{p,\alpha}, L^\infty) \le C_2 M^{-r+\frac{1}{p}-\frac{1}{2}} \Big(\log M\Big)^{(v-1)(r-\frac{1}{p}+1)},$$

$$C_1 M^{-r+\frac{1}{2}} \Big(\log M\Big)^{(v-1)r} \le d_M(W^{\mathbf{r}}_{1,\alpha}, L^\infty) \le C_2 M^{-r+\frac{1}{2}} \Big(\log M\Big)^{(v-1)(r+\frac{1}{2})}.$$

Remark 1.4.2. We remark that except for some special values of parameters $p \le q$, which will be considered in the next chapter, the orders of the linear widths $\delta_n(B^r_p, L^q)$ are still unknown.

1.5 Widths of the function classes with bounded mixed derivative

Let $\mathbb{S}^{d-1,n} = \mathbb{S}^{d-1} \times \cdots \times \mathbb{S}^{d-1}$ (n times) denote the product space equipped with the usual Lebesgue measure $d\sigma(x) = d\sigma(x_1)\cdots d\sigma(x_n)$. For $k = (k_1, \cdots, k_n) \in \mathbb{Z}_+^n$, we define the space $\mathbf{H}_k^{d,n}$ of product spherical harmonics by

$$\mathbf{H}_k^{d,n} = \operatorname{span}\Big\{\varphi_{k,\ell}(\xi) = \prod_{j=1}^n \varphi_{k_j,\ell_j}(\xi_j): \quad 1 \le \ell_j \le a_{k_j}^d = \dim \mathcal{H}_{k_j}^d,\ 1 \le j \le n\},$$

where $\xi_j \in \mathbb{S}^{d-1}$ and $\{\varphi_{k_j,\ell_j}\}_{\ell_j=1}^{a_{k_j}^d}$ denotes the orthonormal system of functions in the space $\mathcal{H}_{k_j}^d$ of spherical harmonics of degree k_j, $j = 1, \cdots, n$.

Obviously, $f \in \mathbf{H}_k^{d,n}$ if and only if $f(\xi_1, \cdots, \xi_n)$ is a spherical harmonic of degree k_j in each variable $\xi_j \in \mathbb{S}^{d-1}$, $j = 1, \cdots, n$, the spaces $\{\mathbf{H}_k^{d,n}\}_{k\in\mathbb{Z}_+^n}$ are mutually orthogonal with respect to the inner product

$$< f, g >= \int_{\mathbb{S}^{d-1,n}} f(x_1, \cdots, x_n)\overline{g(x_1, \cdots, x_n)}\, d\sigma(x_1)\cdots d\sigma(x_n), \quad f, g \in L^2(\mathbb{S}^{d-1,n}).$$

Furthermore,

$$L^2(\mathbb{S}^{d-1,n}) = \bigoplus_{k\in\mathbb{Z}_+^n} \mathbf{H}_k^{d,n}.$$

For $f \in L^2(\mathbb{S}^{d-1,n})$, let

$$\sigma(f) = \sum_{k\in\mathbb{Z}_+^n} \mathbf{Y}_k(f)$$

denote its expansion with respect to the mutually orthogonal spaces $\{\mathbf{H}_k^{d,n}\}_{k\in\mathbb{Z}_+^n}$, where $\mathbf{Y}_k(f)$ is the orthogonal projection of $f \in L^2(\mathbb{S}^{d-1,n})$ onto the space $\mathbf{H}_k^{d,n}$. By the addition formula (0.0.5) for spherical harmonics, we have

$$\mathbf{Y}_k(f)(x_1, \cdots, x_n) = \int_{\mathbb{S}^{d-1,n}} f(y_1, \cdots, y_n) \prod_{j=1}^n c_{k_j}^d \Pi_{k_j}^d(x_j y_j)\, d\sigma(y), \tag{1.5.1}$$

where $k = (k_1, \cdots, k_n)$, $f \in L^2(\mathbb{S}^{d-1,n})$ and $x_j \in \mathbb{S}^{d-1}$, $j = 1, \cdots, n$. Clearly, the expression (1.5.1) gives the definition of $\mathbf{Y}_k(f)$ for all $f \in L(\mathbb{S}^{d-1,n})$.

Denote by $B_p^{\mathbf{r}}$ the class of functions representable in the form

$$f(x_1,\cdots,x_n)=\int_{\mathbb{S}^{d-1,n}}\varphi(y_1,\cdots,y_n)F_{\mathbf{r}}(x_1y_1,\cdots,x_ny_n)\,d\sigma(y_1),\cdots,d\sigma(y_n),\quad \|\varphi\|_p\le 1,$$

where $\mathbf{r}=(r_1,\cdots,r_n)$,

$$F_{\mathbf{r}}(x_1y_1,\cdots,x_ny_n)=\sum_{k\in\mathbb{Z}^n_{>0}}\prod_{j=1}^n(k_j(k_j+2\lambda))^{-\frac{r_j}{2}}c_{k_j}^dP_{k_j}^d(x_jy_j),$$

and $x_j,y_j\in\mathbb{S}^{d-1}$, $j=1,\cdots,n$.

As in the case $n=1$, we write $g=f^{(\mathbf{r})}$ if

$$\sigma(g)=\sum_{k\in\mathbb{Z}^n_+}\Big(\prod_{j=1}^n(k_j(k_j+2\lambda))^{\frac{r_j}{2}}\Big)\mathbf{Y}_k(f).$$

Also, we write

$$W_p^{\mathbf{r}}=\Big\{f\in L^p(\mathbb{S}^{d-1,n}):\quad f^{(\mathbf{r})}\in L^p(\mathbb{S}^{d-1,n})\Big\}.$$

Then, clearly, the class $B_p^{\mathbf{r}}$ can be rewritten as

$$B_p^{\mathbf{r}}=\Big\{f\in W_p^{\mathbf{r}}:\quad \|f^{(\mathbf{r})}\|_p\le 1\Big\}.$$

Throughout the rest of this section, we set $\mathbf{r}=(r_1,\cdots,r_n)$, $\gamma=\frac{\mathbf{r}}{r}$, with

$$0<r=r_1=\cdots=r_v<r_{v+1}\le\cdots\le r_n,$$

The following two lemmas are simple consequences of the corresponding results for $\mathbb{S}^{d-1}$.

Lemma 1.5.1. *Let $T_{N_1,\cdots,N_n}$ denote a polynomial of degree N_j in each variable $x_j\in\mathbb{S}^{d-1}$, $j=1,\cdots,n$. Then*

$$\|T^{(\mathbf{r})}_{N_1,\cdots,N_n}\|_p\le C\Big(\prod_{j=1}^n N_j^{r_j}\Big)\|T_{N_1,\cdots N_n}\|_p,\quad 1\le p\le\infty,$$

$$\|T_{N_1,\cdots N_n}\|_q\le C\Big(\prod_{j=1}^n N_j\Big)^{(d-1)(\frac{1}{p}-\frac{1}{q})}\|T_{N_1,\cdots N_n}\|_p,\quad 1\le p<q\le\infty.$$

Lemma 1.5.2. *Let $t_{m_1,\cdots,m_n}(x_1,\cdots,x_n)$ be a polynomial of degree m_j, $j=1,\cdots,n$, in each variable x_j. Then for $1\le p\le\infty$,*

$$\|t_{m_1,\cdots,m_n}\|_p$$
$$\asymp\Big(\big(\prod_{j=1}^n m_j\big)^{-(d-1)}\sum_{k_1=1}^{(4m_1+1)(2m_1)^{d-2}}\cdots\sum_{k_n=1}^{(4m_n+1)(2m_n)^{d-2}}\Big(\prod_{j=1}^n w_{m_j,k_j}\Big)\Big|t(\xi_{m_1,k_1},\cdots,\xi_{m_n,k_n})\Big|^p\Big)^{\frac1p},$$

with $w_{i,j}$, $\xi_{i,j}$ the same as in Theorem 1.2.5.

Obviously, the weights appearing in Lemma 1.5.2 also have the Kashin type property. Consequently, the weighted Kashin type inequality (Theorem 1.3.1) applies to these weights.

Let $\eta\in C^\infty(\mathbb{R})$ such that supp $\eta\subset[-1,1]$ and $\eta(t)=1$ if $|t|\le\frac12$. Let $\theta_0(t)=\eta(t)$ and $\theta_j(t)=\theta(\frac{t}{2^j})$, $j=1,2,\cdots$, with $\theta(t)=\eta(t)-\eta(2t)$. Then clearly $\sum_{j=0}^\infty\theta_j(t)=1$. For $s=(s_1,\cdots,s_n)\in\mathbb{Z}_+^n$, we define

$$\theta_s(t)=\prod_{j=1}^n\theta_{s_j}(t_j),\quad t=(t_1,\cdots,t_n)$$

and

$$\theta_s(f)=\sum_{k\in\mathbb{Z}_+^n}\theta_s(k)\mathbf{Y}_k(f),\quad f\in L(\mathbb{S}^{d-1,n}).\tag{1.5.2}$$

Lemma 1.5.3. *Let $1<p<\infty$ and $f\in L^p(\mathbb{S}^{d-1,n})$. Then, with the same notations as the above,*

$$\|f\|_p\asymp\Big\|\Big(\sum_{s\in\mathbb{Z}_+^n}|\theta_s(f)|^2\Big)^{\frac12}\Big\|_p.$$

The proof of this lemma runs along the same lines as that of the Littlewood-Paley theorem for the multiple Fourier integrals, (see [St P104–108]). For the sake of completeness, we write the proof here.

Proof. For convenience, we only consider the case $n=2$. For $t_1\in[0,1]$, we define

$$T_{t_1}^1f(x,y)=\sum_{k,\ell=0}^\infty\Big(\sum_{j=0}^\infty\theta_j(k)R_j(t_1)\Big)\mathbf{Y}_{k,\ell}(f)(x,y),$$

where $x, y \in \mathbb{S}^{d-1}$ and $\{R_k(t)\}$ is the system of Rademacher functions on $[0,1]$. An easy computation shows

$$\Big|\Big(\frac{d}{du}\Big)^{\ell}\Big(\sum_{j=0}^{\infty}\theta_j(u)R_j(t_1)\Big)\Big| \le Cu^{-\ell}, \quad \ell = 0, 1, \cdots, \quad u \ge 1.$$

Notice that

$$T_{t_1}^1(f)(x,y) = \sum_{k=0}^{\infty}\Big(\sum_{j=0}^{\infty}\theta_j(k)R_j(t_1)\Big)Y_k(f(\cdot,y))(x).$$

We get by applying Hörmander's multiplier theorem for the spherical harmonics (see [Str]) that for a.e. $y \in \mathbb{S}^{d-1}$,

$$\int_{\mathbb{S}^{d-1}} |T_{t_1}^1 f(x,y)|^p \, d\sigma(x) \le C \int_{\mathbb{S}^{d-1}} |f(x,y)|^p \, d\sigma(x), \quad 1 < p < \infty.$$

Integrating the above inequality with respect to $y \in \mathbb{S}^{d-1}$ yields

$$\|T_{t_1}^1 f\|_p \le C\|f\|_p, \quad 1 < p < \infty.$$

The same inequality of course holds with x replaced by y.

For $t = (t_1, t_2)$, we define

$$T_t(f)(x,y) = \sum_{i,j=0}^{\infty}\sum_{k,\ell=0}^{\infty}\theta_i(k)\theta_j(\ell)r_i(t_1)r_j(t_2)\mathbf{Y}_{k,\ell}(f)(x,y).$$

Noticing that $T_t = T_{t_1}T_{t_2}$, we get

$$\sup_t \|T_t(f)\|_p \le C\|f\|_p, \quad 1 < p < \infty.$$

Now the inequality

$$\Big\|\Big(\sum_{i,j=0}^{\infty}|\theta_{(i,j)}(f)|^2\Big)^{\frac{1}{2}}\Big\|_p \le C_p\|f\|_p \tag{1.5.3}$$

is an immediate consequence of the standard property of Rademacher functions.

The converse inequality can be derived from (1.5.3) by duality, due to the fact

$$\sum_{j,k=0}^{\infty}\chi_{[2^{j-1},2^{j+1}]\times[2^{k-1},2^{k+1}]}(x,y) \le c < \infty.$$

This completes the proof. □

Theorem 1.5.4. *Let* $1 < p \le 2 \le q < \infty$ *and* $r > 2(d-1)^2$. *Then*

$$d_M(B_p^{\mathbf{r}}, L^q) \asymp \Big(\frac{\log^{v-1} M}{M}\Big)^{\frac{r}{d-1}+\frac{1}{p}-\frac{1}{2}}.$$

Proof. We follow the proof in [Tem]. We start by proving the upper estimate. First, note that, the class $B_p^{\mathbf{r}}$ is imbedded in the class $B_2^{\widetilde{\mathbf{r}}}$, with $\widetilde{r}_j = r_j - (d-1)(\frac{1}{p}-\frac{1}{2})$, $j = 1, \cdots, n$, which is a simple consequence of the corresponding result for $\mathbb{S}^{d-1}$, (See [Col]). Therefore, it suffices to prove the estimate for $p = 2$.

Let $\psi \in C^\infty(\mathbb{R})$ such that supp $\psi \subset [\frac{1}{8}, 4]$ and $\psi(x) = 1$ if $\frac{1}{4} \le x \le 2$. Then by Lemma 1.5.3 and a simple duality argument, one can verify that for any sequence $\{g_s\}_{s\in\mathbb{Z}_+^n}$ of functions in $L^q(\mathbb{S}^{d-1,n})$,

$$\Big(\sum_{s\in\mathbb{Z}_+^n} \|\theta_s(f) - g_s\|_q^2\Big)^{\frac{1}{2}} \ge C\|f - \sum_{s\in\mathbb{Z}_+^n} \psi_s(g_s)\|_q, \quad 2 \le q < \infty,$$

where $\psi_s(g_s)$ is defined similarly as in (1.5.2). This implies

$$d_M^2(B_2^{\mathbf{r}}, L^q) \le \sum_{s\in\mathbb{Z}_+^n} d_{M_s}^2(\theta_s, W_2^{\mathbf{r}}, L^q), \tag{1.5.4}$$

with

$$\sum_{s\in\mathbb{Z}_+^n} M_s \le M.$$

Set $a = 2^{d-1}$, $\gamma_j' = \frac{1+\sigma\gamma_j}{1+\sigma}$, $\gamma' = (\gamma_1', \cdots, \gamma_n')$ and let $\kappa = 1 + \frac{1}{\sigma}$, with $\sigma > 0$ a constant such that $r > \frac{(1+\sigma)(d-1)^2}{2}$. Suppose $M \sim a^m m^{v-1}$.

Let

$$M_s = \begin{cases} \dim \bigoplus\limits_{\substack{2^{s_j-1}\le k_j\le 2^{s_j+1}\\ 1\le j\le n}} \mathbf{H}_k^{d,n} \asymp a^{\|s\|_1}, & \text{for } s\gamma' \le m, \\ a^{m-\sigma(s\gamma-m)}, & \text{for } m < s\gamma' \le s\gamma \le \kappa m, \\ 0, & \text{for } s\gamma > \kappa m. \end{cases}$$

A straightforward computation shows

$$\sum_{s\in\mathbb{Z}_+^n} M_s \le C a^m m^{v-1} \asymp M.$$

Thus, by (1.5.4),

$$\begin{aligned} d_M^2(B_2^r, L^q) \le &\sum_{m\le s\gamma'\le s\gamma\le\kappa m} d_{M_s}^2(\theta_s, W_2^r, L^q) + \sum_{s\gamma>\kappa m} \|\theta_s\|^2_{(W_2^r, L^q)} \\ =:&\sigma_1+\sigma_2. \end{aligned} \tag{1.5.5}$$

For $m\le s\gamma'\le\gamma s<\kappa m$, we factor the operator $\theta_s:\ W_2^{\mathbf{r}}\longrightarrow L^q$ as

$$\theta_s:\, W_2^{\mathbf{r}} \xrightarrow{\theta_s} \bigoplus_{2^{s_j-1}\le k_j\le 2^{s_j+1}} \mathbf{H}_k^{d,n}\bigcap L_2 \xrightarrow{U_k} \ell_{2,w_k}^{b_k} \xrightarrow{I} \ell_\infty^{b_k}$$
$$\xrightarrow{\widetilde{U_k^{-1}}} \qquad L_\infty \qquad \xrightarrow{I} L^q.$$

We then get, by Lemma 1.5.2 and Theorem 1.3.1 ,

$$d_{M_s}(\theta_s, W_2^r, L^q)\le Ca^{-\frac{r\gamma s}{d-1}+\frac{\|s\|_1}{2}}\|\delta_s\|_{(2,2)}M_s^{-\frac12}\Big(\frac{a^{\|s\|_1}}{M_s}\Big)^{\frac{d-2}{2}}(1+\log\frac{a^{\|s\|_1}}{M_s}\Big)^{\frac{1}{d-1}},$$

where

$$\delta_s(f)=\sum_{\substack{2^{s_j-1}\le k_j\le 2^{s_j+1}\\ j=1,\cdots,d}}\mathbf{Y}_k(f).$$

Thus

$$\begin{aligned}\sigma_1\le& a^{-(1+\sigma)(d-1)m}\sum_{m\le s\gamma'\le s\gamma\le\kappa m} a^{(1+\sigma)(d-1)\gamma' s-\frac{2r\gamma s}{d-1}}(s\gamma-m+1)^{d-1}\|\delta_s\|^2_{(2,2)}\\ \le& Ca^{-(1+\sigma)(d-1)m}\sum_{\ell=m}^{\kappa m}\max_{\ell\le s\gamma\le\ell+1}a^{(1+\sigma)(d-1)s\gamma'}a^{-\frac{2r\ell}{d-1}}(\ell+1-m)^{d-1},\end{aligned}$$

which, after an easy computation, is controlled by

$$Ca^{-m\frac{2r}{d-1}}\asymp M^{-\frac{2r}{d-1}}\Big(\log M\Big)^{(v-1)\frac{2r}{d-1}}. \tag{1.5.6}$$

For $s\gamma>\kappa m$, by Nikolskii inequality, we have

$$\|\theta_s(f)\|_q\le Ca^{\|s\|_1(\frac12-\frac1q)}2^{-rs\gamma}\|\delta_s\|_{(2,2)}\|f^{(\mathbf{r})}\|_2. \tag{1.5.7}$$

Substituting (1.5.7) into (1.5.5), we conclude that σ_2 is also dominated by (1.5.6). This, together with (1.5.6), gives the required upper estimate for $p=2$.

Next, we turn to the proof of the lower estimate. Without loss of generality, we may assume $v = n$, since $B_p^{(r_1,\cdots,r_v)}(S^{d,v})$ can be regarded as a subspace of $B_p^{(r_1,\cdots,r_n)}(\mathbb{S}^{d-1,n})$. For $s = (s_1, \cdots, s_n) \in \mathbb{Z}_+^n$, write

$$\rho(s) = \Big\{(k,\ell) : \quad 2^{s_j-1} \le k_j < 2^{s_j}, \quad 1 \le \ell_j \le \dim \mathcal{H}_{k_j}^d, \quad j = 1, \cdots, n\Big\}.$$

Let

$$\mu(m) = \bigcup_{\|s\|_1 \le m} \rho(s)$$

and

$$P_m = \Big\{ \sum_{(k,\ell)\in\mu(m)} c_{k,\ell}\varphi_{k,\ell} : \quad c_{k,\ell} \in \mathbb{R}, \quad (k,\ell) \in \mu(m)\Big\}.$$

Set $M = [\frac{|\mu(m)|}{2}]$. Then

$$|\mu(m)| = \dim P_m \asymp 2^{(d-1)m} m^{n-1}.$$

Let $\{\psi_j\}_{j=1}^{|\mu(m)|}$ be an orthonormal system of functions in P_m and $S_M(f)$ denote the M-th Fourier sum of $f \in P_m$ with respect to $\{\psi_j\}$.

Suppose

$$\varphi_{k,\ell}(x) = \sum_{j=1}^{|\mu(m)|} a_{k,\ell}^j \psi_j(x).$$

Write

$$R_{k,\ell}^2 = \|\varphi_{k,\ell} - S_M(\varphi_{k,\ell})\|_2^2 = \sum_{j=M+1}^{|\mu(m)|} |a_{k,\ell}^j|^2.$$

Then, as the standard method shows (see [Tem]), for some $\rho(s) \subset \mu(m)$,

$$\sum_{(k,\ell)\in\rho(s)} R_{k,\ell}^2 \ge \frac{1}{2}|\rho(s)|. \tag{1.5.8}$$

Given a constant $a \in (0,1)$, let $\eta_a \in C^\infty(\mathbb{R})$ such that $0 \le \eta_a(x) \le 1$, supp $\eta_a \subset [\frac{1}{4}, 1]$ and $\eta_a(x) = 1$ if $\frac{1}{2} \le x \le a$.

Define

$$g(x,y) = \sum_{(k,\ell)\in\mu(m)} \prod_{j=1}^{n} \eta_a(\frac{k_j}{2^{s_j}})\varphi_{k,\ell}(x)\varphi_{k,\ell}(y),$$

where $s=(s_1,\cdots,s_n)$ satisfies (1.5.8), $x,y\in\mathbb{S}^{d-1,n}$.

According to the addition formula, one can easily verify by invoking the Abelian transform finite times,

$$\sup_{y\in\mathbb{S}^{d-1,n}}\|g^{(\mathbf{r})}(\cdot,y)\|_p\le c(a)2^{\|s\|_1(r+(d-1)(1-\frac{1}{p}))}.$$

Write

$$R_M(x,y)=g(x,y)-S_M(g(\cdot,y))(x).$$

Then

$$\|R_M(\cdot,y)\|_2^2=\sum_{j=M+1}^{|\mu(m)|}\Big|\sum_{(k,\ell)\in\mu(m)}\Big(\prod_{i=1}^n\theta_a(\frac{k_i}{2^{s_i}}\Big)a_{k,\ell}^j\varphi_{k,\ell}(y)\Big|^2.$$

Integrating the above identity with respect to $y\in\mathbb{S}^{d-1,n}$, we get

$$\begin{aligned}\int\|R_M(\cdot,y)\|_2^2\,dy\ge&\sum_{(k,\ell)\in\rho(s)}\prod_{i=1}^n\eta_a^2(\frac{k_i}{2^{s_i}})R_{k,\ell}^2\\ \ge&\frac{1}{2}|\rho(s)|-\sum_{(k,\ell)\in\rho(s)}\Big(1-\prod_{i=1}^n\eta_a^2(\frac{k_i}{2^{s_i}})\Big)R_{k,\ell}^2.\end{aligned}\tag{1.5.9}$$

Note that $0\le R_{k,\ell}\le 1$ and $\eta_a(x)=1$ whenever $\frac{1}{2}\le x\le a<1$. We get

$$\sum_{(k,\ell)\in\rho(s)}\Big(1-\prod_{i=1}^n\eta_a^2(\frac{k_i}{2^{s_i}})\Big)R_{k,\ell}^2\le C(1-a)|\rho(s)|.\tag{1.5.10}$$

Substituting (1.5.10) into (1.5.9), and letting $a=1-\frac{1}{4C}$, we conclude

$$\int\|R_M(\cdot,y)\|_2^2\,dy\ge\frac{1}{4}|\rho(s)|\asymp 2^{(d-1)\|s\|_1}.$$

Therefore, there exists an element $y^*\in\mathbb{S}^{d-1,n}$ such that

$$\|R_M(\cdot,y^*)\|_2^2\ge c2^{(d-1)\|s\|_1}.$$

Consequently,

$$\begin{aligned}d_M(B_p^{\mathbf{r}},L^q)\ge&d_M(B_p^{\mathbf{r}}\bigcap P_m,L^2)=d_M(B_p^{\mathbf{r}}\bigcap P_m,L^2\bigcap P_m)\\ \ge&C2^{-(d-1)m(\frac{r}{d-1}-\frac{1}{p}+\frac{1}{2})}\\ \asymp&M^{-(\frac{r}{d-1}-\frac{1}{p}+\frac{1}{2})}(\log M)^{(n-1)(\frac{r}{d-1}-\frac{1}{p}+\frac{1}{2})},\end{aligned}$$

which is the required lower estimate and completes the proof. □

Remark 1.5.1. By slightly modifying the above proof, following the technique in [Tem], we can further obtain the orders of the widths $d_M(B_p^{\mathbf{r}}, L^q)$ for $p = 1,\ 2 \leq q < \infty$, and for $1 < p < q \leq 2$. We omit the detail here.

Chapter 2

Linear widths of the function class B_p^r in the space $L^q(\mathbb{S}^{d-1})$, $1 \le p \le 2 \le q \le \infty$

For $1 \le p, q \le \infty$, we define *the linear spherical harmonic width* of B_p^r in the metric L^q by

$$\delta_m^{SH}(B_p^r, L^q) := \inf_{T \in G_m} \sup_{f \in B_p^r} \|f - T(f)\|_q,$$

where

$$G_m = \Big\{T : T \text{ is continuous, linear on } L^q(\mathbb{S}^{d-1}) \text{ and there is}$$
$$\text{a vector space } L \in \mathcal{L}_m \text{ for which } T(L^q) \subset L\Big\},$$

and

$$\mathcal{L}_m := \Big\{\Big(\bigoplus_{k \in A} \mathcal{H}_k^d\Big) : \quad A \text{ is a finite set of non-negative}$$
$$\text{integers for which } \dim\Big(\bigoplus_{k \in A} \mathcal{H}_k^d\Big) \le m\Big\}.$$

The main purpose in this chapter is to investigate the orders of the widths $\delta_m^{SH}(B_p^r, L^q)$. In the case $d = 2$, $\delta_m^{SH}(B_p^r, L^q)$ returns to the well known linear trigonometric width $\delta_m^T(B_p^r, L^q)$ of the Sobolev class on the circle, for which the orders for all $1 \le p \le q \le \infty$ are known, due to the work of Makovoz [Mk, 1984], Maiorov [Mr2, 1986] and Belinskii [Be3,1987].

However, in the case $d \geq 3$, very few investigations have been made on this problem. To the best of our knowledge, so far the orders of the widths $\delta_m^{SH}(B_p^r, L^q)$ for all $1 \leq p \leq 2 \leq q \leq \infty$ have been left unconsidered. The main difficulty in this case lies in the fact that the spaces $\mathcal{H}_k^d$, $k = 0, 1, \cdots$, of the spherical harmonics are much more complicated than the usual trigonometric spaces

$$\mathcal{H}_k^2 = \text{span}\{e^{ikx}, e^{-ikx}\}, \quad k = 0, 1, 2, \cdots$$

In this chapter, we shall prove

Theorem 2.0.5.

$$\delta_m^{SH}(B_p^r, L^q) \asymp \begin{cases} m^{-\frac{r}{d-1}+\frac{1}{2}}, & \textit{for } p = 1, \quad 2 \leq q \leq \infty, \quad r > \frac{d(d-1)}{2}, \\ m^{-\frac{r}{d-1}+\frac{1}{2}}, & \textit{for } 1 \leq p \leq 2, \quad q = \infty, \quad r > \frac{d(d-1)}{2}, \\ m^{-\frac{r}{d-1}+\frac{1}{p}-\frac{1}{2}}, & \textit{for } 1 \leq p \leq 2 \leq q < \frac{2(d-1)p}{dp-2}, \quad r > d-1, \\ m^{-\frac{r}{d-1}+\frac{1}{2}-\frac{1}{q}}, & \textit{for } \frac{2(d-1)q}{dq-2} < p \leq 2 \leq q \leq \infty, \quad r > d-1. \end{cases}$$

We remark that for values of the parameters $p \leq 2 \leq q$ not encompassed by Theorem 2.0.5, the problem on the orders of the corresponding widths remains open.

We point out that as will be shown in the later proof, the probabilistic method, which was used in [Mk] and [Be3] to deal with the problems of the trigonometric widths , does not apply to cope with $\delta_m^{SH}(B_p^r, L^q)$ for $d \geq 3$ and large $q \geq 2$.

The organization of this chapter is as follows. In Section 2.1, we prove an inequality about ultraspherical polynomials, which generalizes a corresponding inequality for trigonometric polynomials due to [Mk] and [Be2]. Such an inequality plays a crucial role in our later proof of Theorem 2.0.5 and will be proved by approximating the subject step by step, which differs from the usual probabilistic method used in the case $d = 2$. In Section 2.2, a Bohr- Favard type inequality for the kernel $F_{r,\mu}(\cos\theta)$ is established. Theorem 2.0.5 is then proved in Section 2.3 by standard method. In Section 2.4 we discuss some further extreme problems in a general background.

We thank Professor Belinskii E. S. for supplying us with some copies of his excellent papers on trigonometric widths, which gave us a better perspective on our own results.

2.1 Approximation of an algebraic polynomial by a polynomial from the spaces spanned by a prescribed number of ultraspherical polynomials

In this section, we prove the following theorem, which plays a crucial role in the proofs of the results in later sections.

Theorem 2.1.1. *Suppose* $0 < \mu < \infty$ *and* $T_N(t) = \sum_{k=0}^{N} c_k C_k^\mu(t)$ *is an algebraic polynomial of degree not exceeding* N. *Then for* $2 \le p \le \infty$ *and* $1 \le M \le N$, *there exists a polynomial representable in the form*

$$T_{\theta_M}(t) = \sum_{k \in \theta_M} b_k C_k^\mu(t)$$

with $\theta_M \subset [0, 2N]$, $|\theta_M| = M$, *such that*

$$\|T_N(t) - T_{\theta_M}(t)\|_p \le C(p,\mu) \max\Big\{ \Big(\frac{N}{M}\Big)^{1/2}, \Big(\frac{N}{M}\Big)^{(1-\frac{2}{p})(\mu+1/2)} \Big(\log(1+\frac{N}{M})\Big)^{\mu(1-\frac{2}{p})} \Big\} \|T_N(t)\|_2,$$

here the norms are computed with respect to the measure $(1-t^2)^{\mu-1/2}\,dt$.

In the case $\mu = 0$, $\{C_k^\mu\}_{k=0}^\infty$ is the system of normalized Tchebicheff polynomials, which after a change of variable reduces to the trigonometric system on the unit circle. In this case Theorem 2.1.1 was established in [Be3] and [Mk] for $2 \le p < \infty$, and in [Be1] for $p = \infty$, by probabilistic method. We remark that the probabilistic method relies essentially on the following property of the orthonormal system $\{e^{ik\theta}\}$:

$$\sup_n \|e^{in\theta}\|_p = 1 < \infty, \quad \text{for } 2 \le p \le \infty.$$

However, in the case $\mu > 0$, as is well known, (see [Sz]), the inequality

$$\sup_k \frac{\|C_k^\mu\|_p}{\|C_k^\mu\|_2} \le C < \infty$$

is valid only for $2 \leq p < 2+\frac{1}{\mu}$. Consequently, the probabilistic method fails in coping with this problem for $2+\frac{1}{\mu} \leq p \leq \infty$, $\mu > 0$.

To overcome such a difficulty, we will prove the inequality step by step. We construct the desired polynomial T_{θ_M} by finite steps. In each step, a new polynomial will be produced, which approximates the polynomial T_N better than the polynomials produced in the previous steps. Then after a finite number of steps, we can select a polynomial with the required approximation property.

We shall use the following notations:

$$T_{\theta_N}(t) = \sum_{k\in\theta_N} b_k C_k^{(\mu)}(t), \quad \text{with } \theta_N \subset \mathbb{Z}_+ \text{ and } |\theta_N| = N,$$

$$dm_\mu(\theta) = \sin^{2\mu}\theta\, d\theta,$$

$$\|f\|_p = \Big(\int_{-1}^{1} |f(x)|^p (1-x^2)^{\mu-1/2}\, dx\Big)^{\frac{1}{p}},$$

$$\|f(\cos\theta)\|_{L^p(dm_\mu(\theta))} = \Big(\int_0^\pi |f(\cos\theta)|^p \sin^{2\mu}\theta\, d\theta\Big)^{\frac{1}{p}}.$$

The proof of Theorem 2.1.1 is based on the following lemmas.

Lemma 2.1.2. *Let $T_{\theta_N}(t) = \sum_{k\in\theta_N} b_k C_k^{(\mu)}(t)$. Then for every $2 \leq p < 2+\frac{1}{\mu}$ and $1 \leq M \leq N$, there exists a polynomial T_{θ_M} with $\theta_M \subset \theta_N$, such that*

$$\|T_{\theta_N}(\cos\theta) - T_{\theta_M}(\cos\theta)\|_{L^p(dm_\mu)} \leq C(\mu)\Big(\frac{1}{1-\mu(p-2)}\Big)^{\frac{1}{p}} \Big(\frac{N}{M}\Big)^{1/2} \|T_{\theta_N}(\cos\theta)\|_{L^2(dm_\mu)},$$

with $C(\mu) > 0$ a constant independent of T_{θ_N}, θ_M and p.

Proof. We follow the idea from [Be1] and [Bo]. Suppose $\frac{N}{M} \sim 2^m$. Let us write

$$\begin{aligned} T_{\theta_N}(\cos\theta) &= \sum_{k\in\theta_N} c_k C_k^{(\mu)}(\cos\theta) \\ &= \sum_{j=1}^{m} \sum_{k\in\theta_N} c_k (1-\varepsilon_k^1)\cdots(1-\varepsilon_k^{j-1})\varepsilon_k^j C_k^{(\mu)}(\cos\theta) \\ &\quad + \sum_{k\in\theta_N} c_k (1-\varepsilon_k^1)\cdots(1-\varepsilon_k^m) C_k^{(\mu)}(\cos\theta) \\ &=: \Phi(\theta,\epsilon) + T_{\theta_{M(\epsilon)}}(\cos\theta), \end{aligned} \tag{2.1.1}$$

where

$$\theta_{M(\epsilon)} = \{k \in \theta_N : \, (1-\varepsilon_k^1)\cdots(1-\varepsilon_k^m) \neq 0\}$$

and $\{\varepsilon_k^j\}_{k\in\theta_N,\ 1\le j\le m}$ are independent ± 1–valued random variables satisfying

$$\int \varepsilon_k^j \, d\epsilon^j = 0, \ k \in \theta_N, \ j = 1, \cdots, m.$$

Then

$$\begin{aligned}
&\int \|\Phi(\theta,\, \epsilon)\|_{L^p(dm_\mu(\theta))} \, d\epsilon \\
&\le \sum_{\ell=1}^m \int \Big\| \sum_{k\in\theta_N} c_k (1-\varepsilon_k^1)\cdots(1-\varepsilon_k^{\ell-1})\varepsilon_k^\ell C_k^{(\mu)}(\cos\theta) \Big\|_{L^p(dm_\mu(\theta))} d\epsilon^1 \cdots d\epsilon^\ell \\
&\le \sum_{\ell=1}^m \int \Big\| \sum_{k\in\theta_N} c_k (1-\varepsilon_k^1)\cdots(1-\varepsilon_k^{\ell-1})\varepsilon_k^\ell C_k^{(\mu)}(\cos\theta) \Big\|_{L^p(dm_\mu(\theta)\bigotimes d\epsilon^\ell)} d\epsilon^1 \cdots d\epsilon^{\ell-1} \\
&\le C\sqrt{p} \sum_{\ell=1}^m \int \Big\| \Big(\sum_{k\in\theta_N} |c_k|^2 (1-\varepsilon_k^1)^2 \cdots (1-\varepsilon_k^{\ell-1})^2 |C_k^{(\mu)}(\cos\theta)|^2 \Big)^{1/2} \Big\|_{L^p(dm_\mu(\theta))} d\epsilon^1 \cdots d\epsilon^{\ell-1},
\end{aligned}$$

the last inequality follows from Khinchine's inequality.

The estimate (0.0.26) and integration with respect to $\epsilon^1, \cdots, \epsilon^{m-1}$ give

$$\begin{aligned}
&\int \|\Phi(\theta,\, \epsilon)\|_{L^p(dm_\mu(\theta))} \, d\epsilon \\
&\le C(\mu) \sum_{\ell=1}^m \int \Big(\sum_{k\in\theta_N} |c_k|^2 (1-\varepsilon_k^1)^2 \cdots (1-\varepsilon_k^{\ell-1})^2 k^{2\mu-2} \Big)^{1/2} d\epsilon^1 \cdots d\epsilon^{\ell-1} \times \\
&\times \Big(\int_0^\pi \frac{1}{\theta^{\mu p}} \frac{1}{(\pi-\theta)^{\mu p}} \sin^{2\mu}\theta \, d\theta \Big)^{\frac{1}{p}} \\
&\le C(\mu) \Big(\frac{1}{1-\mu(p-2)} \Big)^{\frac{1}{p}} \sum_{\ell=1}^m 2^{\frac{\ell}{2}} \Big(\sum_{k\in\theta_N} |c_k|^2 k^{2\mu-2} \Big)^{1/2} \\
&\le C(\mu) \Big(\frac{1}{1-\mu(p-2)} \Big)^{\frac{1}{p}} \Big(\frac{N}{M} \Big)^{1/2} \|T_{\theta_N}(\cos\theta)\|_{L^2(dm_\mu(\theta))}.
\end{aligned} \tag{2.1.2}$$

On the other hand, noticing that

$$|\theta_{M(\epsilon)}| = \frac{1}{2^m} \sum_{k\in\theta_N} (1-\varepsilon_k^1)\cdots(1-\varepsilon_k^m),$$

we get

$$\int |\theta_{M(\epsilon)}|\, d\epsilon = \frac{N}{2^m} \asymp M. \tag{2.1.3}$$

Now a combination of (2.1.1)–(2.1.3) yields the desired result. □

Remark 2.1.1. Some more information can be concluded from the above proof. In fact, let T_{θ_N} and T_{θ_M} as in the above proof. Suppose

$$T_{\theta_N}(t) - T_{\theta_M}(t) = \sum_{k \in \theta_N} c_k C_k^{(\mu)}(t).$$

Then we have $|c_k| \leq \frac{N}{M}|b_k|$ for all $k \in \theta_N$. We will use this fact in the later proof of Theorem 2.0.5 to give an L^2 -estimate.

For the remainder of this section, we write an algebraic polynomial $T_N(t)$ of degree at most N as a linear combination of the ultraspherical polynomials

$$T_N(t) = \sum_{k=0}^{N} c_k C_k^{(\mu)}(t).$$

With this notation, by the well known Nikolskii's inequality for the ultraspherical polynomials, we conclude

Lemma 2.1.3. *For every polynomial $T_N(t)$,*

$$\|T_N(\cos\theta)\|_\infty \leq C(\mu) N^{\mu+\frac{1}{2}} \|T_N(\cos\theta)\|_{L^2(dm_\mu(\theta))}.$$

Lemma 2.1.4. *Let $\eta \in C^\infty(\mathbb{R})$ and $\chi_{[0,1]} \leq \eta \leq \chi_{[0,2]}$. Suppose $f \in L\big([0,\pi], \sin^{2\mu}\theta\, d\theta\big)$ with the expansion $f \sim \sigma(f) = \sum_{k=0}^{\infty} c_k(f) C_k^{(\mu)}(\cos\theta)$. Define*

$$\eta_N(f)(\cos\theta) = \sum_{k=0}^{2N} \eta(\frac{k}{N}) c_k(f) C_k^{(\mu)}(\cos\theta).$$

Then for $1 \leq p \leq \infty$

$$\|\eta_N(f)(\cos\theta)\|_{L^p(dm_\mu)} \leq C(\mu)\|f\|_{L^p(dm_\mu)}.$$

Lemma 2.1.4 can be easily obtained by using the Abelian transform and the uniform boundedness of the Cesàro means of the ultraspherical expansion $\sigma(f)$ of order $\delta > \lambda$ in the L^p metric. (See [BC].)

Lemma 2.1.5. *For every $\alpha > 0$ and $2 < p < q \le \infty$, there exists a decomposition $T_N = T^1_{2N} + T^2_{2N}$, such that*

$$\|T^1_{2N}(\cos\theta)\|_{L^q(dm_\mu(\theta))} \le C(\mu) q^{\frac{1}{q}} \alpha^{\frac{1}{p}-\frac{1}{q}} \|T_N(\cos\theta)\|_{L^p(dm_\mu(\theta))},$$
$$\|T^2_{2N}(\cos\theta)\|_{L^2(dm_\mu(\theta))} \le C(\mu)\Big(\frac{2}{p-2}\Big)^{1/2} \alpha^{\frac{1}{p}-1/2} \|T_N(\cos\theta)\|_{L^p(dm_\mu(\theta))},$$

where $C(\mu)$ is a constant independent of T_N, p and q.

Proof. Without loss of generality, we may assume $\|T_N(\cos\theta)\|_{L^p(dm_\mu(\theta))} = 1$. Let us write $T_N = f_1 + f_2$, where

$$f_1(\theta) = \begin{cases} T_N(\cos\theta), & \text{if } |T_N(\cos\theta)| \le \alpha^{\frac{1}{p}}, \\ \\ \alpha^{\frac{1}{p}} \operatorname{sign}\Big(T_N(\cos\theta)\Big), & \text{otherwise} \end{cases}$$

and

$$f_2(\theta) = T_N(\cos\theta) - f_1(\theta).$$

Then we have

$$\begin{aligned} \int_0^\pi |f_1(\theta)|^q \, dm_\mu(\theta) &= q \int_0^{\alpha^{\frac{1}{p}}} u^{q-1} m_\mu\{\theta \in (0,\ \pi) :\ |T_N(\cos\theta)| > u\}\, du \\ &\le q\alpha^{\frac{1}{p}(q-p)} \int_0^\infty u^{p-1} m_\mu\{\theta \in (0,\ \pi) :\ |T_N(\cos\theta)| > u\}\, du \\ &= q\alpha^{\frac{q}{p}-1} \end{aligned}$$

and

$$\begin{aligned} \|f_2\|^2_{L^2(dm_\mu(\theta))} &= 2\int_0^\infty u m_\mu\Big\{t \in (0,\ \pi) :\ |T_N(\cos t)| > u + \alpha^{\frac{1}{p}}\Big\}\, du \\ &\le 2\int_0^\infty \frac{u\,du}{\Big(u + \alpha^{\frac{1}{p}}\Big)^p} \le \frac{2}{p-2}\alpha^{\frac{2}{p}-1}. \end{aligned}$$

Let $T^1_{2N} = \eta_N f_1$ and $T^2_{2N} = \eta_N f_2$, with η_N the same as in Lemma 2.1.4. Now applying Lemma 2.1.4, noticing that $\eta_N(T_N) = T_N$, we obtain the desired result and complete the proof. □

Proof of Theorem 2.1.1. Theorem 2.1.1 for $2 \le p < 2 + \frac{1}{\mu}$ is a direct consequence of Lemma 2.1.2. It remains to consider the case $\frac{2\mu+1}{\mu} \le p \le \infty$. For any $\varepsilon > 0$, set $p_0 = 2 + \frac{2}{2\mu+\varepsilon} \in (2,\ \frac{2\mu+1}{\mu})$ and define the sequences $\Big\{a_k(M,N)\Big\}$ and $\Big\{c_k\Big\}$ by

$$a_0(M,N) = 0,\ \ a_k(M,N) = b\Big(a_{k-1}(M,N) + \delta(M,N)\Big) + 1/2,$$
$$c_0 = 0,\ \ c_k = bc_{k-1} + \frac{1}{p_0},\ \ \ k = 1, 2, \cdots,$$

where

$$b = (\frac{1}{p_0} - \frac{1}{p})/(1/2 - \frac{1}{p}) \in (0,\ 1),\ \ \delta := \delta(M,N) = \ell\Big(\log\frac{N+1}{M}\Big)^{-1}$$

and $\ell > 0$ is a constant to be specified later. Then one can easily verify that $a_{k+1} > a_k,\ c_{k+1} > c_k,\ k = 1, 2, \cdots$ and

$$\lim_{k\to\infty} a_k(M,\ N) = \frac{b\delta(M,N) + 1/2}{1-b} = \frac{b\delta(M,N)}{1-b} + (1/2 - \frac{1}{p})(2\mu + 1 + \varepsilon),$$
$$\lim_{k\to\infty} c_k = \frac{\frac{1}{p_0}}{1-b} = (2\mu+\varepsilon)(1/2 - \frac{1}{p}).$$

We will prove that for every k, $\varepsilon > 0$, and all N there exists a polynomial $T_{\theta^k_M}$ with $\theta^k_M \subset [0,\ 2N]$, such that

$$\|T_N - T_{\theta^k_M}\|_p \le C_6\Big(\frac{1}{\varepsilon}\Big)^{c_k}\Big(\frac{N}{M}\Big)^{a_k(M,\,N)} N^{(\mu+\frac{1}{2})b^k}\|T_N\|_2, \tag{2.1.4}$$

where $C_6 > 0$ is a constant independent of k, ε, N and M. Then after a finite number of steps we will obtain the desired result by choosing $\varepsilon = (\log\frac{2N}{M})^{-1}$.

(2.1.4) for $k = 0$ is an immediate consequence of Lemma 2.1.3. Assume (2.1.4) for k is valid for all N and $\varepsilon > 0$. We now prove it for $k+1$. We assume $M = 2m$ is an even number. An odd M is treated similarly.

According to Lemma 2.1.2, for $m \in [0,\ N]$, there exists a polynomial T_{θ_m} such that

$$\|T_N - T_{\theta_m}\|_{p_0} \le C_1\Big(\frac{1}{\varepsilon}\Big)^{\frac{1}{p_0}}\Big(\frac{N}{m}\Big)^{1/2}\|T_N\|_2.$$

Applying Lemma 2.1.5, for a parameter $u > 0$ there exists a decomposition

$$T_N - T_{\theta_m} = T^1_{2N} + T^2_{2N},$$

such that

$$\begin{aligned}\|T^1_{2N}\|_p \le & C_2 u^{\frac{1}{p_0}-\frac{1}{p}} \|T_N - T_{\theta_m}\|_{p_0} \le C_2 C_1 \Big(\frac{1}{\varepsilon}\Big)^{\frac{1}{p_0}} u^{\frac{1}{p_0}-\frac{1}{p}} \Big(\frac{N}{m}\Big)^{1/2} \|T_N\|_2,\\ \|T^2_{2N}\|_2 \le & C_3 u^{\frac{1}{p_0}-1/2} \|T_N - T_{\theta_m}\|_{p_0} \le C_3 C_1 \Big(\frac{1}{\varepsilon}\Big)^{\frac{1}{p_0}} u^{\frac{1}{p_0}-1/2} \Big(\frac{N}{m}\Big)^{1/2} \|T_N\|_2.\end{aligned}$$

Using our assumption for k, we approximate the polynomial T^2_{2N} by a polynomial $T_{\theta^k_m}$ in the norm L^p and let $T_{\theta^{k+1}_M} = \eta_N \Big(T_{\theta_m} + T_{\theta^k_m}\Big)$. Then $\theta^{k+1}_M \subset [0,\ 2N]$ and

$$\begin{aligned}&\|T_N - T_{\theta^{k+1}_M}\|_p = \|\eta_N (T_N - T_{\theta_m} - T_{\theta^k_m})\|_p\\ &\le C_4 \|T_N - T_{\theta_m} - T_{\theta^k_m}\|_p = C_4 \|T^1_{2N} + T^2_{2N} - T_{\theta^k_m}\|_p\\ &\le C_4 \|T^1_{2N}\|_p + C_4 \|T^2_{2N} - T_{\theta^k_m}\|_p\\ &\le \Bigg(C_1 C_2 C_4 \Big(\frac{1}{\varepsilon}\Big)^{\frac{1}{p_0}} u^{\frac{1}{p_0}-\frac{1}{p}} \Big(\frac{N}{m}\Big)^{1/2} + C_1 C_3 C_4 C_6 \Big(\frac{1}{\varepsilon}\Big)^{\frac{1}{p_0}+c_k} u^{\frac{1}{p_0}-1/2} \times\\ &\times \Big(\frac{N}{m}\Big)^{1/2} \Big(\frac{2N}{m}\Big)^{a_k(m,2N)} (2N)^{(\mu+\frac{1}{2})b^k}\Bigg) \|T_N\|_2\\ &\le \Bigg(C_1 C_2 C_4 C_5 \Big(\frac{1}{\varepsilon}\Big)^{\frac{1}{p_0}} u^{\frac{1}{p_0}-\frac{1}{p}} \Big(\frac{N}{M}\Big)^{1/2} + C_1 C_3 C_4 C_5 C_6 \Big(\frac{1}{\varepsilon}\Big)^{\frac{1}{p_0}+c_k} u^{\frac{1}{p_0}-1/2} \times\\ &\times \Big(\frac{N}{M}\Big)^{1/2} \Big(\frac{N}{M}\Big)^{a_k(M,N)} N^{(\mu+\frac{1}{2})b^k}\Bigg) \|T_N\|_2.\end{aligned}$$

Set $u = \Big(\frac{1}{\varepsilon}\Big)^{c_k/(1/2-\frac{1}{p})} \Big(\frac{N}{M}\Big)^{\big(a_k(M,N)+\delta(M,N)\big)/(1/2-\frac{1}{p})} N^{(\mu+\frac{1}{2})b^k/(1/2-\frac{1}{p})}$. Then the right-hand side can be rewritten in the form

$$\begin{aligned}& C_6 \Big(\frac{1}{\varepsilon}\Big)^{c_{k+1}} \Big(\frac{N}{M}\Big)^{a_{k+1}} N^{(\mu+\frac{1}{2})b^{k+1}} \|T_N\|_2 \Big(\frac{C_1 C_2 C_4 C_5}{C_6} + C_1 C_3 C_4 C_5 \Big(\frac{N}{M}\Big)^{-\delta}\Big)\\ =& C_6 \Big(\frac{1}{\varepsilon}\Big)^{c_{k+1}} \Big(\frac{N}{M}\Big)^{a_{k+1}} N^{(\mu+\frac{1}{2})b^{k+1}} \|T_N\|_2 \Big(\frac{C_1 C_2 C_4 C_5}{C_6} + C_1 C_3 C_4 C_5 e^{-\ell}\Big).\end{aligned}$$

Taking $C_6 = 3C_1C_2C_4C_5$ and $\ell = \log(2C_1C_3C_4C_5)$, we get (2.1.4) for $k+1$ and finish the proof. □

2.2 Approximation of the functions $F_{r,\mu}(\cos\theta)$

For $f \in L^p$, with the same notations as in the previous sections, we define

$$e_m(f,\, L^p) = \inf_{\theta_m} \inf_{T_{\theta_m}} \|f(\theta) - T_{\theta_m}(\cos\theta)\|_p.$$

The quantity $e_m(f, L^p)$ was originally introduced by Stechkin, who used it in the criterion for absolute convergence of orthogonal series. This characteristic has become popular after Ismagilov found nontrivial estimates for $e_m(|x|, L^\infty)$ and gave interesting and important applications to the widths of Sobolev classes. The Ismagilov method was developed in a series of papers. (See [Be1],[Be3].)

The main task in this section is to estimate the sharp orders of the quantities $e_m(F_{r,\mu}, L^p)$ for $2 \le p \le \infty$, where the functions

$$F_{r,\mu}(\cos\theta) = \sum_{k=1}^{\infty} \big(k(k+2\mu)\big)^{-\frac{r}{2}} \frac{\Gamma(\mu)(k+\mu)}{2\pi^{\mu+1}} C_k^{(\mu)}(\cos\theta),\ \ \mu > 0$$

are analogous of the Bernoulli kernels of the ultraspherical polynomials. The estimates will be used to derive Theorem 2.0.5 in the next section and can be stated as follows.

Theorem 2.2.1. *Suppose $r > 2\mu + 1$ and $2 \le p \le \infty$. Then*

$$e_m(F_{r,\mu},\ L^p) \asymp m^{-r+\mu+1/2}.$$

Proof. The lower estimate follows from the following inequalities:

$$e_m(F_{r,\mu},\ L^p) \ge e_m(F_{r,\mu},\ L^2) = \inf_{\theta_m} \Big\| \sum_{k \notin \theta_m} (k(k+2\mu))^{-\frac{r}{2}} \frac{\Gamma(\mu)(k+\mu)}{2\pi^{\mu+1}} C_k^{(\mu)}(\cos\theta) \Big\|_2$$

$$\asymp \inf_{\theta_m} \Big(\sum_{k \notin \theta_m} k^{-2r+2\mu} \Big)^{1/2} \ge C \Big(\sum_{k=m+1}^{\infty} \frac{1}{k^{2r-2\mu}} \Big)^{1/2} \ge C m^{-r+\mu+1/2}.$$

To estimate the upper bound, it suffices to consider the case $p = \infty$. Suppose $m \sim 2^\ell$ and $r > 2\mu + 1 + \varepsilon$ for some sufficiently small $\varepsilon > 0$. Let us write

$$F_{r,\mu}(\cos\theta) = \sum_{j=0}^{\ell-1} \sigma_j(F_{r,\mu}, \theta) + \sum_{j=\ell}^{[\gamma\ell]} \sigma_j(F_{r,\mu}, \theta) + \sum_{j=[\gamma\ell]+1}^{\infty} \sigma_j(F_{r,\mu}, \theta),$$

with

$$\sigma_j(F_{r,\mu},\,\theta)=\sum_{k=2^j}^{2^{j+1}-1}(k(k+2\mu))^{-\frac{r}{2}}\frac{\Gamma(\mu)(k+\mu)}{2\pi^{\mu+1}}C_k^{(\mu)}(\cos\theta),$$
$$\gamma=\frac{r-\mu-1/2-\varepsilon}{r-2\mu-1-\varepsilon}>1.$$

For $\ell\le j\le[\gamma\ell]$, using Theorem 2.1.1 , we approximate $\sigma_j(F_{r,\mu},\theta)$ by a polynomial $T_{\theta_{k_j}}$ with

$$k_j=\left[2^{-j\frac{r-2\mu-1-\varepsilon}{\mu+1/2}}2^{\ell\frac{r-\mu-1/2-\varepsilon}{\mu+1/2}}\right].$$

Let

$$T_{\Theta_m}(\cos\theta)=\sum_{j=0}^{\ell-1}\sigma_j(F_{r,\mu},\,\theta)+\sum_{j=\ell}^{[\gamma\ell]}T_{\theta_{k_j}}(\cos\theta).$$

Then a straightforward computation shows

$$|\Theta_m|\le 2^{\ell-1}+\sum_{j=\ell}^{[\gamma\ell]}k_j\asymp 2^\ell\asymp m.$$

By Theorem 2.1.1 and Nikolskii inequality for the ultraspherical polynomials, we have

$$\begin{aligned}&\|F_{r,\mu}(\cos\theta)-T_{\Theta_m}(\cos\theta)\|_\infty\\ \le&C\sum_{j=\ell}^{[\gamma\ell]}\Big(\frac{2^j}{k_j}\Big)^{\mu+1/2}\Big(\log(1+\frac{2^j}{k_j})\Big)^{\mu}\|\sigma_j(F_{r,\mu},\,\theta)\|_2+C\sum_{j=[\gamma\ell]+1}^{\infty}2^{j(\mu+1/2)}\|\sigma_j(F_{r,\mu},\,\theta)\|_2,\end{aligned}$$

which, by the fact $\|\sigma_j(F_{r,\mu},\,\theta)\|_2\asymp 2^{-j(r-\mu-\frac{1}{2})}$, is dominated by

$$\le Cm^{-r+\mu+1/2}.$$

This gives the required upper estimate and completes the proof. □

2.3 The linear spherical harmonic widths

We now turn to the proof of Theorem 2.0.5. We start by proving the first estimate. The lower estimate follows from Kamzolov's result. To show the upper bound, it suffices to consider the special case $q=\infty$.

Suppose $2^{(d-1)\ell} \le m < 2^{(d-1)(\ell+1)}$ and $N = 2^{\ell+1}$. Then according to Theorem 2.2.1 and its proof, there exists a polynomial

$$T_{\theta_N}(\cos\theta) = \sum_{k\in\theta_N} b_k \frac{(k+\lambda)\Gamma(\lambda)}{2\pi^{\lambda+1}} C_k^{(\lambda)}(\cos\theta),$$

for which

$$\|F_r(\cos\theta) - T_{\theta_N}(\cos\theta)\|_\infty \le CN^{-r+\frac{d-1}{2}}, \tag{2.3.1}$$

where

$$\theta_N = \{1, \cdots, 2^{\ell-1}-1; n_1^{(\ell)}, \cdots, n_{k_\ell}^{(\ell)}; \cdots, n_1^{(\gamma\ell)}, \cdots; n_{k_{\gamma\ell}}^{(\gamma\ell)}\},$$

$$\{n_1^{(j)}, \cdots, n_{k_j}^{(j)}\} \subset [0,\ 2^{j+2}],\quad k_j = [2^{\ell\frac{2r-d+1-2\varepsilon}{d-1}} 2^{-j\frac{2r-2d+2-2\varepsilon}{d-1}}],\quad j = \ell, \cdots, [\gamma\ell]$$

and

$$\gamma = \frac{r - \frac{d-1}{2} - \varepsilon}{r-d+1-\varepsilon},\quad r > \frac{d(d-1)}{2} + \varepsilon.$$

Let $L_m = \bigoplus_{k\in\theta_N} \mathcal{H}_k$. Then observing $\dim \mathcal{H}_k^d \asymp k^{d-2}$, we get

$$\dim L_m \le C\sum_{j=1}^{2^{\ell-1}} j^{d-2} + C\sum_{j=\ell}^{[\gamma\ell]} 2^{(j+2)(d-2)} k_j \asymp m.$$

It follows from (2.3.1) and Minkowski's inequality that for $\varphi \in L^1$,

$$\|\varphi * F_r - \varphi * T_{\theta_N}\|_\infty \le \|F_r(\cos\theta) - T_{\theta_N}(\cos\theta)\|_\infty \|\varphi\|_1 \le Cm^{-\frac{r}{d-1}+1/2}\|\varphi\|_1,$$

which, together with the definition of B_1^r and the obvious fact $\varphi * T_{\theta_N} \in L_m$, gives the required upper estimate.

Next, we prove the third estimate. As before, we assume $2^{(d-1)\ell} \le m < 2^{(d-1)(\ell+1)}$ and let $N = 2^{\ell+1}$. The lower bound also follows from Kamzolov's result [Ka2]. To estimate the upper bound, instead of Theorem 2.1.1, we use Lemma 2.1.1 and Remark 2.1.1. Then, as in the proof of the first estimate, we construct a polynomial

$$T_{\theta_N}(t) = \sum_{k\in\theta_N} b_k \frac{\Gamma(\lambda)(k+\lambda)}{2\pi^{\lambda+1}} C_k^\lambda(t),$$

with

$$\begin{aligned}\theta_N =&\{1,\ldots,2^{\ell-1},n_1^{(\ell)},\ldots,n_{k_\ell}^{(\ell)},\ldots,n_1^{(\gamma\ell)},\ldots,n_{k_{\gamma\ell}}^{(\gamma\ell)}\},\\ &\{n_1^{(j)},\ldots,n_{k_j}^{(j)}\}\subset[2^j,\,2^{j+1}],\\ k_j =&[2^{(\ell-j)r+j}],\ j=\ell,\ldots,[\gamma\ell],\quad \gamma=\frac{r-\frac{d-1}{2}}{r-\frac{d}{2}},\end{aligned}\tag{2.3.2}$$

for which the following two inequalities are satisfied:

$$\|(F_r-T_{\theta_N})*\varphi\|_{q_1}\le Cm^{-\frac{r}{d-1}+\frac12}\|\varphi\|_1,\quad \text{for some } q_1\in[2,\frac{2(d-1)}{d-2})$$
$$\|(F_r-T_{\theta_N})*\varphi\|_2\le C\sup_{\ell\le j\le\gamma\ell}\frac{2^{j-jr}}{k_j}\|\varphi\|_2\asymp N^{-r}\|\varphi\|_2.$$

Let $L_m=\bigoplus\limits_{k\in\theta_N}\mathcal{H}_k^n$. Then from (2.3.2), we get, by easy computation,

$$\dim L_m\asymp m.$$

Now the required upper estimate follows immediately from applying the Riesz–Thorin interpolation theorem.

Finally, noticing that the second and the fourth estimates can be derived from the other two estimates by duality, we conclude the proof.

2.4 Concluding remarks

First, we remark that the argument in Section 2.1 also applies to some general settings. Let $\{X,\Sigma,d\mu\}$ denote a probabilistic space and $\Phi=\{\varphi_k\}_{k=0}^\infty$ an orthonormal system of functions in $L^2(X)$. We make the following assumptions on the orthonormal system Φ:

1.

$$\varphi_k\in L^\infty(X),\quad \|\varphi_k\|_\infty\le C(k+1)^a\|\varphi_k\|_2,\tag{2.4.1}$$

for $k\in\mathbb{Z}_+$ and a given fixed positive constant a.

2.

$$\sup_k\frac{\|\varphi_k\|_q}{\|\varphi_k\|_2}\le C<\infty,\quad \text{for}\quad 2\le q<q_0,$$

where $q_0 > 2$ is a fixed constant.

3. There exists a sequence of linear operators $\{V_k\}_{k=0}^{\infty}$ on $L(X)$ with the following three properties:

(i) For $N \in \mathbb{Z}_+$, $f \in L(X)$,

$$V_N f \in \operatorname{span}\{\varphi_k : \quad 0 \le k \le CN\}, \quad \text{for some fixed constant} \quad C > 1.$$

(ii) For $N \in \mathbb{Z}_+$ and $f \in \operatorname{span}\{\varphi_k : \quad 0 \le k \le N\}$,

$$V_N f = f.$$

(iii)

$$\sup_k \|V_k f\|_p \le C\|f\|_p, \quad \text{for } f \in L^p(X), \quad 2 \le p \le \infty.$$

Under the above assumptions, we have

Theorem 2.4.1. *For a function* $T_N = \sum_{k=0}^{N} c_k \varphi_k$, $1 \le M \le N$, *there exists a function representable in the form*

$$T_{\theta_M}(t) = \sum_{k \in \theta_M} b_k \varphi_k$$

with $\theta_M \subset [0, 2N]$, $\quad |\theta_M| = M$, *such that*

$$\|T_N - T_{\theta_M}\|_\infty \le C\Big(\frac{N}{M}\Big)^\alpha \Big(\log(1 + \frac{N}{M})\Big)^\beta \|T_N\|_2,$$

where $\alpha = \alpha(a, q_0), \beta = \beta(a, q_0) > 0$ *are constants independent of* N, M *and* T_N.

Next, for $r > 0$, define

$$W_{2,\Phi}^r := \Big\{\sum_{k=0}^{\infty} a_k \varphi_k : \quad a_0 = 0, \quad \sum_{k=0}^{\infty} |a_k|^2 k^{2r} \le 1\Big\}.$$

As before, for $\theta_k \subset \mathbb{Z}_+$ with $|\theta_k| = k$, let T_{θ_k} denote a function representable in the form

$$T_{\theta_k}(x) = \sum_{k \in \theta_k} a_k \varphi_k(x), \quad a_k \in \mathbb{R}.$$

We consider the following extreme problem:

$$E_N(W_{2,\Phi}^r, L^q) = \sup_{f \in W_{2,\Phi}^r} \inf_{\theta_N} \|f - T_{\theta_N}\|_q, \quad 2 \le q \le \infty.$$

Theorem 2.4.2. *Let $r > \alpha(a, q_0)$ and $2 \le q \le \infty$. Then, under the above assumptions,*

$$E_N(W^r_{2,\Phi}, L^q) \asymp N^{-r}.$$

In the case $\Phi = \{e^{in\theta}\}$, Theorem 2.4.2 was due to [Be3]. Here the proof is similar to that of [Be3]. (The only differences are that here we should use Theorem 2.4.1 and the inequality

$$\|T_N\|_\infty \le CN^\alpha \log^\beta(1+N)\|T_N\|_2,$$

with $T_N \in \text{span}\{\varphi_k\}_{k=0}^N$, which can be deduced from Theorem 2.4.1 by taking $M = 1$.) We omit the detail.

Finally, we give some additional remarks on spherical harmonics. For sake of convenience, we only consider $d = 3$. We introduce the system of polar coordinates on $\mathbb{S}^2$ as in Section 1.1. Let

$$P_k^m(x) = (-1)^m \frac{(2m)!}{2^m m!}(1-x^2)^{\frac{m}{2}} C_{k-m}^{m+\frac{1}{2}}(x).$$

It is well known (see [An-As-R, P457]) that the set consisting of the following $2k+1$ functions

$$\sqrt{\frac{2k+1}{4\pi}}P_k(\cos\theta), \quad A_m \cos m\phi P_k^m(\cos\theta), \quad A_m \sin m\phi P_k^m(\cos\theta), \quad m = 1, \cdots, k,$$

forms an orthonormal basis of the space $\mathcal{H}_k^3$ of spherical harmonics of degree k in three variables, where

$$A_m = \sqrt{\frac{(k-m)!(2k+1)}{(k+m)!2\pi}}.$$

For this system, up to now, we are not able to prove that it satisfies the condition (P1) above. If the condition (P1) was satisfied, then some similar results for the class $B_p^r(\mathbb{S}^2)$ could also be obtained.

Chapter 3

Strong equivalences between the K-functionals and some well known linear operators

We first describe briefly the problem in a general setting. Let Ω be a nonempty set endowed with a non-negative completely additive measure μ and B a Banach space of μ-measurable functions on Ω with a norm designated by $\|\cdot\|_B$. Suppose $\mathcal{S}$ is a linear space contained B and $D:\ \mathcal{S}\longrightarrow\mathcal{S}$ is linear, unbounded on B. Let $W(B,D)$ denote the function space

$$W(B,D):=\Big\{f\in B: Df\in B\Big\}.$$

We assume $W(B,D)\neq\{0\}$. To each operator D, we associate a K-functional $K(f,D,t)_B$ defined by

$$K(f,D,t)_B=\inf\Big\{\|f-g\|_B+t\|Dg\|_B:\ g\in W(B,D)\Big\},\quad t>0,\quad f\in B.$$

It will be convenient to use the notation

$$A(f,t)\approx B(f,t),$$

which means there is a $C>0$, independent of f and t, such that

$$C^{-1}A(f,t)\le B(f,t)\le CA(f,t).$$

Given a family $\{T_t\}_{t>0}$ of linear continuous operators on B, the following two types of equivalences are usually considered:

$$\begin{aligned} \mathrm{I}: \quad & K(f,D,t)_B \approx \|f - T_t f\|_B, \\ \mathrm{II}: \quad & K(f,D,t)_B \approx \|f - T_t f\|_B + \|f - T_{\theta t} f\|_B, \end{aligned}$$

where $\theta \in (0,1)$ is independent of f and t. Taking into account the monotonicity of the K-functional $K(f,D,\cdot)_B$, we have, under the equivalence of type I,

$$\|f - T_t f\|_B \approx \|f - T_{\theta t} f\|_B.$$

Hence, we conclude that the equivalence of type I is stronger than that of type II. We call the equivalence of type I the strong equivalence. We say $\{T_t\}$ is a realization of the K-functional $K(f,D,\cdot)_B$ if the strong equivalence is satisfied. An inequality of the form

$$\|f - T_t f\|_B \leq C_B K(f,D,t)_B$$

is called a direct inequality while that of the form

$$K(f,D,t)_B \leq C_B \|f - T_t f\|_B$$

a strong converse inequality of type A. (SCIA). Generally, the SCIA is more difficult to establish than the following weak type inequality:

$$K(f,D,t)_B \leq C\|f - T_t f\|_B + \|f - T_{\theta t} f\|_B. \tag{3.0.1}$$

We call an inequality of the form (3.0.1) a strong converse inequality of type B. (SCIB).

Fore more background information of the above notations, we refer the reader to [Di1], [C-Di], [Di-Iv].

In this chapter, the linear space $\mathcal{S}$ and the unbounded operator D we usually concern with are the space of tempered distributions and the fractional differential operator, generally defined in the distributional sense. It is our aim, as suggested in the title, to establish the strong equivalences between the K-functionals and some families of well known operators, such as Cesàro operators, Riesz operators, average operators, Steklov means, etc. For most of the operators we consider, the best previously known results are the equivalences of type II.

We organize this chapter as follows. In Section 3.1, we consider the Riesz means and the Cesàro means in a general setting introduced in [Di1] and [C-Di]. Equivalence between each of these operators and the corresponding K-functional is established. In Section 3.2, we confine ourselves again to the case of unit sphere $\mathbb{S}^{d-1}$. Equivalences between the K-functionals and the average operators, the K-functionals and the Steklov type means, are established in Sections 3.2.2 and 3.2.4 respectively. A conjecture on the norm of the derivative of the average operator raised in [Di-Ru1] is solved in Section 3.2.3. In Section 3.2.5, we apply the technique developed in Sections 3.2.2–3.2.3 to obtain a strong equivalence between the K-functional and the modulus of continuity on $\mathbb{S}^{d-1}$. As a consequence, the well known Jackson type inequality on $\mathbb{S}^{d-1}$ is deduced and its original proofs in [Ru2] and [Ri-Wa] are simplified. In section 3.3, we obtain some similar results as in Section 3.2 for the Jacobi expansions, which also improve some previously known results. A strong converse inequality for the average operator of high order and the K-functional related to the Laplacian on $\mathbb{R}^d$ is obtained in Section 3.4, which proves a conjecture raised in [Di-Ru2].

We thank professor Z. Ditzian for supplying us with some preprints of his excellent papers.

3.1 Riesz mean, Cesàro mean and the K-functional

3.1.1 General notations and assumptions

We first introduce some notations from [Di1]. Let S be a nonempty set equipped with a positive measure μ and let $L^p(S)$, $(1 \le p \le \infty)$ denote the space of functions on S with the usual norm

$$\|f\|_p = \Big(\int_S |f|^p \, d\mu(x)\Big)^{\frac{1}{p}}.$$

Suppose $P(D)$ is a self adjoint, unbounded operator on $L^2(S)$. We make the following assumptions on $P(D)$:

(i) $P(D)$ has only discrete spectrum $\{-\psi(k)\}_{k=0}^{\infty}$ and each eigenvalue $-\psi(k)$ corresponds to a finite dimensional eigenspace H_k.

(ii) $0 = \psi(0) < \psi(1) < \cdots < \psi(k) < \cdots$ and each $\psi(k)$ is a polynomial in k.

(iii) For some fixed $p \in [1, \infty]$, $H_k \subset L^p(S) \bigcap L^{p'}(S)$ and

$$\overline{\text{span} \bigcup_k H_k} = L^p(S).$$

For the remainder of this section, we assume p, B, $\|\cdot\|$ denote a fixed number for which assumption (iii) is satisfied, the space $L^p(S)$ and the norm $\|\cdot\|_p$ respectively.

Since,under the above assumptions, the projection $P_k f$ on H_k is well defined for all $f \in B$, we have the formal expansion

$$f \sim \sum_{k=0}^{\infty} P_k f.$$

Accordingly, *the ℓ-th order Cesàro mean* of the expansion can be defined (as usual) by

$$\sigma_N^{\ell}(f) = \sum_{k=0}^{N} \frac{A_{N-k}^{\ell}}{A_N^{\ell}} P_k f.$$

We make an additional assumption on the Cesàro means:

(iv) For some positive integer $\ell = \ell(B)$,

$$\sup_N \|\sigma_N^{\ell}(f)\| \le C(\ell, B)\|f\|. \tag{3.1.1}$$

Assumptions (i)–(iv) were made by Ditzian [Di1]. As pointed out in [Di1] and [C-Di], these assumptions are very natural ones. In fact, many well known differential operators and the expansions related to them, such as spherical harmonics and the Laplace- Beltrami operator, Jacobi expansions and the Jacobi operator, and Hermite and Laguerre expansions and their operators, etc, satisfy them.

Next, we define the fractional differential operator $P(D)^\alpha$ (for a given α), in the sense of distributions, by

$$P(D)^\alpha f \sim \sum_{k=0}^{\infty} (\psi(k))^\alpha P_k f.$$

We write $P(D)^\alpha f = f^{(\alpha)}$ if $P(D)^\alpha f \in B$. To each operator $P(D)^\alpha$ is associated with a K-functional

$$K_\alpha(f,t) := K(f, P(D)^\alpha, t^\alpha) := \inf\Big\{\|f-g\| + t^\alpha \|g^{(\alpha)}\| : \ g^{(\alpha)} \in B\Big\}.$$

For further detail of the background information, we refer the reader to [Di1] and [C-Di].

3.1.2 Riesz means

For $\psi > 0, \ \alpha > 0$ and $\ell \in \mathbb{N}$, the generalized Riesz mean, which was introduced in [Di1], is defined by

$$R_{\psi,\alpha,\ell}(f) = \sum_{\psi(k)<\psi} \Big(1 - \Big(\frac{\psi(k)}{\psi}\Big)^\alpha\Big)^\ell P_k(f).$$

It follows from [Di1] that

$$\sup_{\psi>0} \|R_{\psi,\alpha,\ell}(f)\| \le C\|f\|, \tag{3.1.2}$$

with C independent of f.

In this subsection, we shall prove the following theorem, which was conjectured in [Di1] under the hypothesis (3.1.2).

Theorem 3.1.1. *Suppose $\ell \in \mathbb{N}$ and (3.1.1) is satisfied. Then for $\psi > 0, \ \alpha > 0$ and $m \in \mathbb{N}$,*

$$\|(R_{\psi,\alpha,\ell} - I)^m f\| \approx K_{\alpha m}(f, \psi^{-1}).$$

Theorem 3.1.1 for $\ell = 1$ was due to [Di1]. For $\ell \ge 2$, Ditzian [Di1] proved the following weak type equivalence (of type II):

$$\|(R_{\psi,\alpha,\ell} - I)^m f\| + \psi^{-\alpha m}\Big\|\Big(R_{\psi,\alpha,\ell,m} f\Big)^{(\alpha m)}\Big\| \ \approx K_{\alpha m}(f, \psi^{-1}), \tag{3.1.3}$$

where the operator $R_{\psi,\alpha,\ell,m}$ is defined by

$$R_{\psi,\alpha,\ell,m} = I - (I - R_{\psi,\alpha,\ell})^m = \sum_{k=1}^{m} (-1)^{k-1} \binom{m}{k} R^k_{\psi,\alpha,\ell}. \tag{3.1.4}$$

To prove Theorem 3.1.1, we need a series of lemmas.

Lemma 3.1.2. *Suppose $\ell \in \mathbb{N}$ is as in assumption (iv) and $\eta \in C^{(\ell+1)}(\mathbb{R}_+)$ is of compact support. For $\psi > 0$, define*

$$V_\psi(f) = \sum_{k=0}^{\infty} \eta\Big(\frac{\psi(k)}{\psi}\Big) P_k(f).$$

Then

$$\|V_\psi(f)\| \le C_\eta \|f\|,$$

with $C_\eta > 0$ independent of f and $\psi > 0$.

Proof. Suppose $\operatorname{supp}\eta \subset [0, a]$ with $a > 0$ depending only on η and suppose $\psi(n_0 - 1) \le a\psi < \psi(n_0)$ with n_0 a positive integer.

Noticing

$$\|P_k f\| = \Big\| \overleftarrow{\triangle}^{\ell+1} \binom{k+\ell}{\ell} \sigma^\ell_k(f) \Big\| \le C(k+1)^\ell \|f\|,$$

according to assumption (ii), without loss of generality, we may assume the function $\psi(x)$ is strictly increasing on $[0, \infty)$.

By (3.1.1) and the Abelian transform, it suffices to prove

$$\sum_{k=0}^{n_0} \Big| \triangle^{\ell+1} \eta(\frac{\psi(k)}{\psi}) \Big| k^\ell \le C_\eta. \tag{3.1.5}$$

Let $\varphi(x) = \eta(\frac{\psi(x)}{\psi})$. Then a straightforward computation shows

$$|\varphi^{(\ell+1)}(x)| \le C_\eta \sum_{i=1}^{\ell+1} \Big(\frac{\psi(x)}{\psi}\Big)^i \frac{1}{(x+1)^{\ell+1}}. \tag{3.1.6}$$

Noticing

$$\triangle^{\ell+1} \eta(\frac{\psi(k)}{\psi}) = \varphi^{(\ell+1)}(\theta_k)$$

for some $\theta_k \in [k, k+\ell+1]$, we get from (3.1.6)

$$\Big|\triangle^{\ell+1}\eta(\frac{\psi(x)}{\psi})\Big| \le C_\eta \sum_{i=1}^{\ell+1}\Big(\frac{\psi(\theta_k)}{\psi}\Big)^i \frac{1}{\theta_k^{\ell+1}}. \tag{3.1.7}$$

Now substituting (3.1.7) into the left-hand side of (3.1.5), taking into account the monotonicity of $\psi(x)$, we obtain (3.1.5) and complete the proof. □

Lemma 3.1.3. *Suppose $\psi > 0$ and $R_{\psi,\alpha,\ell,m}f$ is defined by (3.1.4). Then*

$$\Big\|\Big(\frac{1}{\psi}\Big)^{\alpha m}\Big(R_{\psi,\alpha,\ell,m}f\Big)^{(\alpha m)}\Big\| \le C\|(I - R_{\psi,\alpha,\ell})^m f\|,$$

with $C > 0$ independent of ψ and f.

Proof. By (3.1.4) and (3.1.2), it is sufficient to prove

$$\Big\|(\frac{1}{\psi})^{\alpha m}(R_{\psi,\alpha,\ell}f)^{(\alpha m)}\Big\| \le C\|(I - R_{\psi,\alpha,\ell})^m f\|. \tag{3.1.8}$$

We begin by fixing η, a C^∞ function of compact support, defined on $\mathbb{R}$, with the properties that $\eta(x) = 1$ for $|x| \le \frac{1}{4}$ and $\eta(x) = 0$ for $|x| \ge \frac{1}{2}$.

Let

$$a(k,\psi) = \Big(1 - \Big(\frac{\psi(k)}{\psi}\Big)^\alpha\Big)^\ell.$$

We decompose the operator $\big(\frac{1}{\psi}\big)^{\alpha m}(R^b_{\psi,\alpha,\ell}f)^{(\alpha m)}$ as

$$(\frac{1}{\psi})^{\alpha m}(R^b_{\psi,\alpha,\ell}f)^{(\alpha m)} = T^1_\psi f + T^2_\psi f, \tag{3.1.9}$$

where

$$T^1_\psi f = \sum_{\psi(k)<\psi} a(k,\psi)\Big(\frac{\psi(k)}{\psi}\Big)^{\alpha m}\eta\Big(\frac{\psi(k)}{\psi}\Big)P_k f,$$
$$T^2_\psi f = \sum_{\psi(k)<\psi} a(k,\psi)\Big(\frac{\psi(k)}{\psi}\Big)^{\alpha m}\Big(1 - \eta(\frac{\psi(k)}{\psi})\Big)P_k f.$$

First, we prove

$$\|T^1_\psi f\| \le C\|(I - R_{\psi,\alpha,\ell})^m f\|, \tag{3.1.10}$$

with $C>0$ independent of ψ and f. To this end, let us rewrite $T_\psi^1 f$ as

$$T_\psi^1 f = \sum_{\psi(k)<\psi} a(k,\psi)\xi(\frac{\psi(k)}{\psi})P_k(h), \tag{3.1.11}$$

where

$$\xi(t) = \frac{\eta(t)t^{\alpha m}}{(1-(1-t^\alpha)^\ell)^m}, \quad \text{and} \quad h = (I - R_{\psi,\alpha,\ell})^m f. \tag{3.1.12}$$

Noticing

$$\xi(t) = \frac{\eta(t)}{(\ell + \sum_{j=1}^{\ell-1}(-1)^j \binom{\ell}{j+1} t^{\alpha j})^m} \in C_0^\infty(\mathbb{R}_+)$$

with $\operatorname{supp}\xi \subset \{t:\ 0 \le t \le \frac{1}{2}\}$, we obtain (3.1.10), by Lemma 3.1.2, (3.1.2) and (3.1.11) –(3.1.12).

Next, we prove (3.1.10) with $T_\psi^1 f$ replaced by $T_\psi^2 f$.

Define

$$U_\psi(g) = \sum_{\psi(k)<\psi} \Big(1-\eta(\frac{\psi(k)}{\psi})\Big)\frac{a(k,\psi)}{(1-a(k,\psi))^m}P_k(g).$$

Below we will prove

$$\|U_\psi(g)\| \le C\|g\|, \tag{3.1.13}$$

with $C>0$ independent of ψ and g.

We rewrite $U_\psi(g)$ as

$$U_\psi(g) = U_\psi^1(g) + U_\psi^2(g),$$

with

$$U_\psi^1(g) = \sum_{\psi(k)<\psi} \Big(1-\eta(\frac{\psi(k)}{\psi})\Big)a(k,\psi)\Big[1+ma(k,\psi)\Big]P_k(g),$$
$$U_\psi^2(g) = \sum_{\psi(k)<\psi} \Big(1-\eta(\frac{\psi(k)}{\psi})\Big)a(k,\psi)\Big[\frac{1}{(1-a(k,\psi))^m} - 1 - ma(k,\psi)\Big]P_k(g).$$

By Lemma 3.1.2 and (3.1.2), one can easily verify

$$\|U_\psi^1(g)\| \le C\|g\|. \tag{3.1.14}$$

To deal with U_ψ^2, we set

$$\varphi(t) = \begin{cases} (1-\eta(t))(1-t^\alpha)^\ell\Big[\frac{1}{\big(1-(1-t^\alpha)^\ell\big)^m} - 1 - m(1-t^\alpha)^\ell\Big], & \text{if } 0 \le t \le 1, \\ 0, & \text{if } t \ge 1. \end{cases}$$

Then, a straightforward computation shows

$$\varphi(t) = \big(1-\eta(t)\big)\frac{(1-t^\alpha)^{3\ell}\sum_{j=0}^{(m-1)\ell} C_{m,\ell,j}t^{\alpha j}}{\big(1-(1-t^\alpha)^\ell\big)^m}, \quad \frac{1}{4} \le t \le 1,$$

where the $C_{m,\ell,j}$ are constants depending only on m, ℓ and j. This clearly implies $\varphi \in C_0^{(\ell+1)}(\mathbb{R}_+)$. Noticing

$$U_\psi^2(g) = \sum_{k=0}^\infty \varphi\big(\frac{\psi(k)}{\psi}\big)P_k(g),$$

by Lemma 3.1.2 , we get (3.1.14), with $U_\psi^2(g)$ in place of $U_\psi^1(g)$. Putting this together, we get (3.1.13).

Now noticing

$$T_\psi^2(f) = \Big(\frac{1}{\psi}\Big)^{\alpha m}\Big(U_\psi\big(I - R_{\psi,\alpha,\ell}\big)^m f\Big)^{(\alpha m)},$$

by Bernstein's inequality (see [Di1]), we get from (3.1.2) and (3.1.13)

$$\|T_\psi^2(f)\| \le C\|(I - R_{\psi,\alpha,\ell})^m f\|,$$

which, together with (3.1.10) and (3.1.9), yields (3.1.8). This completes the proof. □

Now Theorem 3.1.1 is an immediate consequence of (3.1.3) Lemma 3.1.3.

3.1.3 Cesàro means

In this subsection, we will prove

Theorem 3.1.4. *Suppose $\ell \in \mathbb{N}$ and (3.1.1) is satisfied. Then*

$$\|f - \sigma_N^\ell(f)\| \approx K(f, P(D)^{\alpha_0}, \frac{1}{N}),$$

where $\alpha_0 = \big(\deg\psi(x)\big)^{-1}$ and $\psi(x)$ is as in assumption (ii).

Since the coefficients of the Cesàro mean are more complicated than those of Riesz mean, the proof of Theorem 3.1.4 is a little more difficult than that of Theorem 3.1.1. We prove Theorem 3.1.4 on the basis of the following lemmas.

Lemma 3.1.5. *Suppose $\varphi(x), \phi(x)$ are two algebraic polynomials of the same degree. Assume for some positive integer n_0, $\phi(x)\varphi(x) > 0$ for $x \geq n_0$. For a given $r > 0$, define*

$$T(f) = \sum_{k=n_0}^{\infty} \Big(\frac{\varphi(k)}{\phi(k)}\Big)^r P_k f.$$

Then

$$\|T(f)\| \leq C(\varphi, \phi, r, n_0)\|f\|.$$

Proof. Let

$$\Psi(x) = \Big(\frac{\varphi(x)}{\phi(x)}\Big)^r, \quad x \geq n_0.$$

Noticing that $\varphi(x)$ and $\phi(x)$ are polynomials of the same degree, one can easily verify

$$|\Psi^{(\ell+1)}(x)| \leq C\Big(\frac{1}{x+1}\Big)^{\ell+2}, \ x \geq n_0. \tag{3.1.15}$$

Now let us define

$$\mu_k = \begin{cases} \Psi(k), & k \geq n_0, \\ 0, & 0 \leq k < n_0. \end{cases}$$

Using the Abelian transform $\ell + 1$ times, taking into account (3.1.1), we obtain

$$\|T(f)\| \leq C_B \sum_{k=0}^{\infty} |\triangle^{\ell+1}\mu_k| k^\ell \|f\|,$$

which, by (3.1.15), implies the desired result. ∎

Lemma 3.1.6.

$$\|f - \sigma_N^\ell(f)\| \leq CK(f, P(D)^{\alpha_0}, \frac{1}{N}),$$

with $C > 0$ independent of N and f.

Proof. We get the idea from [Di1]. Let $g = R_{\psi(\frac{N}{2}),\alpha_0,\ell,1}f$ with $R_{\psi(\frac{N}{2}),\alpha_0,\ell,1}$ defined as in (3.1.4). Observing that $g \in \bigoplus_{k=0}^{\frac{N}{2}} H_k$ and $\Big(\psi(\frac{N}{2})\Big)^{\alpha_0} \sim N$, we get from (3.1.3) that

$$\|f-g\| + \frac{1}{N}\|g^{(\alpha_0)}\| \leq CK(f, P(D)^{\alpha_0}, \frac{1}{N}).$$

On the other hand, by (3.1.1),

$$\begin{aligned}\|\sigma_N^\ell(f) - f\| \leq & \|\sigma_N^\ell(f) - \sigma_N^\ell(g)\| + \|\sigma_N^\ell(g) - g\| + \|g-f\| \\ \leq & C\|f-g\| + \|\sigma_N^\ell(g) - g\|.\end{aligned}$$

Hence, it suffices to prove

$$\|\sigma_N^\ell(g) - g\| \leq C\frac{1}{N}\|g^{(\alpha_0)}\|. \tag{3.1.16}$$

From assumptions (iii) and (iv), it follows

$$\lim_{N\to\infty} \|\sigma_N^\ell(g) - g\| = 0,$$

which implies

$$\begin{aligned}\sigma_N^\ell(g) - g = & \sum_{k=N}^{\infty}\Big(\sigma_k^\ell(g) - \sigma_{k+1}^\ell(g)\Big) \\ = & -\sum_{k=N}^{\infty}\frac{1}{(k+1+\ell)(k+1)}\sum_{j=0}^{\frac{N}{2}}\frac{A_{k-j}^\ell}{A_k^\ell}\frac{j\ell(k+1)}{k-j+1}P_j(g).\end{aligned} \tag{3.1.17}$$

Let $\eta \in C_0^\infty(\mathbb{R}_+)$ be a C^∞ function, defined on $\mathbb{R}$, with the properties that $\eta(x) = 1$ for $|x| \leq \frac{1}{2}$ and $\eta(x) = 0$ for $|x| \geq \frac{3}{4}$. Then, noticing $g \in \bigoplus_{k=0}^{\frac{N}{2}} H_k$, by (3.1.1) and Lemma 3.1.5, we have

$$\begin{aligned}\Big\|\sum_{j=0}^{\frac{N}{2}}\frac{A_{k-j}^\ell}{A_k^\ell}\frac{j\ell(k+1)}{k-j+1}P_j(g)\Big\| = & \Big\|\sum_{j=0}^{\frac{N}{2}}\frac{A_{k-j}^\ell}{A_k^\ell}\frac{j\ell(k+1)}{k-j+1}\eta(\frac{j}{N})P_j(g)\Big\| \\ \leq & C\Big\|\sum_{j=0}^{\frac{N}{2}}\frac{k+1}{k-j+1}\eta(\frac{j}{N})P_j(g^{(\alpha_0)})\Big\| \\ \leq & C\sum_{j=0}^{\frac{3}{4}N}\Big|\overrightarrow{\triangle}^{\ell+1}\Big(\eta(\frac{j}{N})\frac{k+1}{k+1-j}\Big)\Big|(j+1)^\ell\Big\|g^{(\alpha_0)}\Big\|.\end{aligned} \tag{3.1.18}$$

A straightforward computation shows

$$\Big|\overrightarrow{\triangle}^{\ell+1}\Big(\eta(\frac{j}{N})\frac{k+1}{k+1-j}\Big)\Big| \leq C\Big(\frac{1}{N}\Big)^{\ell+1}, 0 \leq j \leq \frac{3}{4}N \leq \frac{3}{4}k. \tag{3.1.19}$$

Now substituting (3.1.19) into (3.1.18), taking account into (3.1.17), we obtain (3.1.16) and complete the proof.

□

Lemma 3.1.7.

$$\Big(\frac{1}{\psi(N)}\Big)^{\alpha_0}\Big\|\Big(\sigma_N^\ell(f)\Big)^{(\alpha_0)}\Big\| \leq C\|f-\sigma_N^\ell(f)\|,$$

with C independent of N and f.

To prove Theorem 3.1.7, we have to use some properties of the coefficients of the Cesàro mean, which are contained in Lemmas 3.1.8–3.1.10 below.

Lemma 3.1.8. *Let*

$$a_k = \begin{cases} \frac{N}{k}\big(1-\frac{A_{N-k}^\ell}{A_N^\ell}\big), & 1 \leq k \leq N, \\ 0, & k \geq N+1. \end{cases}$$

Then for $i = 0, 1, \cdots, \ell+1$,

$$|\overrightarrow{\triangle}^i a_k| \leq C\Big((\frac{1}{N})^i + (\frac{1}{k+1})^{i+1}\Big),$$

with $1 \leq k \leq [\frac{3}{4}N]+1$,

Proof. We rewrite a_k as

$$a_k = b_k + c_k, \tag{3.1.20}$$

with

$$\begin{aligned} b_k &= \frac{N}{k}\Big(1-(1-\frac{k}{N})^\ell\Big) \\ c_k &= \frac{N}{k}\Big((1-\frac{k}{N})^\ell - \frac{A_{N-k}^\ell}{A_N^\ell}\Big). \end{aligned} \tag{3.1.21}$$

Let $\varphi(t) = \frac{1}{t}(1-(1-t)^\ell)$. Noticing that

$$\varphi(t) = \sum_{j=1}^{\ell} \binom{\ell}{j} (-1)^{j-1} t^{j-1} \in C^\infty[0,\infty]$$

and $b_k = \varphi(\frac{k}{N})$, we get for $i \in \mathbb{Z}_+$,

$$|\overrightarrow{\triangle}^i b_k| \le C\Big(\frac{1}{N+1}\Big)^i, \ \ 1 \le k \le N.$$

Hence, by (3.1.20), it remains to show for $i = 0, 1, \cdots, \ell+1$,

$$|\overrightarrow{\triangle}^i c_k| \le C\Big(\frac{1}{k+1}\Big)^{i+1}, \ \ 1 \le k \le [\frac{3}{4}N] + 1.$$

On account of (3.1.21), it suffices to prove for $0 \le i \le \ell + 1$,

$$\Big|\overrightarrow{\triangle}^i\Big((1-\frac{k}{N})^\ell - \frac{A^\ell_{N-k}}{A^\ell_N}\Big)\Big| \le C_i \frac{1}{(N+1)^{i+1}}, \ \ 1 \le k \le [\frac{3}{4}N]+1. \tag{3.1.22}$$

Noticing that

$$\overrightarrow{\triangle} A^\delta_{N-k} = A^{\delta-1}_{N-k}, \ \ \delta > -1, \tag{3.1.23}$$

and (see [Zy, P77, (1.18)]),

$$A^\delta_k = \frac{k^\delta}{\Gamma(\delta+1)}\Big(1 + O\big(\frac{1}{k}\big)\Big), \ \ \delta > -1, \ \ k \in \mathbb{N},$$

we get for $0 \le k \le [\frac{3}{4}N] + 1$,

$$\overrightarrow{\triangle}^i \frac{A^\ell_{N-k}}{A^\ell_N} = \begin{cases} \frac{A^{\ell-i}_{N-k}}{A^\ell_N} = \frac{\Gamma(\ell+1)}{\Gamma(\ell+1-i)} \frac{(N-k)^{\ell-i}}{N^\ell}\Big(1 + O(\frac{1}{N})\Big), & \text{if } 0 \le i \le \ell, \\ 0, & \text{if } i = \ell+1. \end{cases} \tag{3.1.24}$$

On the other hand, it is easy to verify that for $0 \le k \le [\frac{3}{4}N] + 1$,

$$\overrightarrow{\triangle}^i (1-\frac{k}{N})^\ell = \begin{cases} \frac{\Gamma(\ell+1)}{\Gamma(\ell-i+1)}\Big(\frac{1}{N}\Big)^i (1 - \frac{k+\theta_{i,k}}{N})^{\ell-i}, & \text{if } 0 \le i \le \ell, \\ 0, & \text{if } i = \ell+1, \end{cases} \tag{3.1.25}$$

with

$$0 < \theta_{i,k} < i, \ \ i = 0, 1, \cdots, \ell.$$

Now combining (3.1.24) with (3.1.25), we get for $0 \le i \le \ell$, $0 \le k \le [\frac{3}{4}N]+1$,

$$\begin{aligned}\overrightarrow{\triangle}^i\Big((1-\frac{k}{N})^\ell - \frac{A^\ell_{N-k}}{A^\ell_N}\Big) =& \frac{\Gamma(\ell+1)}{\Gamma(\ell-i+1)}\frac{1}{N^i}\Big((1-\frac{k+\theta_i}{N})^{\ell-i} - (1-\frac{k}{N})^{\ell-i} + O(\frac{1}{N})\Big)\\ =& O\Big((\frac{1}{N})^{i+1}\Big),\end{aligned}$$

and for $i=\ell+1$, $0\le k\le [\frac{3}{4}N]+1$,

$$\overrightarrow{\triangle}^{\ell+1}\Big((1-\frac{k}{N})^\ell - \frac{A^\ell_{N-k}}{A^\ell_N}\Big) = 0.$$

which gives (3.1.22) and completes the proof. □

Lemma 3.1.9. *Suppose* $a_k \ge \delta > 0,\ k=0,1,\cdots$. *Then*

$$\Big|\overrightarrow{\triangle}^n \frac{1}{a_k}\Big| \le C(\delta,n)\sup\Big\{\Big|\overrightarrow{\triangle}^{i_1}a_{k+j_1}\cdots\overrightarrow{\triangle}^{i_m}a_{k+j_m}\Big| : 1\le i_u,\ j_u\le n,\ 1\le u\le m\le n,\\ i_1+i_2+\cdots+i_m = n\Big\}.$$

Lemma 3.1.9 can be easily obtained by induction on $n=0,1,\cdots$ and using the following two identities:

$$\begin{aligned}\overrightarrow{\triangle}\frac{1}{a_k} &= -\frac{\overrightarrow{\triangle}a_k}{a_k a_{k+1}},\\ \overrightarrow{\triangle}^{n+1}\frac{1}{a_k} &= -\overrightarrow{\triangle}^n\Big(\frac{\overrightarrow{\triangle}a_k}{a_k a_{k+1}}\Big)\\ &= -\sum_{j=0}^n\binom{n}{j}\Big(\overrightarrow{\triangle}^{n-j+1}a_{k+j}\Big)\Big(\sum_{i=0}^j\binom{j}{i}(\overrightarrow{\triangle}^i\frac{1}{a_k})\overrightarrow{\triangle}^{j-i}(\frac{1}{a_{k+1+i}})\Big).\end{aligned}$$

Lemma 3.1.10. *Let*

$$\mu_k = \begin{cases}\frac{A^\ell_{N-k}}{A^\ell_N}, & 0\le k\le N,\\ 0, & k\ge N+1.\end{cases}$$

Then for $i=0,1,\cdots,\ell+1$,

$$\Big|\overrightarrow{\triangle}^i(\frac{\mu_k^2}{1-\mu_k})\Big| \le C(\frac{1}{N})^i, \tag{3.1.26}$$

with $\frac{N}{8}\le k\le N$.

Proof. First, we prove for $i = 0, 1, \cdots, \ell$,

$$\Big|\overrightarrow{\triangle}^i \frac{1}{1-\mu_k}\Big| \le C\Big(\frac{1}{N}\Big)^i, \ k \ge \frac{N}{8}, \tag{3.1.27}$$

and for $i = \ell + 1$

$$\Big|\overrightarrow{\triangle}^{\ell+1} \frac{1}{1-\mu_k}\Big| \le \begin{cases} C\Big(\frac{1}{N}\Big)^{\ell+1}, & \text{if } \frac{N}{8} \le k \le N-\ell-2, \\ C\Big(\frac{1}{N}\Big)^{\ell}, & \text{if } N-\ell-1 \le k \le N+\ell+1. \end{cases} \tag{3.1.28}$$

Notice that

$$\overrightarrow{\triangle}^i a_k = \overrightarrow{\triangle}^i a_{N+1+\ell} + \sum_{j=k}^{N+\ell} \overrightarrow{\triangle}^{i+1} a_j, \ i \ge 0. \tag{3.1.29}$$

It is sufficient to prove (3.1.28). By (3.1.23), it is easy to verify that for $0 \le i \le \ell$,

$$|\overrightarrow{\triangle}^i \mu_k| \le C\Big(\frac{1}{N}\Big)^i, \ k \ge 0, \tag{3.1.30}$$

and for $i = \ell + 1$

$$\overrightarrow{\triangle}^{\ell+1}\mu_k = \begin{cases} \frac{1}{A_N^\ell} = O(\frac{1}{N^\ell}), & \text{if } k = N \\ 0, & \text{otherwise.} \end{cases} \tag{3.1.31}$$

A straightforward computation shows

$$1 - \mu_k \ge 1 - \frac{A^\ell_{\frac{7N}{8}}}{A_N^\ell} \ge \frac{1}{16} \tag{3.1.32}$$

whenever $k \ge \frac{N}{8}$.

Now combining (3.1.30)–(3.1.32), on account of Lemma 3.1.9, we derive (3.1.28) and hence (3.1.27).

Next, we prove for $i = 0, \cdots, \ell + 1$,

$$|\overrightarrow{\triangle}^i \mu_k^2| \le C\Big(\frac{1}{N}\Big)^i, \quad k \ge \frac{N}{8}. \tag{3.1.33}$$

By (3.1.29), it is enough to consider the case $i = \ell + 1$. In fact, (3.1.33) for $\ell + 1$ is a consequence of (3.1.19), (3.1.31) and the following identity

$$\overrightarrow{\triangle}^{\ell+1}\mu_k^2 = \mu_k \overrightarrow{\triangle}^{\ell+1}\mu_k + \mu_k \overrightarrow{\triangle}^{\ell+1}\mu_{k+\ell+1} + \sum_{i=1}^{\ell} \binom{\ell+1}{i} \overrightarrow{\triangle}^i \mu_k \overrightarrow{\triangle}^{\ell+1-i}\mu_{k+i}.$$

Finally, we prove (3.1.26).

Again, by (3.1.29), we only need to prove (3.1.26) for $i = \ell + 1$.

We write

$$\Big|\overrightarrow{\triangle}^{\ell+1}(\frac{\mu_k^2}{1-\mu_k})\Big| = \Big|\frac{\overrightarrow{\triangle}^{\ell+1}\mu_k^2}{1-\mu_{k+\ell+1}} + \mu_k^2\overrightarrow{\triangle}^{\ell+1}\Big(\frac{1}{1-\mu_k}\Big) + \sum_{i=1}^{\ell}\binom{\ell+1}{i}\overrightarrow{\triangle}^{i}\mu_k^2\overrightarrow{\triangle}^{\ell+1-i}\Big(\frac{1}{1-\mu_{k+i}}\Big)\Big|,$$

which, by (3.1.30), (3.1.31) and (3.1.33), is estimated by

$$C\Big(\frac{1}{N}\Big)^{\ell+1},$$

which gives (3.1.26) and completes the proof. □

Proof of Lemma 3.1.7. Without loss of generality, we may assume $P_0(f) = 0$. Let $\eta \in C_0^\infty(\mathbb{R}_+)$ such that $\eta(x) = 1$ for $|x| \le \frac{1}{4}$ and $\eta(x) = 0$ for $|x| \ge \frac{1}{2}$. We decompose $\Big(\frac{1}{\psi(N)}\Big)^{\alpha_0}\Big(\sigma_N^\ell(f)\Big)^{(\alpha_0)}$ as the sum

$$T_N^1(f) + T_N^2(f), \tag{3.1.34}$$

with

$$T_N^1(f) := \sum_{k=0}^{N}\Big(\frac{\psi(k)}{\psi(N)}\Big)^{\alpha_0}\frac{A_{N-k}^\ell}{A_N^\ell}\eta(\frac{k}{N})P_k(f),$$
$$T_N^2(f) := \sum_{k=0}^{N}\Big(\frac{\psi(k)}{\psi(N)}\Big)^{\alpha_0}\frac{A_{N-k}^\ell}{A_N^\ell}(1-\eta(\frac{k}{N}))P_k(f).$$

First, we prove

$$\|T_N^1(f)\| \le C\|f - \sigma_N^\ell(f)\|. \tag{3.1.35}$$

By Berntein's inequality and (3.1.1), we get

$$\|T_N^1(f)\| \le C\Big\|\sum_{k=1}^{N}\frac{k}{N}\eta(\frac{k}{N})P_k(f)\Big\|. \tag{3.1.36}$$

Define

$$G_N^1(g) = \sum_{k=1}^{N}\eta(\frac{k}{N})\frac{\frac{k}{N}}{1-\frac{A_{N-k}^\ell}{A_N^\ell}}P_k(g). \tag{3.1.37}$$

Observe that for $1 \le k \le N$,

$$\frac{N}{k}(1-\frac{A_{N-k}^\ell}{A_N^\ell}) \ge \frac{N}{k}\Big(1-(1-\frac{\ell}{N+\ell})^k\Big) \ge \frac{1}{2}\frac{1}{\ell+1} > 0.$$

From Lemmas 3.1.8 and 3.1.9, we get,

$$\Big|\overrightarrow{\triangle}^{\ell+1}\Big(\eta(\frac{k}{N})\frac{1}{\frac{N}{k}(1-\frac{A_{N-k}^{\ell}}{A_N^{\ell}})}\Big)\Big| \leq C\Big(\Big(\frac{1}{k+1}\Big)^{\ell+2}+\Big(\frac{1}{N}\Big)^{\ell+1}\Big),$$

which, by (3.1.1), implies

$$\|G_N^1(g)\| \leq C\|g\|. \tag{3.1.38}$$

Now combining (3.1.36)–(3.1.38), we obtain (3.1.35).

Next, we prove

$$\|T_N^2(f)\| \leq C\|f-\sigma_N^{\ell}(f)\|. \tag{3.1.39}$$

To this end, we define

$$G_N^2(g) = \sum_{k=0}^{N}\frac{A_{N-k}^{\ell}}{A_N^{\ell}}\frac{1}{1-\frac{A_{N-k}^{\ell}}{A_N^{\ell}}}(1-\eta(\frac{k}{N}))P_k(g).$$

We decompose G_N^2 as

$$G_N^2(g) = G_N^{2,1}(g) + G_N^{2,2}(g),$$

with

$$G_N^{2,1}(g) := \sum_{k=0}^{N}\frac{\big(\frac{A_{N-k}^{\ell}}{A_N^{\ell}}\big)^2}{1-\frac{A_{N-k}^{\ell}}{A_N^{\ell}}}(1-\eta(\frac{k}{N}))P_k(g),$$

$$G_N^{2,2}(g) := \sum_{k=0}^{N}\frac{A_{N-k}^{\ell}}{A_N^{\ell}}(1-\eta(\frac{k}{N}))P_k(g).$$

From Lemma 3.1.10 and the Abelian transform, it follows

$$\|G_N^{2,1}(g)\| \leq C\|g\|.$$

On the other hand, by assumption (iv) and Lemma 3.1.2, it is easy to verify $\|G_N^{2,2}(g)\| \leq C\|g\|$.

Thus

$$\|G_N^2(g)\| \leq C\|g\|.$$

Observing

$$T_N^2(f) = \Big\|\big(\frac{1}{\psi(N)}\big)^{\alpha_0}\Big(G_N^2(f-\sigma_N^{\ell}(f))\Big)^{(\alpha_0)}\Big\|,$$

by Bernstein's inequality (see [Di1]), we derive (3.1.39).

Now combining (3.1.34), (3.1.35) and (3.1.39), we obtain Lemma 3.1.7 and complete the proof. □

Now Theorem 3.1.4 follows directly from Lemmas 3.1.6 and 3.1.7 and the fact $\psi(N)^{\alpha_0} \sim N$.

Remark 3.1.1. With a slight modification of the proofs in the above two subsections, we can show that Theorems 3.1.1 and 3.1.4 remain valid with ℓ replaced by a positive number δ for which the assumption (iv) is satisfied.

3.1.4 Special results for $\mathbb{S}^{d-1}$

Now we return to the case of unit sphere $\mathbb{S}^{d-1}$, for which the fractional differential operator D^r, the K -functional $K_r(f,t)_p$, the Cesàro mean σ_N^δ are defined by (0.0.14), (0.0.15), (0.0.7) respectively, and the generalized Riesz mean $R_{\psi,\alpha,\delta}$ is defined by

$$R_{\psi,\alpha,\delta}(f) = \sum_{k(k+2\lambda)<\psi} \Big(1 - (\frac{k(k+2\lambda)}{\psi})^{\alpha}\Big)^{\delta} Y_k(f).$$

With a slight change of the symbols in Sections 3.1.1–3.1.3, on account of Remark 3.1.1 and (0.0.3)–(0.0.5), we obtain the following two theorems.

Theorem 3.1.11. *Let* $\delta_p = \max\Big\{(d-1)\big|\frac{1}{p} - \frac{1}{2}\big| - \frac{1}{2}, 0\Big\}$ *and* $J(d)$ *as in (0.0.10). Then the equivalence*

$$\|\sigma_N^\delta(f) - f\|_p \approx K_1(f, \frac{1}{N})_p$$

is valid if one of the following conditions is satisfied:

(i) $\delta > \lambda,\ 1 \le p \le \infty,$

(ii) $\delta = \lambda,\ \ 1 < p < \infty,$

(iii) $0 \le \delta < \lambda,\ \ \delta > \delta(p),\ \ p \in J(d).$

Theorem 3.1.12. *With the same notations as in Theorem* 3.1.11*, for* $\alpha > 0$*, the equivalence*

$$\|R_{\psi,\alpha,\delta}(f) - f\|_p \approx K_{2\alpha}(f, \frac{1}{\sqrt{\psi}})_p$$

is valid if one of the conditions (i)–(iii) in Theorem 3.1.11 *is satisfied.*

Remark 3.1.2. With a slight modification of the proofs in the above sections, we can easily extend Theorems 3.1.11 and 3.1.12 to the case of Hardy space $H^p(\mathbb{S}^{d-1})$. This will be discussed in Chapter 4 of this thesis.

3.2 Realization of the K-functional related to the Laplace-Beltrami operator on $\mathbb{S}^{d-1}$

3.2.1 Two basic lemmas on Jacobi polynomials

Throughout the rest of this chapter, we assume $\alpha > \beta > -\frac{1}{2}$ and $R_k^{(\alpha,\beta)}(t)$ denotes the following normalized Jacobi polynomial:

$$R_k^{(\alpha,\beta)}(x) := \frac{P_k^{(\alpha,\beta)}(x)}{P_k^{(\alpha,\beta)}(1)}.$$

From [Sz] we have

$$P_k^{(\alpha,\beta)}(1) = \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)\Gamma(\alpha+1)}. \tag{3.2.1}$$

For a sequence $\{u_k\}_{k=0}^\infty$ of complex numbers, recall that $\triangle u_k = u_k - u_{k+1}$ and $\triangle^{\ell+1} u_k = \triangle^\ell(\triangle u_k),\ \ell \in \mathbb{N}$.

The main goal in this subsection is to prove the following two lemmas, which will be used repeatedly in the later sections.

Lemma 3.2.1. *Let $\varepsilon \in (0, \frac{\pi}{4})$ and $\theta \in (0, \pi - \varepsilon)$. Then for $\ell, k \in \mathbb{Z}_+$,*

$$|\triangle^\ell R_k^{(\alpha,\beta)}(x)| \le C(\ell,\alpha,\beta,\varepsilon) \begin{cases} \theta^\ell, & 0 < \theta \le k^{-1}, \\ (\frac{1}{k\theta})^{\alpha+\frac{1}{2}}\theta^\ell, & k^{-1} \le \theta \le \pi - \varepsilon, \end{cases}$$

with $C(\ell,\alpha,\beta,\varepsilon) > 0$ independent of k and θ.

Proof. We use the formula ([An-As-R], p. 304, (6.4.20))

$$P_k^{(\alpha+1,\beta)}(\cos\theta) = \frac{2}{2k+\alpha+\beta+2} \frac{(k+\alpha+1)P_k^{(\alpha,\beta)}(x) - (k+1)P_{k+1}^{(\alpha,\beta)}(\cos\theta)}{1-\cos\theta}.$$

We then get from (3.2.1)

$$\triangle R_k^{(\alpha,\beta)}(\cos\theta) = R_k^{(\alpha,\beta)}(\cos\theta) - R_{k+1}^{(\alpha,\beta)}(\cos\theta) = (1-\cos\theta)\frac{2k+\alpha+\beta+2}{2(\alpha+1)}R_k^{(\alpha+1,\beta)}(\cos\theta).$$

Thus, for $\ell \in \mathbb{N}$,

$$\begin{aligned}\triangle^{\ell+1} R_k^{(\alpha,\beta)}(\cos\theta) =& \frac{1-\cos\theta}{\alpha+1}\triangle^{\ell}\Big((k+\frac{\alpha+\beta}{2}+1)R_k^{(\alpha+1,\beta)}(\cos\theta)\Big)\\ =& \frac{1-\cos\theta}{\alpha+1}\Big((k+\frac{\alpha+\beta}{2}+1)\triangle^{\ell}R_k^{(\alpha+1,\beta)}(\cos\theta) - \ell\triangle^{\ell-1}R_{k+1}^{(\alpha+1,\beta)}(\cos\theta)\Big).\end{aligned}$$

Now induction on ℓ and invoking (0.0.24),(3.2.1) give the required estimates for $\triangle^{\ell}R_k^{(\alpha,\beta)}(\cos\theta)$. This completes the proof.

□

Lemma 3.2.2. *Let m be a fixed positive integer, $\varepsilon \in (0, \frac{\pi}{4})$ and $\theta \in [0, \pi-\varepsilon]$. Then there exists a $\tau := \tau(\alpha,\beta,m,\varepsilon) > 0$ such that for $j \geq m$ and $k\theta \geq \tau$,*

$$\Big|\triangle^{\ell}\Big(R_k^{(\alpha,\beta)}(x)\Big)^j\Big| \leq C(\alpha,\beta,m,\varepsilon)\Big(\frac{1}{2}\Big)^j\Big(\frac{1}{k\theta}\Big)^{m(\alpha+\frac{1}{2})}\theta^{\ell}, \tag{3.2.2}$$

with $\ell = 0, 1, \cdots, [\alpha]+2$.

Proof. Let $\tau = \tau(\alpha,\beta,m,\varepsilon)$ be a specified constant. It will, in fact, be convenient to give its precise definition later.

(3.2.2) for $j = m$, $\ell = 0$ and $k\theta \geq \tau$ is an immediate consequence of (0.0.24) and (3.2.1). (3.2.2) for $j = m$, $1 \leq \ell \leq [\alpha]+2$ and $k\theta \geq \tau$ follows from Lemma 3.2.1 and the following identity

$$\begin{aligned}\triangle^{\ell}(\mu_k)^m = & \sum_{\substack{v_1+\cdots+v_{m-1}\leq \ell\\ 0\leq v_i\leq \ell\\ 1\leq i\leq m-1}} \binom{\ell}{v_1}\binom{\ell-v_1}{v_2}\cdots\binom{\ell-v_1-\cdots-v_{m-2}}{v_{m-1}}\times\\ & \times\prod_{i=1}^{m-1}\Big(\triangle^{i}\mu_{k+v_i}\Big)\triangle^{\ell-v_1-\cdots-v_{m-1}}\mu_k,\end{aligned}$$

which can be easily obtained by induction on ℓ.

Now assume (3.2.2) holds for $j = N \geq m,\ 1 \leq \ell \leq [\alpha]+2,\ k\theta \geq \tau$. More precisely, we assume for $1 \leq \ell \leq [\alpha]+2,\ k\theta \geq \tau$

$$\Big|\triangle^\ell\Big(R_k^{(\alpha,\beta)}(\cos\theta)\Big)^N\Big| \leq C_2(\alpha,\beta,\varepsilon,m)\Big(\frac{1}{2}\Big)^N\Big(\frac{1}{k\theta}\Big)^{m(\alpha+\frac{1}{2})}\theta^\ell. \tag{3.2.3}$$

Below we will prove inequality (3.2.3), with $N+1$ in place of N, remains valid.

We use the identity:

$$\triangle^\ell\Big(R_k^{(\alpha,\beta)}(\cos\theta)\Big)^{N+1} = \sum_{v=0}^{\ell}\binom{\ell}{v}\triangle^v R_k^{(\alpha,\beta)}(\cos\theta)\triangle^{\ell-v}\Big(R_{k+v}^{(\alpha,\beta)}(\cos\theta)\Big)^N,$$

which, together with Lemma 3.2.1 and our assumption for $j = N$, gives

$$|\triangle^\ell(R_k^{(\alpha,\beta)}(\cos\theta))^{N+1}| \leq C_2(\alpha,\beta,m,\varepsilon)(\frac{1}{2})^{N+1}(\frac{1}{k\theta})^{m(\alpha+\frac{1}{2})}\theta^\ell\Big(C_1(\alpha,\beta,m,\varepsilon)(\frac{1}{k\theta})^{\alpha+\frac{1}{2}}\Big),$$

where $C_2(M,\alpha,\beta,\varepsilon)$ is the same as in (3.2.3) and $C_1(M,\alpha,\beta,\varepsilon)$ is a constant independent of N,k,θ.

Now taking

$$\tau = 2\Big(C_1(\alpha,\beta,m,\varepsilon)\Big)^{\frac{1}{\alpha+\frac{1}{2}}},$$

noting $\alpha > -\frac{1}{2}$, we obtain (3.2.3), with $N+1$ in place of N, for $k\theta \geq \tau,\ 0 \leq \ell \leq [\alpha]+2$. This completes the proof. □

3.2.2 Average on the cap of $\mathbb{S}^{d-1}$

For $t \in (0,\pi)$, let

$$\Phi(t) = \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}\int_0^t \sin^{d-2} u\, du.$$

Given $f \in L(\mathbb{S}^{d-1})$, the average $B_t(f)$ on the cap of $\mathbb{S}^{d-1}$ is defined by

$$B_t(f,x) := \frac{1}{\Phi(t)}\int_{\{y\in\mathbb{S}^{d-1}:\ xy\geq\cos t\}} f(y)\, d\sigma(y),\ x \in \mathbb{S}^{d-1}, \tag{3.2.4}$$

while the *r-th order average* $B_{t,r}(f)$ (for a given $r > 0$) is defined by

$$B_{t,r}(f)(x) = \sum_{j=1}^{\infty}\binom{\frac{r}{2}}{j}(-1)^{j+1}B_t^j(f)(x). \tag{3.2.5}$$

The following equivalence was proved in [Di-Ru1]:

$$K(f, D, t^2)_p \approx \|f - B_t f\|_p + \|f - B_{\frac{t}{M}} f\|_p, \tag{3.2.6}$$

where $f \in L^p(\mathbb{S}^{d-1}),\ 1 \le p \le \infty,\ t \in [0, \pi]$ and $M > 0$ is independent of f and t.

The main result in this subsection is the following theorem, which, in the special case $r = 2$, improves (3.2.6) and answers a problem raised in [Di-Ru1].

Theorem 3.2.3. *Let $t \in (0, \pi)$, $r > 0$ and $1 \le p \le \infty$. Then for $f \in L^p(\mathbb{S}^{d-1})$,*

$$K(f, D^{\frac{r}{2}}, t^r)_p \approx \|f - B_{t,r} f\|_p.$$

The proof of Theorem 3.2.3 is based on the following lemmas.

Lemma 3.2.4. *For $k \in \mathbb{Z}_+,\ t > 0$, let*

$$a(k, t) = \frac{1}{\int_0^t \sin^{d-2}\theta} \int_0^t P_k^d(\cos\theta) \sin^{d-2}\theta \, d\theta. \tag{3.2.7}$$

Then the following two statements hold:

(i) For every $f \in L^1(\mathbb{S}^{d-1})$,

$$Y_k(B_t(f))(x) = a(k, t) Y_k(f)(x). \tag{3.2.8}$$

(ii)

$$a(k, t) = \frac{\sin^{d-1} t}{(d-1)\int_0^t \sin^{d-2}\theta \, d\theta} P_{k-1}^{d+2}(\cos t). \tag{3.2.9}$$

Proof. (3.2.8) is a direct consequence of (3.2.4), while (3.2.9) can be obtained by using the following formula twice:

$$P_k^d(\cos\theta) = \frac{2^\lambda \Gamma(\lambda + \frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(\lambda)} (\sin\theta)^{1-2\lambda} \int_0^\theta \cos(k+\lambda)u(\cos u - \cos\theta)^{\lambda-1} \, du. \tag{3.2.10}$$

This completes the proof. □

Lemma 3.2.5. *Suppose $\varepsilon \in (0, \frac{\pi}{4}),\ t \in (0, \pi - \varepsilon),\ r > 0$ and $1 \le p \le \infty$.Let η be a C^∞ function on $\mathbb{R}$ with the properties that $\eta(x) = 1$ for $0 \le |x| \le 1,\ \ \eta(x) = 0$ for $|x| \ge 2$ and $0 \le \eta(x) \le 1$ for all $x \in \mathbb{R}$. Define the operator V_z (for a given $z > 0$) by*

$$V_z(f)(x) = \sum_{k=0}^\infty \eta(\frac{k}{z}) Y_k(f)(x).$$

Then there exists a constant $\gamma := \gamma(d,\varepsilon)$, which is sufficiently large and independent of t, such that for any $f \in L^p(\mathbb{S}^{d-1})$,

$$\|f - V_{[\frac{\beta}{t}]}f\|_p \le C(p,d,\varepsilon,r)\|f - B_{t,r}(f)\|_p$$

holds whenever $\beta \in [\gamma,\gamma^2]$.

Proof. It follows from (3.2.5) and (3.2.8) that

$$Y_k(f - B_{t,r}f)(x) = \Big(1 - a(k,t)\Big)^{\frac{r}{2}} Y_k(f). \tag{3.2.11}$$

For simplicity, we write $N = [\frac{\beta}{t}]$. By (3.2.11), we can decompose $f - V_N(f)$ as follows:

$$f(x) - V_N(f)(x) = \sum_{j=0}^{\infty} \frac{\Gamma(j+\frac{r}{2})}{j!\Gamma(\frac{r}{2})}\Big(\frac{1}{2}\Big)^j T^j(f - B_{t,r}(f))(x),$$

with

$$T^j(g)(x) = \sum_{k=N}^{\infty} \Big(1 - \eta(\frac{k}{N})\Big)(2a(k,t))^j Y_k(g)(x).$$

On account of the fact

$$\frac{\Gamma(j+\frac{r}{2})}{j!\Gamma(\frac{r}{2})} = O(j^{\frac{r}{2}-1}),$$

it suffices to prove

$$\|T^j(g)\|_p \le C_p\|g\|_p, \tag{3.2.12}$$

with $C_p > 0$ independent of j and N.

(3.2.12) for $j = 0, 1$ is an immediate consequence of the following two facts:

$$\|B_t\|_{(p,p)} = 1,$$

$$\sup_k \|V_k\|_{(p,p)} \le C_p < \infty.$$

Now assume $j \ge 2$. Noticing that

$$0 < \frac{\sin^{d-1} t}{(d-1)\int_0^t \sin^{d-2}\theta\, d\theta} < 1,\ \ t \in (0,\pi),$$

we invoke Lemma 3.2.2 with $m = 2$ to obtain

$$\Big|\triangle^\ell\big(2a(k,t)\big)^j\Big| \le C_d'\Big(\frac{1}{kt}\Big)^{2\lambda+2} t^\ell,\quad \ell = 0, 1, \cdots, [\lambda]+2,$$

where $kt \geq \tau := \tau(d,\varepsilon)$. We choose $\gamma := \gamma(d,\varepsilon)$, sufficiently large, so that $([\frac{\beta}{t}]-d-1)t > \tau$ whenever $\beta \in [\gamma, \gamma^2]$. Then using the formula

$$\triangle^{\ell}(a_k b_k) = \sum_{v=0}^{\ell} C_{\ell}^{v} (\triangle^{v} a_k) \triangle^{\ell-v} b_{k+v}, \tag{3.2.13}$$

we conclude

$$\sum_{k=N+1-d}^{\infty} \Big| \triangle^{[\lambda]+2} \Big((1-\eta(\frac{k}{N}))(2a(k,t))^{j} \Big) \Big| k^{[\lambda]+1} \leq C_d < \infty,$$

which, by Lemma 0.0.17, implies (3.2.12) and completes the proof. □

For the remainder of this section, we assume $V_z(f)$ always denotes the operator defined as in Lemma 3.2.5.

Lemma 3.2.6. *Suppose $\varepsilon \in (0, \frac{\pi}{4})$, $t \in (0, \pi - \varepsilon)$ and $1 \leq p \leq \infty$. Let a be a given positive number and $N \in [\frac{a}{t}, \frac{a^2}{t}]$ a non-negative integer. Then for $f \in L^p(\mathbb{S}^{d-1})$*

$$\frac{1}{N^r} \Big\| \big(V_N f\big)^{(r)} \Big\|_p \leq C \| f - B_{t,r}(f) \|_p,$$

with $C > 0$ independent of f and t.

Proof. It follows from (3.2.11) and (3.2.7) that

$$\frac{1}{N^r} \Big(V_N f\Big)^{(r)}(x) = O(1) \sum_{k=0}^{2N} \eta(\frac{k}{N}) \Big(\frac{1}{b(k,t)} \Big)^{\frac{r}{2}} Y_k(f - B_{t,r}(f))(x),$$

where

$$b(k,t) = \frac{N^{d+1}}{k(k+2\lambda)} \int_0^t (1 - P_k^d(\cos\theta)) \sin^{d-2}\theta \, d\theta. \tag{3.2.14}$$

Thus, by Lemma 0.0.17, it will suffice to prove

$$\Big| \triangle^{\ell} \Big(\frac{1}{b(k,t)} \Big)^{\frac{r}{2}} \Big| \leq C \Big(\frac{1}{N} \Big)^{\ell}, \quad \ell = 0, 1, \cdots, d. \tag{3.2.15}$$

From Lemma 3.1.9, (3.2.15) is a consequence of the following two inequalities:

$$b(k,t) \geq C_d > 0, \quad \text{for} \quad 0 \leq k \leq 2N, \tag{3.2.16}$$

$$|\triangle^{i} b(k,t)| \leq C_d \frac{1}{(N+1)^{i}}, \quad \text{for} \quad 0 \leq k \leq 2N, \ i = 0, 1, \cdots, d. \tag{3.2.17}$$

We first prove (3.2.16). We use the following two facts about the ultraspherical polynomials P_k^d:

$$|P_k^d(u)| \le P_k^d(1) = 1, \tag{3.2.18}$$

$$\frac{d}{du}P_k^d(u) = \frac{k(k+d-2)}{d-1}P_{k-1}^{d+2}(u). \tag{3.2.19}$$

We then get

$$\begin{aligned} b(k,t) &\ge \frac{N^{d+1}}{k(k+2\lambda)}\int_0^{\min(\frac{a}{N},\frac{1}{8N})}(1-P_k^d(\cos\theta))\sin^{d-2}\theta\,d\theta \\ &= N^{d+1}\int_0^{\min(\frac{a}{N},\frac{1}{8N})}\int_0^{\theta}P_{k-1}^{d+2}(\cos u)\sin u\,du\sin^{d-2}\theta\,d\theta. \end{aligned} \tag{3.2.20}$$

Noticing that $P_{k-1}^{d+2}(1) = 1$, invoking Bernstein's inequality for trigonometric polynomials, we get for $0 < \theta < \frac{1}{2(k-1)}$,

$$P_{k-1}^{d+2}(\cos\theta) \ge \frac{1}{2}. \tag{3.2.21}$$

Substituting (3.2.21) into (3.2.20) yields for $0 \le k \le 2N$,

$$b(k,t) \ge \frac{N^{d+1}}{2}\int_0^{\min(\frac{a}{N},\frac{1}{2N})}\int_0^{\theta}\sin u\,du\sin^{d-2}\theta\,d\theta \ge C_d > 0,$$

which is (3.2.16).

Next, we prove (3.2.17). We apply the formula (3.2.10) to obtain

$$b(k,t) = C(\lambda)N^{d+1}\int_0^t\sin^{d-2}\theta\int_0^{\theta}(\sin u)^{-(d-2)}\int_0^u\cos(k+\lambda)v(\cos v-\cos u)^{\lambda}\,dv\,du\,d\theta, \tag{3.2.22}$$

which obviously gives the definition of $b(k,t)$ for all $k \in \mathbb{R}$. Hence, $b(k,t)$ can be regarded as a C^∞ function of $k \in \mathbb{R}$. From (3.2.22), it is easy to check that for $0 \le k \le 2N$,

$$|(\frac{d}{dk})^{\ell}b(k,t)| \le C(\lambda)(\frac{1}{N})^{\ell}, \quad \ell \in \mathbb{N}, \tag{3.2.23}$$

which implies (3.2.17) and completes the proof. □

Lemma 3.2.7. *Suppose $t \in (0,\pi)$, $r > 0$ and $1 \le p \le \infty$. Then for $f \in W_p^r$,*

$$\|f - B_{t,r}(f)\|_p \le Ct^{-r}\|f^{(r)}\|_p,$$

with $C > 0$ independent of t and f.

Proof. Suppose $\frac{1}{t} \sim N$. Then

$$\begin{aligned}\|f - B_{t,r}f\|_p \le& \|f - V_N f\|_p + \|V_N f - B_{t,r}(V_N f)\|_p + \|B_{t,r}(V_N f) - B_{t,r}f\|_p \\ \le& C\|f - V_N f\|_p + \|V_N f - B_{t,r}(V_N f)\|_p. \end{aligned} \tag{3.2.24}$$

By Lemma 0.0.17 , one can easily verify

$$\|f - V_N f\|_p \le C t^r \|f^{(r)}\|_p. \tag{3.2.25}$$

Hence, it remains to prove

$$\|V_N f - B_{t,r}(V_N f)\|_p \le C t^r \|f^{(r)}\|_p. \tag{3.2.26}$$

We rewrite $V_N(f) - B_{t,r}(V_N f)$ as

$$V_N f - B_{t,r}(V_N f) = t^r \sum_{k=0}^{2N} \Big(\frac{1 - a(k,t)}{k(k+d-2)t^2}\Big)^{\frac{r}{2}} \eta(\frac{k}{N}) Y_k(f^{(r)}). \tag{3.2.27}$$

It follows from the proof of Lemma 3.2.6 that for $\ell \in \mathbb{Z}_+$, $0 \le k \le 2N$,

$$\Big(\frac{d}{dk}\Big)^{\ell} \Big(\frac{1 - a(k,t)}{k(k+d-2)t^2}\Big)^{\frac{r}{2}} \le C \frac{1}{N^{\ell}}. \tag{3.2.28}$$

Now invoking Lemma 0.0.17 and (3.2.27)–(3.2.28), we get (3.2.26) and conclude the proof. □

Proof of Theorem 3.2.3. The direct inequality

$$\|f - B_{t,r}f\|_p \le C K(f, D^{\frac{r}{2}}, t^r)_p$$

is a consequence of Lemma 3.2.7. It remains to prove the converse inequality

$$K(f, D^{\frac{r}{2}}, t^r)_p \le C\|f - B_{t,r}f\|_p. \tag{3.2.29}$$

Let $\varepsilon > 0$ be a specified sufficiently small constant. (We will give its precise definition later). (3.2.29) for $t \in (0, \pi - \varepsilon)$ is an immediate consequence of the definition the K-functional and Lemmas 3.2.5 and 3.2.6. So, it remains to prove (3.2.29) for $t \in [\pi - \varepsilon, \pi]$.

Without loss of generality, we may assume $\int_{\mathbb{S}^{d-1}} f(x)d\sigma(x) = 0$. Then

$$B_t(f)(x) = \frac{\displaystyle\int_0^t S_\theta(f)(x)\sin^{d-2}\theta d\theta}{\displaystyle\int_0^t \sin^{d-2}\theta d\theta} = \frac{\displaystyle\int_t^\pi S_\theta(f)(x)\sin^{d-2}\theta d\theta}{\displaystyle\int_0^t \sin^{d-2}\theta d\theta}.$$

Noting that $\|S_\theta\|_{(p,p)} = 1$, we get

$$\|B_t f\|_p \le \beta(d,\varepsilon)\|f\|_p,$$

with

$$\beta(d,\varepsilon) = \frac{\int_{\pi-\varepsilon}^{\pi}\sin^{d-2}\theta\, d\theta}{\int_0^{\pi-\varepsilon}\sin^{d-2}\theta\, d\theta} \in (0,1). \tag{3.2.30}$$

Hence, by (3.2.5), we have

$$\|B_{t,r}f\|_p \le \sum_{j=1}^{\infty}\left|\binom{\frac{r}{2}}{j}\right|\beta(d,\varepsilon)^j\|f\|_p. \tag{3.2.31}$$

By (3.2.30), we may select an $\varepsilon > 0$ sufficiently small, so that,

$$\sum_{j=1}^{\infty}\left|\binom{\frac{r}{2}}{j}\right|\beta(d,\varepsilon)^j \le \gamma_d < 1.$$

We then get from (3.2.31)

$$\|f - B_{t,r}f\|_p \ge \|f\|_p - \|B_{t,r}f\|_p \ge (1-\gamma_d)\|f\|_p \ge (1-\gamma_d)K(f, D^{\frac{r}{2}}, t^r)_p,$$

which gives (3.2.29) for $t \in (0, \pi - \varepsilon)$ and completes the proof. □

3.2.3 On the derivative of the average operator: another proof of Theorem 3.2.3

The main goal in this subsection is to prove the following result, which was conjectured in [Di-Ru1].

Theorem 3.2.8. *Let $1 \le p \le \infty$, $\theta \in [0,\pi]$, D the Laplace -Beltrami operator and B_θ the average operator defined by (3.2.4). Then*

$$\lim_{m\to\infty}\sup_{\theta\in[0,\pi]}\|\theta^2 D B_\theta^m\|_{(p,p)} = 0.$$

As pointed out in [Di-Ru1], Theorem 3.2.3 for $r = 2$ in last subsection follows from Theorem 3.2.8. Hence the following proof also gives a proof of Theorem 3.2.8 for $r = 2$, which is different from that in Section 3.2.2.

Proof. First step. Prove

$$\lim_{m\to\infty} \sup_{\theta\in[0,\frac{2}{3}\pi]} \|\theta^2 DB_\theta^m\|_{(2,2)} = 0. \tag{3.2.32}$$

It follows from Lemma 3.2.4 that

$$\theta^2 DB_\theta^m f \sim \sum_{k=0}^{\infty} u_k(m,\theta) Y_k(f), \tag{3.2.33}$$

where

$$u_k(m,\theta) = -k(k+d-2)\theta^2 \left(\frac{\displaystyle\int_0^\theta P_k^d(\cos t)\sin^{d-2} t\, dt}{\displaystyle\int_0^\theta \sin^{d-2} t\, dt} \right)^m. \tag{3.2.34}$$

Hence (3.2.32) is equivalent to the following statement:

$$\lim_{m\to\infty} \sup\Big\{ |u_k(m,\theta)| : \quad k\in\mathbb{N}, \theta\in[0,\frac{2\pi}{3}] \Big\} = 0. \tag{3.2.35}$$

To prove (3.2.35), we consider three cases:

Case 1. $0 \le k\theta \le \frac{1}{2}$.

In this case, we prove

$$|u_k(m,\theta)| \le \frac{C_d}{m+1}. \tag{3.2.36}$$

By Bernstein's inequality for trigonometric polynomials, it is easy to verify that for $0 \le t \le \frac{1}{2k}$,

$$\frac{1}{2} \le P_k^d(\cos t) \le 1 - \frac{1}{\pi^2(d-2)} k^2 t^2. \tag{3.2.37}$$

Substituting (3.2.37) into (3.2.34) yields

$$\Big| u_k(m,\theta) \Big| \le C_d (k\theta)^2 \Big(1 - (c_d k\theta)^2 \Big)^m, \tag{3.2.38}$$

where $0 \le \theta \le \frac{1}{2k}$ and $C_d \in (0,1)$ is an inessential constant. In fact, by a more delicate computation, one may choose

$$C_d = \frac{2^{d-2}(d-1)}{\pi^d (d+1)(d-2)}.$$

Now combining (3.2.38) with the following identity

$$\max_{0 \le t \le 1} t(1-t)^m = \frac{1}{m+1}\Big(1 - \frac{1}{m+1}\Big)^m,$$

gives (3.2.36).

Case 2. $C^{-1} \le k\theta \le C$, with $C > 2$ a constant, which is independent of k, θ and m and may be sufficiently large.

In this case, we shall prove the following inequality:

$$|u_k(m,\theta)| \le C_d r^m, \tag{3.2.39}$$

where $C_d > 0$ and $r \in (0,1)$, both of which are independent of k, m, θ.

Let

$$a_k(\theta) = \frac{1}{\int_0^\theta \sin^{d-2} t\, dt} \int_0^\theta P_k^d(\cos t) \sin^{d-2} t\, dt. \tag{3.2.40}$$

Then clearly, (3.2.39) is an immediate consequence of the following two statements:

$$1 - a_k(\theta) \ge c_d > 0, \tag{3.2.41}$$

$$1 + a_k(\theta) \ge c_d > 0. \tag{3.2.42}$$

We first prove (3.2.41). To this end, we use (3.2.18) and (3.2.19) to obtain

$$1 - a_k(\theta) \ge \frac{1}{\int_0^\theta \sin^{d-2} t\, dt} \int_0^{\frac{1}{2k}} \Big(1 - P_k^d(\cos t)\Big) \sin^{d-2} t\, dt, \tag{3.2.43}$$

which, by (3.2.19), equals to

$$\frac{k(k+2\lambda)}{\int_0^\theta \sin^{d-2} t\, dt} \int_0^{\frac{1}{2k}} \int_0^t P_{k-1}^{d+2}(\cos v) \sin v\, dv \sin^{d-2} t\, dt. \tag{3.2.44}$$

However, by Bernstein's inequality for trigonometric polynomials, the fact $P_{k-1}^{d+2}(1) = 1$, and a straightforward computation, it is easy to verify that

$$(3.2.44) \geq \frac{1}{2} \frac{k(k+2\lambda)}{\int_0^\theta \sin^{d-2} t\, dt} \int_0^{\frac{1}{2k}} \int_0^t \sin v\, dv \sin^{d-2} t\, dt \geq C_d > 0. \tag{3.2.45}$$

Now combining (3.2.43)–(3.2.45) gives (3.2.41).

The proof of (3.2.42) is similar and simpler. In fact, again by Bernstein's inequality and the fact $P_k^d(1) = 1$,

$$\begin{aligned} 1 + a_k(\theta) &= \frac{1}{\int_0^\theta \sin^{d-2} t\, dt} \int_0^\theta \Big(1 + P_k^d(\cos t)\Big) \sin^{d-2} t\, dt \\ &\geq \frac{3}{2} \frac{1}{\int_0^\theta \sin^{d-2} t\, dt} \int_0^{\frac{1}{2k}} \sin^{d-2} t\, dt, \end{aligned}$$

which clearly has a low bound $C_d > 0$. This gives (3.2.42).

Case 3. $\theta \in [0, \frac{2\pi}{3}]$, $k\theta \geq C_2$, *with* $C_2 > 0$ *a constant, which is independent of* m, k, θ *and sufficiently large.*

In this case we prove

$$|u_k(m,\theta)| \leq C(\frac{1}{2})^m, \tag{3.2.46}$$

whenever $C_2 > 0$ is sufficiently large.

It follows from Lemma 3.2.4 that

$$u_k(m,\theta) = -\theta^2 \left(\frac{\sin^{d-1}\theta}{(d-1)\int_0^\theta \sin^{d-2} t\, dt} \right)^m k(k+d-2)\Big(P_{k-1}^{d+2}(\cos\theta)\Big)^m. \tag{3.2.47}$$

Invoking Lemma 3.2.1 , with $\ell = 0$, yields

$$|u_k(m,\theta)| \leq C\Big(\frac{C}{kt}\Big)^{m(\lambda+\frac{3}{2})-2},$$

which, obviously implies (3.2.46).

Now combining the above three cases, putting (3.2.36), (3.2.39) and (3.2.46) together, we conclude

$$|u_k(m,\theta)| \leq C_d \frac{1}{m+1},$$

for all $k, m \in \mathbb{Z}_+$ and $\theta \in [0, \frac{2}{3}\pi]$, which implies (3.2.32).

Second step. Prove

$$\lim_{m\to\infty} \sup_{[0,\frac{2}{3}\pi]} \|\theta^2 D B_\theta^m\|_{(p,p)} = 0, \quad 1 \le p \le \infty. \tag{3.2.48}$$

Fix a number $\varepsilon > 0$, which may be sufficiently small. Write $N = [\frac{\varepsilon}{2\theta}]$. Let η be a C^∞ function, defined on $[0, \infty)$, with the properties that $\eta(x) = 1$ for $0 \le x \le 1$ and $\eta(x) = 0$ for $x \ge 2$. By (3.2.33), we may decompose $\theta^2 D B_\theta^m f$ as

$$\theta^2 D B_\theta^m f = T^1_{m,\theta,\varepsilon} f + T^2_{m,\theta,\varepsilon} f,$$

where

$$T^1_{m,\theta,\varepsilon} f = \sum_{k=0}^{2N} \eta(\frac{k}{N}) u_k(m,\theta) Y_k(f),$$
$$T^2_{m,\theta,\varepsilon} f = \sum_{k=N}^{\infty} \big(1 - \eta(\frac{k}{N})\big) u_k(m,\theta) Y_k(f).$$

First, we deal with $T^1_{m,\theta,\varepsilon}(f)$. We rewrite it as

$$T^1_{m,\theta,\varepsilon}(f) = \theta^2 B_\theta^m \Big(\big(\eta_N f\big)^{(2)} \Big).$$

Then we use the fact $\|B_\theta\|_{(p,p)} = 1$, and the Bernstein's inequality for spherical harmonics to obtain

$$\|T^1_{m,\theta,\varepsilon}(f)\|_p \le C(N\theta)^2 \|f\|_p \le C\varepsilon^2 \|f\|_p. \tag{3.2.49}$$

Next we deal with $T^2_{m,\theta,\varepsilon}(f)$. It follows from Lemma 0.0.17 that

$$\|T^2_{m,\theta,\varepsilon} f\|_p \le C_p \|f\|_p \sum_{k=N}^{\infty} \Big| \triangle^{d+1} \Big(u_k(m,\theta)(1 - \eta(\frac{k}{N})) \Big) \Big| k^d. \tag{3.2.50}$$

By induction on $\ell = 0, 1, \cdots, d$, it is easy to verify

$$\Big| \triangle^\ell u_k(m,\theta) \Big| \le C(\varepsilon) m^\ell \theta^\ell \sup_{0 \le i,j \le \ell} \Big| u_{k+j-1}(m-i,\theta) \Big|, \tag{3.2.51}$$

which, by invoking (3.2.39) and Lemma 3.2.2 with $\alpha = \beta = \lambda + \frac{1}{2}$, is bounded by

$$C(\varepsilon) m^{\ell} \theta^{\ell} \Big(\frac{1}{k\theta} \Big)^{d+2} (a(\varepsilon))^m, \tag{3.2.52}$$

with $a(\varepsilon) \in (0,1)$.

Combining (3.2.49), (3.2.50) and (3.2.52), we get

$$\sup_{\theta \in [0, \frac{2}{3}\pi]} \|\theta^2 D B_\theta^m\|_{(p,p)} \le C\varepsilon^2 + C(\varepsilon) m^{d+1} a(\varepsilon)^m.$$

Now letting $m \to \infty$ first and then letting $\varepsilon \to 0$, we obtain (3.2.48).

Final step. *Prove*

$$\lim_{m\to\infty} \sup_{\theta \in [\frac{2}{3}\pi, \pi]} \|\theta^2 D B_\theta^m\|_{(p,p)} = 0, \quad \textit{for} \ \ 1 \le p \le \infty. \tag{3.2.53}$$

Since

$$D B_\theta^m f = D B_\theta^m \Big(f - \int_{\mathbb{S}^{d-1}} f \, dy \Big),$$

without loss of generality, we may assume $\int_{\mathbb{S}^{d-1}} f \, dy = 0$. Then

$$B_\theta(f)(x) = \frac{1}{\int_0^\theta \sin^{d-2} t \, dt} \int_\theta^\pi S_t(f)(x) \sin^{d-2} t \, dt, \tag{3.2.54}$$

which, by the fact $\|S_\theta\|_{(p,p)} = 1$, implies

$$\|B_\theta(f)\|_p \le \gamma_d \|f\|_p, \tag{3.2.55}$$

with

$$\gamma_d = \frac{\int_{\frac{2\pi}{3}}^{\pi} \sin^{d-2} t \, dt}{\int_0^{\frac{\pi}{2}} \sin^{d-2} t \, dt} \in (0,1). \tag{3.2.56}$$

Observing

$$\theta^2 D B_\theta^m = \theta^2 B_\theta^{m-2} D B_\theta^2,$$

we invoke (3.2.55) $m-2$ times to get

$$\|\theta^2 D (B_\theta)^m f\| \le \gamma_d^{m-2} \|\theta^2 D B_\theta^2(f)\|_p. \tag{3.2.57}$$

But applying Theorem 3.2 of [Di-Ru1] with $r = 1,\ \theta_1 = \tau_2 = \theta$ gives

$$\sup_{\theta\in(0,\pi)} \|\theta^2 DM_\theta(f)\|_p \le C_p\|f\|_p. \tag{3.2.58}$$

Substituting (3.2.58) into (3.2.57) yields (3.2.53) and concludes the proof. □

3.2.4 The Steklov type means

For $t > 0$, let

$$\begin{aligned} \Phi(t) &= \frac{2\pi^{\frac{d-1}{2}}}{\Gamma(\frac{d-1}{2})}\int_0^t \sin^{d-2} u\, du, \\ \mu(t) &= \int_0^t \frac{\Phi(\theta)}{(\sin\theta)^{d-2}}\, d\theta. \end{aligned} \tag{3.2.59}$$

Given $f \in L(\mathbb{S}^{d-1})$, its *Stekelov -type means* $A_t f(x)$ are defined (as in [Di-Ru1]) by

$$A_t(f)(x) = \frac{1}{\mu(t)}\int_0^t \frac{\Phi(\theta)}{(\sin\theta)^{d-2}} B_\theta(f,x)\, d\theta, \tag{3.2.60}$$

while the *r-th order Stekelov means* $A_{t,r}f$ (for a given $r > 0$) are defined by

$$A_{t,r}(f)(x) = \sum_{j=1}^{\infty} \binom{\frac{r}{2}}{j}(-1)^{j+1} A_t^j(f)(x). \tag{3.2.61}$$

The main goal in this subsection is to prove the following theorem, which, in the special case $r = 2$, improves the weak type equivalence in [Di-Ru1].

Theorem 3.2.9. *Let $t \in (0,\pi)$, $r > 0$ and $1 \le p \le \infty$. Then for $f \in L^p(\mathbb{S}^{d-1})$,*

$$K(f, D^{\frac{r}{2}}, t^r)_p \approx \|f - A_{t,r}f\|_p.$$

The proof of Theorem 3.2.9 is based on the following lemmas.

Lemma 3.2.10. *Let*

$$a(k,t) = \begin{cases} 1, & \text{if } k = 0, \\ \dfrac{\Gamma(\frac{d-1}{2})}{2\pi^{\frac{d-1}{2}}\mu(t)}\displaystyle\int_0^t \frac{1}{\sin^{d-2}\theta}\int_0^\theta P_k^d(\cos u)\sin^{d-2} u\, du\, d\theta, & \text{if } k \ge 1, \end{cases} \tag{3.2.62}$$

where $\mu(t)$ is given by (3.2.59). Then the following statements hold:

(i) For $k \geq 1$,

$$a(k,t) = M(t)\frac{1 - P_k^d(\cos t)}{k(k+d-2)t^2}, \tag{3.2.63}$$

$$= \frac{M(t)}{(d-1)t^2}\int_0^t P_{k-1}^{d+2}(\cos\theta)\sin\theta\, d\theta \tag{3.2.64}$$

$$= \frac{C(d)M(t)}{t^2}\int_0^t\int_0^u \cos(k+\lambda)v(\cos v - \cos u)^{\lambda}\, dv(\sin u)^{-(d-2)}\, du, \tag{3.2.65}$$

where

$$M(t) = \frac{t^2}{\displaystyle\int_0^t \frac{1}{\sin^{d-2}\theta}\int_0^{\theta}\sin^{d-2}u\, du\, d\theta}. \tag{3.2.66}$$

(ii) For $k \in \mathbb{Z}_+$,

$$Y_k(A_t(f))(x) = a(k,t)Y_k(f)(x). \tag{3.2.67}$$

Proof. (i) (3.2.63), (3.2.64) follow from (3.2.18) and (3.2.19), while (3.2.65) is a consequence of the formula (3.2.10).

(ii) The identity (3.2.67) is an immediate consequence of (3.2.7) and the definition (3.2.60). This completes the proof. □

Lemma 3.2.11. *Let $t \in (0,\pi)$, $\gamma_d > 0$ a given constant and $a(k,t)$ defined by (3.2.65) for all $k \in \mathbb{R}$. Then the following inequalities hold:*

$$\Big|\Big(\frac{d}{dk}\Big)^{\ell}\Big(\frac{1-a(k,t)}{k(k+2\lambda)t^2}\Big)\Big| \leq C_d t^{\ell}, \quad \textit{for } 0 < k \leq [\frac{\gamma_d}{t}] + 1,\ \ell = 0,\cdots,d, \tag{3.2.68}$$

$$\frac{1-a(k,t)}{k(k+2\lambda)t^2} > c_d > 0, \quad \textit{for } 0 < k \leq [\frac{\gamma_d}{t}] + 1. \tag{3.2.69}$$

Proof. First, we prove (3.2.68). By the identity (3.2.65), it is easy to verify that for $k \in \mathbb{R}$, $0 \leq \ell \leq d$,

$$\Big|(\frac{d}{dk})^{\ell}a(k,t)\Big| \leq C(\ell)t^{\ell}. \tag{3.2.70}$$

On the other hand, it follows from (3.2.63) and (3.2.62) that

$$\frac{1-a(k,t)}{k(k+2\lambda)t^2} = \frac{M(t)}{t^4}\int_0^t \frac{1}{\sin^{d-2}\theta}\int_0^{\theta} a(k,u)u^2\sin^{d-2}u\, du\, d\theta. \tag{3.2.71}$$

Combining (3.2.71) with (3.2.70) implies (3.2.68).

Next, we prove (3.2.69). We invoke (3.2.62) to get

$$\frac{1-a(k,t)}{k(k+2\lambda)t^2}=\frac{M(t)}{k(k+2\lambda)t^4}\int_0^t\frac{1}{\sin^{d-2}\theta}\int_0^\theta(1-P_k^d(\cos u))\sin^{d-2}u\,du\,d\theta,$$

which, by the Bernstein's inequality for trigonometric polynomials, implies (3.2.69). This completes the proof. □

Lemma 3.2.12. *Let $t\in(0,\pi)$, $r>0$, $1\le p\le\infty$ and $N\sim\frac{1}{t}$. Then for $f\in L^p(\mathbb{S}^{d-1})$,*

$$\left\|\left(V_N(f)\right)^{(r)}\right\|_p\le C_{p,r}t^r\|f-A_{t,r}(f)\|_p. \tag{3.2.72}$$

Proof. It follows from Lemma 3.2.10 and (3.2.61) that for $k\ge 0$,

$$Y_k(f-A_{t,r}f)(x)=(1-a(k,t))^rY_k(f). \tag{3.2.73}$$

The rest of the proof runs along the same lines as that of Lemma 3.2.6. (This time we use Lemma 3.2.11). We omit the details. □

Lemma 3.2.13. *Let $t\in(0,\pi)$, $r>0$, $1\le p\le\infty$ and γ_d a given constant, which is independent of t and sufficiently large. Then for any $\tau\in[\gamma_d,\gamma_d^2]$,*

$$\|f-V_{[\frac{\tau}{t}]}f\|_p\le C_{p,r}\|f-A_{t,r}(f)\|_p.$$

Proof. Write $N=[\frac{\tau}{t}]$. By (3.2.73) and Lemma 3.2.10, we decompose $f(x)-V_N(f)(x)$ as follows:

$$f(x)-V_N(f)(x)=\sum_{j=0}^\infty\frac{\Gamma(j+\frac{r}{2})}{j!\Gamma(\frac{r}{2})}T^j(f-A_{t,r}(f))(x),$$

where

$$T^j(g)(x)=\sum_{k=N}^\infty(1-P_k^d(\cos t))^j(1-\eta(\frac{k}{N}))\Big(\frac{M(t)}{k(k+d-2)t^2}\Big)^jY_k(g)(x).$$

Thus, it suffices to prove for $j=0,1,\cdots$,

$$\|T^j(g)\|_p\le C_pj^{d+1}\Big(\frac{1}{2}\Big)^j\|g\|_p. \tag{3.2.74}$$

To this end, define

$$U_j(g) = \sum_{k=N}^{\infty} \Big(\frac{M(t)}{k(k+d-2)t^2} \Big)^j Y_k(g)(x).$$

Observing

$$0 < M(t) < 2(d-1)\pi,$$

by Lemma 0.0.17, we get

$$\begin{aligned}
\|U_j(g)\|_p \le & C_p \Big(\Big(\frac{2(d-1)\pi}{t^2} \Big)^j \sum_{k=N}^{\infty} \Big| \triangle^{d+1} \Big(\frac{1}{k(k+d-2)} \Big)^j \Big| A_k^d \Big) \|g\|_p \\
\le & C_p \Big(\frac{2(d-1)\pi}{t^2} \Big)^j \sum_{k=N}^{\infty} \frac{j^{d+1}}{k^{2j+d+1}} k^d \|g\|_p \\
\le & C_p \Big(\frac{2(d-1)\pi}{\tau^2} \Big)^j j^{d+1} \|g\|_p.
\end{aligned}$$

Now choosing $\gamma_d \ge \sqrt{4(d-1)\pi}$, we conclude that for $\tau \in [\gamma_d, \gamma_d^2]$,

$$\|U_j(g)\|_p \le C_p j^{d+1} \Big(\frac{1}{4} \Big)^j \|g\|_p.$$

Observe

$$T^j(g) = (I - S_t)^j (I - V_N) U_j(g),$$

where I is the identity operator. On account of the fact $\|S_\theta\|_{(p,p)} = 1$, we obtain (3.2.74) and complete the proof.

□

Lemma 3.2.14. *Suppose $t \in (0, \pi)$, $r > 0$ and $1 \le p \le \infty$. Then for $f \in W_p^r$,*

$$\|f - A_{t,r}(f)\|_p \le C_p t^{-r} \|f^{(r)}\|_p. \tag{3.2.75}$$

Proof. Suppose $\frac{1}{t} \sim N$. By lemma 3.2.10, we may rewrite $V_N f - A_{t,r}(V_N f)$ as

$$V_N f - A_{t,r}(V_N f) = t^r \sum_{k=0}^{2N} \Big(\frac{1 - a(k,t)}{k(k+d-2)t^2} \Big)^{\frac{r}{2}} \eta(\frac{k}{N}) Y_k(f^{(r)}), \tag{3.2.76}$$

with $a(k,t)$ defined by (3.2.62). Then we use Lemma 0.0.17 to get

$$\|V_N f - A_{t,r}(V_N f)\|_p \le C t^{-r} \|f^{(r)}\|_p.$$

This, together with the fact

$$\|f - V_N f\|_p \le C\frac{1}{N^r}\|f^{(r)}\|_p,$$

gives (3.2.75) and completes the proof. □

Now Theorem 3.2.9 follows immediately from Lemmas 3.2.12– 3.2.14.

3.2.5 The modulus of continuity: a simple proof of Jackson type inequality

Given $r > 0$, the *rth order difference operator* Δ_t^r *with step* $t > 0$ is defined by

$$\Delta_t^r = \sum_{k=0}^{\infty} \binom{\frac{r}{2}}{k} (S_t)^k, \tag{3.2.77}$$

while *the r-th order modulus of continuity* of f in the L^p $(1 \le p \le \infty)$ metric is defined by

$$\omega_r(f,t)_p = \sup_{0<\theta\le t} \|\Delta_t^r f\|_p.$$

For $N \in \mathbb{Z}_+$, let $E_N(f)_p$ denote the best approximation of a function $f \in L^p(\mathbb{S}^{d-1})$, $1 \le p \le \infty$, by spherical harmonics of degree $\le N$:

$$E_N(f)_p = \inf\Big\{\|f - T_N\|_p : \ T_N \in \bigoplus_{k=0}^{N} \mathcal{H}_k^d\Big\}.$$

One of the basic problems in the approximation theory on the sphere $\mathbb{S}^{d-1}$ is to establish the following Jackson type inequality:

$$E_N(f)_p \le C_{p,r}\omega_r(f\frac{1}{N})_p, \quad N \in \mathbb{N}, \ r > 0. \tag{3.2.78}$$

The investigation of this problem on the sphere began with the work of Kušnirenko [Kush], who proved (3.2.78) on the two sphere $\mathbb{S}^2$ for $r = 2$, $p = \infty$. Later, (3.2.78) was proved by Butzer and Jansche [Bu-Ja] for $r = 2, 1 \le p \le \infty$, also by Pawelke [Paw] for $r = 2, 1 \le p \le \infty$, by Lizorkin and Nikolskii [Li-Ni] for $r > 0$, $p = 2$ and by Kalyabin [Ka] for $r > 0$, $1 < p < \infty$. It was Rustamov [Rus2,1993], [Rus3, 1992] who first declared to prove (3.2.78) for all the remaining $r > 0$, $1 \le p \le \infty$. However, as pointed out by Riemenschneider and Wang K.Y. in [Ri-Wa, 1995], his proof does not include the cases $p = 1, \infty$. In fact, the proofs of his key lemmas (Lemma 3.9 of [Rus2] and Lemma 3 of Rus1]) are based on the following functional statement: *the unit ball* $B_p^r := \left\{ f \in L^p(\mathbb{S}^{d-1}) : \|f^{(r)}\|_p \le 1 \right\}$, $(1 \le p \le \infty)$ *is a compact subset of* W_p^r *in the sense of weak-type topology of* $L^p(\mathbb{S}^{d-1})$. Obviously, this statement is not valid for $p = 1, \infty$. A simple counterexample to it for $p = 1$ is as follows: Let

$$f_N(x) = \sum_{k=0}^{N} \frac{A_{N-k}^d}{A_N^d} c_{d,k} \frac{P_k^d(x\mathbb{1})}{(k(k+2\lambda))^{\frac{r}{2}}},$$

where $\mathbb{1} = (1, 0, \cdots, 0)$. Since f_N is essentially the Cesàro Kernel of order d of the Fourier-Laplace series, from [BC], it follows

$$\sup_N \|f_N^{(r)}\|_1 < \infty.$$

But, as is easily shown,

$$\lim_{N \to \infty} f_N^{(r)} = \delta \sim \sum_{k=0}^{\infty} c_{d,k} P_k^d(x\mathbb{1})$$

in the weak topology of L^1, where δ is the unit Dirac mass at the point $\mathbb{1} = (1, 0, \cdots, 0)$, which is obviously not an L^1 function.

It was Riemenschneider and Wang K Y [Ri-Wa] who finally gave the correct proof of (3.2.78) for all the remaining cases $r > 0$, $p = 1, \infty$. Their proof was based on some difficult estimates of oscillatory integrals and consequently was much complicated. The main purpose in this subsection is to give a simple proof of (3.2.78) for all $r > 0$, $1 \le p \le \infty$. Our proof is motivated by [Ri-Wa].

The main result can be stated as follows.

Theorem 3.2.15. *Let* $1 \le p \le \infty$, $r > 0$. *Then there exists a constant* $\tau_d > 0$, *which depends only on d and may be sufficiently small, such that for* $f \in L^p(\mathbb{S}^{d-1})$, $t \in (0, \tau_d)$,

$$\|\Delta_t^r f\|_p \approx K(f, D^{\frac{r}{2}}, t^r)_p.$$

As a consequence, we have

Corollary 3.2.16. *Let $t \in (0,\pi)$,$r > 0$ and $1 \le p \le \infty$. Then for $f \in L^p(\mathbb{S}^{d-1})$,*

$$K(f, D^{\frac{r}{2}}, t^r)_p \approx \omega_r(f,t)_p.$$

From Corollary 3.2.16, we deduce

Corollary 3.2.17. *(Jackson type inequality) Let $r > 0$, $1 \le p \le \infty$ and $N \in \mathbb{N}$. Then for $f \in L^p(\mathbb{S}^{d-1})$,*

$$E_N(f)_p \le C_{r,p}\omega_r(f, \frac{1}{N})_p.$$

As far as we know, the best previously known result is Corollary 3.2.16, which, as will be shown, is weaker than Theorem 3.2.15.

Now we start by proving Theorem 3.2.15 . The direct inequality

$$\|\Delta_t^r f\|_p \le C_{p,r} K(f, D^{\frac{r}{2}}, t^r)_p$$

is a simple consequence of the following inequality (see [Wa-Wa]):

$$\|\Delta_t^r g\|_p \le C_{p,r} t^r \|g^{(r)}\|_p.$$

The main part of the proof is to establish the corresponding converse inequality, which is based on the following lemmas.

Lemma 3.2.18. *Let $r > 0$, $t \in (0, \frac{\pi}{2})$, $1 \le p \le \infty$ and $\gamma_d > 0$ a given constant, which is sufficiently large. Suppose $\tau \in [\gamma_d, \gamma_d^2]$. Then for $f \in L^p(\mathbb{S}^{d-1})$,*

$$\|f - V_{[\frac{\tau}{t}]}(f)\|_p \le C_{p,r}\|\Delta_t^r f\|_p.$$

Proof. Without loss of generality, we may assume $f \in C^\infty(\mathbb{S}^{d-1})$. Write $N = [\frac{\tau}{t}]$. By (3.2.77), we may decompose $f - V_N(f)$ as

$$f - V_N(f) = \sum_{j=0}^{\infty} \frac{\Gamma(j+\frac{r}{2})}{j!\Gamma(\frac{r}{2})} T^j(\Delta_t^r f),$$

where

$$T^j(g) = \sum_{k=0}^{\infty}(1-\eta(\frac{k}{N}))\Big(P_k^d(\cos t)\Big)^j Y_k(g).$$

It suffices to prove

$$\|T^j(g)\|_p \le C_p\Big(\frac{1}{2}\Big)^j \|g\|_p. \tag{3.2.79}$$

According to Lemma 3.2.2, we conclude that for $\ell = 0, 1, \cdots, [\lambda]+2$, $j \ge 6$,

$$\Big|\triangle^\ell\Big(P_k^d(\cos t)\Big)^j\Big| \le C_d\Big(\frac{1}{2}\Big)^j\Big(\frac{1}{kt}\Big)^{6\lambda} t^\ell,$$

where $kt \ge \gamma_d$ and γ_d is sufficiently large. Invoking Lemma 0.0.17, we get (5.2) for $j \ge 6$, while (5.2) for $0 \le j \le 6$ is a simple consequence of the fact $\|S_\theta\|_{(p,p)} = 1$. This completes the proof. □

Lemma 3.2.19. *Let $1 \le p \le \infty$, $\gamma_d' > 0$ a given constant and $\gamma_d' \le \alpha \le \gamma_d'^2$. Then there exists a number $\tau_d > 0$, which is sufficiently small, such that, for $f \in L^p(\mathbb{S}^{d-1})$ and $t \in (0, \tau_d)$,*

$$\Big\|\Big(V_{[\frac{\alpha}{t}]}(f)\Big)^{(r)}\Big\|_p \le C_{p,r}t^r\Big\|\Delta_t^r f\Big\|_p.$$

Proof. Write $N = [\frac{\alpha}{t}]$. Define

$$T_N(g)(x) = \sum_{k=0}^{2N}\eta(\frac{k}{N})\Big(\frac{1}{\widetilde{a}(k,t)}\Big)^{\frac{r}{2}} Y_k(g)(x),$$

where

$$\widetilde{a}(k,t) = \frac{1-P_k^d(\cos t)}{k(k+d-2)t^2}.$$

By (3.2.77), it is sufficient to prove that

$$\|T_N(g)\|_p \le C_{p,r}\|g\|_p.$$

According to Lemma 0.0.17 , it suffices to prove the following two statements:

$$\Big|\Big(\frac{d}{dk}\Big)^\ell \widetilde{a}(k,t)\Big| \le C\Big(\frac{1}{N}\Big)^\ell, \quad \ell \in \mathbb{N},\ 0 \le k \le CN, \tag{3.2.80}$$

$$\widetilde{a}(k,t) \ge C_d > 0, \quad 1 \le k \le 4N,\ 0 < t < \tau_d, \tag{3.2.81}$$

where $\tau_d > 0$ is sufficiently small and will be specified later. (3.2.80) is a consequence of (3.2.65). To prove (3.2.81), as in the proof of Theorem 3.2.8, we consider the following three cases:

Case 1. $0 < t < \frac{1}{2k}$.

In this case, (3.2.81) follows from the Bernstein's inequality for trigonometric polynomials and the following identity:

$$\widetilde{a}(k,t) = \frac{1}{(d-1)t^2} \int_0^t P_{k-1}^{d+2}(\cos u) \sin u \, du,$$

which can be easily derived from (3.2.10).

Case 2. $\frac{1}{2} \le kt < C_1^2$, $d \ge 4$.

We use (3.2.18) and (3.2.19) to obtain

$$P_k^d(\cos t) = \frac{d-3}{\sin^{d-3} t} \int_0^t P_{k+1}^{d-2}(\cos u) \sin^{d-4} u \, du. \tag{3.2.82}$$

By (3.2.18)and Bernstein's inequality for trigonometric polynomials, it is easy to verify

$$\frac{1}{\int_0^t \sin^{d-4} u \, du} \int_0^t \Big(1 - P_{k+1}^{d-2}(\cos u)\Big) \sin^{d-4} u \, du \ge a_d > 0. \tag{3.2.83}$$

Combining (3.2.82) and (3.2.83) yields

$$1 - P_k^d(\cos t) \ge 1 - (1 - a_d) \frac{(d-3) \int_0^t \sin^{d-4} u \, du}{\sin^{d-3} t}.$$

This, together with the fact

$$\lim_{t \to 0+} \frac{(d-3) \int_0^t \sin^{d-4} u \, du}{\sin^{d-3} t} = 1,$$

implies (3.2.81), for a sufficiently small $\tau_d > 0$.

Case 3. $\frac{1}{2} \le kt < C_1^2$, $d = 3$.

This time we use (3.2.10) to obtain

$$1-P_k^d(\cos t) = 1 - \frac{\sqrt{2}}{\pi}\int_0^t (\cos u - \cos t)^{-\frac{1}{2}}\,du$$
$$+\frac{\frac{\sqrt{2}}{\pi}\int_0^t(\cos u - \cos t)^{-\frac{1}{2}}\,du}{\int_0^t(\cos u - \cos t)^{-\frac{1}{2}}\,du}\int_0^t\Big(1-\cos(k+\frac{1}{2})u\Big)(\cos u - \cos t)^{-\frac{1}{2}}\,du. \qquad (3.2.84)$$

A simple calculation shows

$$\lim_{\theta\to 0}\frac{\sqrt{2}}{\pi}\int_0^\theta(\cos u - \cos\theta)^{-\frac{1}{2}}\,du = 1,$$

which, together with (3.2.84), implies (3.2.81) for sufficiently small $\tau_d > 0$.

Combining the above cases, we prove the second statement and therefore conclude the proof. □

Now the converse inequality follows directly from Lemmas 3.2.18 and 3.2.19. This completes the proof of Theorem 3.2.15 .

Proof of Corollary 3.2.16. Corollary 3.2.16 for $0 < t < \tau_d$ follows immediately from Theorem 3.2.15 and the monotonicity of the K-functional $K(f, D^{\frac{r}{2}}, \cdot)$. Now suppose $\tau_d < t < \pi$. Without loss of generality, we may assume $\int_{\mathbb{S}^{d-1}} f(x)\,dx = 0$. For $g \in W_p^r$ with $\int_{\mathbb{S}^{d-1}} g(x)\,dx = 0$, we have

$$\omega_r(f,t)_p \le C_r\|f\|_p \le C_{n,r}\Big(\|f-g\|_p + t^r\|g\|_p\Big), \quad t > \tau_d, \qquad (3.2.85)$$

which, together with the inequality (see [Col])

$$\|g\|_p \le C\|g^{(r)}\|_p,$$

implies

$$\omega_r(f,t)_p \le C_{p,r}K(f, D^{\frac{r}{2}}, t^r)_p. \qquad (3.2.86)$$

Conversely, since $t > \tau_d$,

$$\omega_r(f,t)_p \ge \omega_r(f,\tau_d)_p \ge C_{p,r}K(f, D^{\frac{r}{2}}, \tau_d^r)_p \ge C'_{p,r}K(f, D^{\frac{r}{2}}, t^r)_p, \qquad (3.2.87)$$

with $C_{p,r},\ C'_{p,r} > 0$. Combining (3.2.86) and (3.2.87), we complete the proof. □

Corollary 3.2.17 is an immediate consequence of Corollary 3.2.16 and Lemma 3.2.18.

3.3 K-functionals and the averages induced by the Jacobi Expansions

In this section, we will extend the results in Sections 3.2 to the setting of Jacobi expansions. First, we introduce some notations. Let $\alpha > -1, \beta > -1$ and $\omega(x) = \omega^{(\alpha,\beta)}(x) := (1-x)^{\alpha}(1+x)^{\beta}$. Let $P(D)$ denote the differential operator

$$P(D) = P^{(\alpha,\beta)}(D) := \frac{1}{\omega(x)}\frac{d}{dx}(1-x^2)\omega(x)\frac{d}{dx}.$$

Then it follows from [Sz] that

$$P^{(\alpha,\beta)}(D)R_k^{(\alpha,\beta)}(x) = -\lambda(k)R_k^{(\alpha,\beta)}(x), \tag{3.3.1}$$

with

$$\lambda(k) := k(k+\alpha+\beta+1).$$

Suppose $L_p^{(\alpha,\beta)}[-1,1]$ denotes the space

$$L_p^{(\alpha,\beta)}[-1,1] = \{f : [-1,1] \longrightarrow \mathbb{R} |\ \ f \text{ measurable and } \ \|f\|_p < \infty\},$$

with the norm

$$\|f\|_p = \|f\|_{p,\alpha,\beta} := \Big(\int_{-1}^{1} |f(u)|^p \omega(u)\, du\Big)^{\frac{1}{p}}.$$

Given a function $f \in L_p^{(\alpha,\beta)}[-1,1]$, its Jacobi expansion is denoted by

$$f \sim \sigma^{(\alpha,\beta)}(f) = \sum_{k=0}^{\infty} H_k^{(\alpha,\beta)}(f)(x),$$

where

$$H_k^{(\alpha,\beta)}(f)(x) := \frac{1}{\|R_k^{(\alpha,\beta)}\|_2^2}\int_{-1}^{1} f(y)R_k^{(\alpha,\beta)}(y)\omega(y)\, dy R_k^{(\alpha,\beta)}(x).$$

As in Chapter 0, denote by $\sigma_N^{(\alpha,\beta),\,\delta}(f)(x)$ the Cesàro mean of the expansion $\sigma^{(\alpha,\beta)}(f)$ of order $\delta > -1$. It follows from [Ga] that for $\delta > \alpha + \frac{1}{2}$, $\alpha \geq \beta > -1$ and $\alpha + \beta \geq -1$,

$$\sup_N \|\sigma_N^{(\alpha,\beta),\,\delta}(f)\|_p \leq C_p\|f\|_p, \quad 1 \leq p \leq \infty.$$

Motivated by (3.3.1), given $r > 0$, we define the *r-th order differential operator* $P(D)^r$ by

$$P(D)^r f \sim \sum_{k=0}^{\infty} \lambda(k)^r H_k^{(\alpha,\beta)}(f).$$

Associated with the differential operator $P(D)^r$ is the *rth order K-functional* $K(f, P(D)^r, u)_p$, which is defined by

$$K(f, P(D)^r, u)_p = \inf_{g \in \mathcal{A}} \Big\{ \|f - g\|_p + u\|P(D)^r g\|_p \Big\},$$

where

$$\mathcal{A} = \Big\{ g \in L_p^{(\alpha,\beta)}[-1,1] : \; P(D)^r g \in L_p^{(\alpha,\beta)}[-1,1] \Big\}.$$

The *generalized translation* τ_t (for a given $t \in [-1,1]$) is defined by

$$H_k^{(\alpha,\beta)}(\tau_t^{(\alpha,\beta)} f)(x) := R_k^{(\alpha,\beta)}(t) H_k^{(\alpha,\beta)}(f)(x), \quad x \in [-1,1]. \tag{3.3.2}$$

Gasper [Ga] showed in 1972 that

$$\sup_{t \in [-1,1]} \|\tau_t(f)\|_p \le C_p \|f\|_p, \quad 1 \le p \le \infty$$

holds if and only if $\alpha \ge \beta, \; \alpha + \beta \ge -1$.

The following average operator $A_t = A_t^{(\alpha,\beta)}$ was introduced and discussed by Butzer, Jansche, and Stens [Bu- Ja- St]:

$$A_t^{(\alpha,\beta)}(f)(x) := \frac{1}{\xi(t)} \int_t^1 \int_t^u \frac{1}{(1-y)^{\alpha+1}(1+y)^{\beta+1}} \, dy (\tau_u^{(\alpha,\beta)} f)(x) \omega^{(\alpha,\beta)}(u) \, du, \tag{3.3.3}$$

where $x \in [-1,1], \quad t \in [-1,1]$, and

$$\xi(t) = \int_t^1 \int_t^u \frac{1}{(1-y)^{\alpha+1}(1+y)^{\beta+1}} \, dy \omega^{(\alpha,\beta)}(u) \, du. \tag{3.3.4}$$

It follows from [Di-Fe] that $A_t^{(\alpha,\beta)} \mathbb{1} = \mathbb{1}$ and for $f \in L_p^{(\alpha,\beta)}[-1,1]$,

$$A_t^{(\alpha,\beta)} f \sim \sum_{k=1}^{\infty} \frac{1 - R_k^{(\alpha,\beta)}(t)}{\lambda(k)\xi(t)} H_k^{(\alpha,\beta)}(f). \tag{3.3.5}$$

For more detail, we refer the reader to [Di-Fe]

In the rest of this section, we always assume $\alpha \geq \beta > -1$, $\alpha+\beta \geq -1$. And for convenience, we will abbreviate $R_k^{(\alpha,\beta)}$, $H_k^{(\alpha,\beta)}$, $\tau_t^{(\alpha,\beta)}, \cdots$ by R_k, H_k, $\tau_t, \cdots$.

Our first result in this section is:

Theorem 3.3.1. *Suppose* $t \in (-1,1)$. *Then*

$$\|f - A_t f\|_p \approx K(f, P(D), 1-t)_p$$

with $1 \leq p \leq \infty$.

As pointed out in [Di-Fe], the following weak type equivalence can be obtained by following the technique developed in [Di-Iv]:

$$\|f - A_t f\|_p + \|f - A_{\rho t+1-\rho} f\|_p \approx K(f, P(D), 1-t)_p.$$

Sketch of the proof of Theorem 3.3.1. It remains to prove the strong converse inequality. The proof runs along the same lines as that of Theorem 3.2.3. Suppose $t = \cos\theta$, $\theta \in [0, \frac{3\pi}{4}]$ and

$$C_\alpha \leq (N-\alpha-2)\theta < C_\alpha^2. \tag{3.3.6}$$

Let

$$a_k(\theta) = \begin{cases} 1, & k = 0, \\ \frac{1-R_k^{(\alpha,\beta)}(\cos\theta)}{\lambda(k)\xi(\cos\theta)}, & k \neq 0. \end{cases} \tag{3.3.7}$$

Without loss of generality, we may assume $f \in C^\infty[-1,1]$. Then by (3.3.5),

$$A_t f = \sum_{k=1}^{\infty} a_k(\theta) H_k(f).$$

It suffices to verify the following statements:

i) If the constant C_α in (3.3.6) is sufficiently large, then for $k \geq N$,

$$|a_k(\theta)| \leq \frac{1}{2}. \tag{3.3.8}$$

ii) For $0 \leq k \leq 4N$,

$$1 - a_k(\theta) \geq C_1 \lambda(k) \theta^2. \tag{3.3.9}$$

iii) If $0 \le k \le 4N,\ \ell \in \mathbb{Z}_+$, then

$$\Big|\triangle^\ell\Big(\frac{1-R_k(\cos\theta)}{\lambda(k)\theta^2}\Big)\Big| \le CN^{-\ell}, \tag{3.3.10}$$

$$\Big|\triangle^\ell\Big(\frac{1-a_k(\theta)}{\lambda(k)\theta^2}\Big)\Big| \le CN^{-\ell}. \tag{3.3.11}$$

We first prove statement i). By (3.3.4) and a straightforward computation, we have

$$\begin{aligned}\xi(\cos\theta) = C\int_0^\theta\int_u^\theta(\sin\frac{y}{2})^{-1-2\alpha}(\cos\frac{y}{2})^{-2\beta-1}\,dy(\sin\frac{u}{2})^{2\alpha+1}(\cos\frac{u}{2})^{2\beta+1}\,du\\ \sim\theta^2\sim 1-t.\end{aligned} \tag{3.3.12}$$

Substituting (3.3.12) into (3.3.7) gives

$$|a_k(\theta)| \le 2\frac{1}{\lambda(k)\xi(\cos\theta)} \le \frac{1}{2}\frac{C}{k^2\theta^2}, \tag{3.3.13}$$

which obviously implies statement ii).

Next, we prove statement i). By (3.3.3) and (3.3.5), one can easily verify

$$\begin{aligned}1-a_k(\theta) =& C_{\alpha,\beta}\frac{1}{\xi(\cos\theta)}\int_0^\theta\int_u^\theta(\sin\frac{v}{2})^{-2\alpha-1}(\cos\frac{v}{2})^{-2\beta-1}\,dv\times\\ &\times(1-R_k(\cos u))(\sin\frac{u}{2})^{2\alpha+1}(\cos\frac{u}{2})^{2\beta+1}\,du,\end{aligned} \tag{3.3.14}$$

with $C_{\alpha,\beta} > 0$. On the other hand, from Bernstein's inequality for the trigonometric polynomials, it follows that for $0 \le u \le \frac{1}{2k}$,

$$R_{k-1}^{(\alpha+1,\beta+1)}(\cos u) \ge \frac{1}{2}.$$

Thus for $0 < u < \frac{1}{2k}$,

$$\begin{aligned}1-R_k(\cos u) = R_k'(\cos\xi)(1-\cos u) = \frac{\lambda(k)}{2(\alpha+1)}R_{k-1}^{(\alpha+1,\beta+1)}(\cos\xi)(1-\cos u)\\ \ge Ck^2u^2.\end{aligned} \tag{3.3.15}$$

Now substituting (3.3.15) into (3.3.14) yields (3.3.9).

Finally, we prove statement iii). We start by proving (3.3.10). Notice that

$$\begin{aligned}\frac{1-R_k(\cos\theta)}{\lambda(k)\theta^2} &= \frac{1}{\lambda(k)\theta^2}\int_{\cos\theta}^1 R_k'(u)\,du\\ &=\frac{1}{\theta^2}\frac{1}{2(\alpha+1)}\int_0^\theta R_{k-1}^{(\alpha+1,\beta+1)}(\cos u)\sin u\,du.\end{aligned}$$

We get

$$\Big|\triangle^{\ell}\frac{1-R_k(\cos\theta)}{\lambda(k)\theta^2}\Big| \le \frac{C_\alpha}{\theta^2}\int_0^\theta \Big|\triangle^{\ell}R_{k-1}^{(\alpha+1,\beta+1)}(\cos u)\Big|\sin u\,du,$$

which, by Lemma 2.1, implies (3.3.10).

To prove (3.3.11), we use (3.3.14) to obtain

$$\frac{1-a_k(\theta)}{\lambda(k)\theta^2}$$

$$=\frac{\theta^2}{\xi(\theta)}\frac{1}{\theta^4}\int_0^\theta\int_u^\theta(\sin\frac{v}{2})^{-2\alpha-1}(\cos\frac{v}{2})^{-2\beta-1}u^2\frac{1-R_k(\cos u)}{\lambda(k)u^2}(\sin\frac{u}{2})^{2\alpha+1}(\cos\frac{u}{2})^{2\beta+1}\,dvdu. \tag{3.3.16}$$

Combining (3.3.10) with (3.3.16), we get (3.3.11). This gives statement iii) and concludes the proof. □

For $r>0$, we define the *r-th order average operator* by

$$A_{t,r}=\sum_{j=0}^{\infty}(-1)^{j+1}\binom{r}{j}A_t^j.$$

Then with a slight modification of the above proof, we obtain

Theorem 3.3.2. *Suppose $t\in(-1,1)$, $r>0$ and $1\le p\le\infty$. Then for $f\in L^p$,*

$$\|f-A_{t,r}f\|_p\approx K(f,P(D)^r,(1-t)^r)_p.$$

Let us define *the rth order modulus of continuity* $\omega_r(f,\theta)_p$ (for a given $r>0$) by

$$\omega_r(f,\theta)_p=\sup_{0<t<\theta}\|(I-\tau_{cost})^{\frac{r}{2}}f\|_p,$$

where τ_t denotes the translation operator. For $f\in L_p^{(\alpha,\beta)}$, we denote by $E_N(f)_p$ its best approximation by algebraic polynomials of degree not exceeding N in the metric $L_p^{(\alpha,\beta)}[-1,1]$.

Analogous to the proof of Theorem 3.2.15, we obtain

Theorem 3.3.3. *Let $r>0$, $\theta\in[0,\pi]$ and $1\le p\le\infty$. Then for $f\in L_p^{(\alpha,\beta)}[-1,1]$,*

$$\omega_r(f,\theta)_p\approx K(f,P(D)^{\frac{r}{2}},\theta^r)_p.$$

As an immediate consequence, we have the following

Corollary 3.3.4. *Let $r > 0$, $N \in \mathbb{N}$, $1 \le p \le \infty$. Then for $f \in L_p^{(\alpha,\beta)}[-1,1]$,*

$$E_N(f)_p \le C_{p,r}\omega_r(f, \frac{1}{N})_p.$$

Remark 3.3.1. Theorem 3.3.3 for $1 < p < \infty$ was due to Rustamov [Rus4]. The statement of Theorem of [Rus4] contains the end points $p = 1, \infty$. However, the author's proof obviously does not include these end points. In fact, the proof in [Rus4] runs along the same lines as that of [Rus2]. Both of them used a same functional argument, which, as already pointed out in Section 3.2.5, is incorrect for these end points.

Finally, we extend a result in [Di-Fe]. Given a function $f \in L^1$ and a number $t \in [-1,1]$, the operator $B_t(f)$ is defined by

$$B_t(f)(x) = 2A_{\frac{1+t}{2}}f - A_t f,$$

while the ℓ-th order operator is defined by

$$B_{t,\ell} = \sum_{j=1}^{\ell}(-1)^{j+1}\binom{\ell}{j}A_{1-\frac{j(1-t)}{\ell}}f.$$

Using the same technique as in previous sections, we obtain

Theorem 3.3.5. *Suppose $\ell \in \mathbb{Z}_+$, $t \in [-1,1]$ and $f \in L_p^{(\alpha,\beta)}[-1,1]$ with $1 \le p \le \infty$. Then there exists a constant $\rho \in (0,1]$, independent of f and t, such that*

$$K(f, P(D)^\ell, (1-t)^\ell) \approx \|f - B_{t,\ell}f\|_p + \|f - B_{\rho t+1-\rho,\ell}f\|_p.$$

Theorem 3.3.5 for $\ell = 2$ was due to [Di-Fe].

3.4 The averages and the K-functionals related to the Laplacian on $\mathbb{R}^d$

Given a function $f \in L(\mathbb{R}^d)$, its Fourier transform is defined by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-ix\xi}\,dx, \quad \xi \in \mathbb{R}^d.$$

For a given positive integer ℓ, the ℓ-th order Laplacian $\widetilde{\triangle}^\ell$ is defined, in the sense of distributions, by

$$(\widetilde{\triangle}^\ell f)^\wedge(\xi) = (-1)^\ell |\xi|^{2\ell} \widehat{f}(\xi).$$

As is well known, (see, for example, [St, p. 117]), for a function f which is sufficiently smooth and small at infinity, $\widetilde{\triangle}^\ell f$ coincides with $\left(\sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}\right)^\ell f$.

To each operator $\widetilde{\triangle}^\ell$, we associate a K-functional

$$K_{2\ell}(f,t)_p = \inf\Big\{ \|f-g\|_p + t^{2\ell} \|\triangle^\ell g\|_p : \ g, \widetilde{\triangle}^\ell g \in L^p(\mathbb{R}^d) \Big\}, \tag{3.4.1}$$

where $t > 0$, $1 \le p \le \infty$ and $\|\cdot\|_p$ denotes the usual L^p-norm on $\mathbb{R}^d$.

Let V_d denote the volume of the unit ball of $\mathbb{R}^d$. For $t > 0$ and a locally integrable function f, as in [Di-Ru2], the average $B_t f$ is defined by

$$B_t(f)(x) = \frac{1}{t^d V_d} \int_{\{u \in \mathbb{R}^d : \ |u| \le t\}} f(x+u)\, du,$$

while the ℓ-th order average $B_{\ell,t} f$ (for a given positive integer ℓ) is defined by

$$B_{\ell,t}(f)(x) = \frac{-2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell - j} B_{jt}(f,x). \tag{3.4.2}$$

We refer the reader to [Di-Ru2] for more background information.

Our main goal in this section is to prove the following theorem, which was conjectured in [Di-Ru2, p. 138].

Theorem 3.4.1. *Let $\ell \in \mathbb{N}$, $1 \le p \le \infty$ and $f \in L^p(\mathbb{R}^d)$. Then for $t > 0$,*

$$\|f - B_{\ell,t}(f)\|_p \approx K_{2\ell}(f,t)_p.$$

Theorem 3.4.1 for $\ell = 1$ was proved in [Di-Ru2] and for $d = 1$, ℓ small, can be obtained by following the technique developed in [Di–Iv]. (The latter was pointed out in [Di-Ru2, p. 138]). For $\ell \ge 2$, $d \ge 2$, the following weaker equivalence was established in [Di-Ru2, Theorems 4.8 and 5.7]:

$$K_{2\ell}(f,t)_p \approx \|f - B_{\ell,t} f\|_p + \|f - B_{\ell,t\rho} f\|_p, \ 1 \le p \le \infty, \tag{3.4.3}$$

for some $\rho > 1$.

We remark that with a slight modification of the proof below, similar result for the periodic case can also be obtained.

The following lemma can be easily obtained by a straightforward computation.

Lemma 3.4.2. *Let $\chi_{B(0,1)}(x)$ denote the characteristic function of the unit ball*

$$B(0,1) := \{x = (x_1, \cdots, x_d) \in \mathbb{R}^d : \ x_1^2 + \cdots + x_d^2 \leq 1\},$$

V_d the volume of $B(0,1)$ and $I(x) = \frac{1}{V_d}\chi_{B(0,1)}(x)$. Then

$$\widehat{I}(x) = \gamma_d \int_0^1 \cos(u|x|)(1-u^2)^{\frac{d-1}{2}}\, du \tag{3.4.4}$$

with

$$\gamma_d = \Big(\int_0^1 (1-u^2)^{\frac{d-1}{2}}\, du\Big)^{-1}. \tag{3.4.5}$$

Lemma 3.4.3. *Let $B_{\ell,t}$ be defined by (3.4.2) and $I(x)$ defined as in Lemma 3.4.2. Then for $f \in L(\mathbb{R}^d)$,*

$$\widehat{B_{\ell,t}f}(x) = m_\ell(t|x|)\widehat{f}(x), \tag{3.4.6}$$

where

$$m_\ell(|x|) = \frac{-2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell - j} \widehat{I}(jx) \tag{3.4.7}$$

$$= 1 - A_\ell(|x|), \tag{3.4.8}$$

$$A_\ell(|x|) = \gamma_d \frac{4^\ell}{\binom{2\ell}{\ell}} \int_0^1 (1-u^2)^{\frac{d-1}{2}} (\sin \frac{u|x|}{2})^{2\ell}\, du, \tag{3.4.9}$$

and γ_d is given by (3.4.5).

Proof. For $t > 0$, write

$$I_t(x) = \frac{1}{t^d} I(\frac{x}{t}).$$

Then from the definition (3.4.2), it follows

$$(B_{\ell,t}f)(x) = \frac{-2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell - j} (f * I_{jt})(x),$$

which implies (3.4.7). Substituting (3.4.4) into (3.4.7) yields

$$m_\ell(|x|) = \frac{-2\gamma_d}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell - j} \int_0^1 \cos(ju|x|)(1-u^2)^{\frac{d-1}{2}}\, du, \tag{3.4.10}$$

with γ_d as in (3.4.5). Now (3.4.10), together with the following identity

$$(\sin\frac{x}{2})^{2\ell} = \frac{\binom{2\ell}{\ell}}{4^\ell} + \frac{2}{4^\ell}\sum_{j=1}^{\ell}(-1)^j \binom{2\ell}{\ell - j} \cos jx,$$

gives (3.4.8) and (3.4.9). This completes the proof. □

Lemma 3.4.4.

$$\Big|(\frac{d}{dx})^j \int_0^1 \cos(ux)(1-u^2)^{\frac{d-1}{2}}\, du\Big| \le C_j \Big(\frac{1}{x+1}\Big)^{\frac{d+1}{2}}, \tag{3.4.11}$$

with $j = 0, 1, \cdots,\ x \ge 0$.

Proof. From formula (4.7.5) of [An-As-R], it follows

$$\int_0^1 \cos(ux)(1-u^2)^{\frac{d-1}{2}}\, du = 2^{\frac{d-2}{2}} \sqrt{\pi}\,\Gamma(\frac{d+1}{2}) \frac{J_{\frac{d}{2}}(x)}{x^{\frac{d}{2}}}, \tag{3.4.12}$$

where $J_\alpha(x)$ is the Bessel function of the first kind of order α. Now (3.4.11) follows directly from (3.4.12) and the following two identities on Bessel functions:

$$\frac{d}{dx} x^{-\alpha} J_\alpha(x) = -x^{-\alpha} J_{\alpha+1}(x), \qquad ([\text{An-As-R}, (4.6.2)\ ]),$$
$$J_\alpha(x) = O\Big(\frac{1}{(x+1)^{\frac{1}{2}}}\Big), \qquad ([\text{An-As-R}, (4.8.5)]).$$

This concludes the proof. □

Lemma 3.4.5. *Suppose* a *is a* C^∞ *function defined on* $[0, \infty)$ *with the property*

$$\Big|\Big(\frac{d}{du}\Big)^j a(u)\Big| \le C\Big(\frac{1}{1+u}\Big)^{d+1}, \quad u \ge 0,\ j = 0, \cdots, d+1. \tag{3.4.13}$$

For $t > 0$, *define the operator* T_t, *in the sense of distributions, by*

$$\Big(T_t(f)\Big)^\wedge(\xi) = a(t|\xi|)\hat{f}(\xi),\ \xi \in \mathbb{R}^d.$$

Then for $1 \le p \le \infty$ *and* $f \in L^p(\mathbb{R}^d)$,

$$\sup_{t>0} \|T_t(f)\|_p \le C_p \|f\|_p.$$

Proof. Let

$$K(x) = \int_{\mathbb{R}^d} e^{ix\xi} a(|\xi|)\, d\xi. \tag{3.4.14}$$

Since

$$T_t(f)(x) = f * K_t(x),$$

with

$$K_t(x) = \frac{1}{t^d} K(\frac{x}{t}),$$

it is sufficient to prove

$$\|K\|_{L^1(\mathbb{R}^d)} < \infty. \tag{3.4.15}$$

By (3.4.14), we get for $\gamma = (\gamma_1, \cdots, \gamma_d) \in \mathbb{Z}_+^d$,

$$(-x)^\gamma K(x) = \int_{\mathbb{R}^d} e^{ix\xi} \Big(\frac{\partial}{\partial \xi}\Big)^\gamma \Big(a(|\xi|\Big)\, d\xi,$$

which, by (3.4.13), clearly implies

$$|x^\gamma K(x)| \le C \int_{\mathbb{R}^d} \frac{d\xi}{(1+|\xi|)^{d+1}} < \infty,$$

with $|\gamma| = \gamma_1 + \cdots + \gamma_d \le d+1$. Now taking the supremum over all $\gamma,\ |\gamma| = d+1$ yields

$$|K(x)| \le \frac{C}{|x|^{d+1}},$$

which, together with the fact $K \in C(\mathbb{R}^d)$, implies (3.4.15) and completes the proof. $\square$

Proof of Theorem 3.4.1. On account of the equivalence (3.4.3), it will suffice to prove the converse inequality

$$K_{2\ell}(f,t)_p \le C_{\ell,p} \|f - B_{\ell,t}(f)\|_p. \tag{3.4.16}$$

We start with the identity (3.4.10). We then use Lemma 3.4.4 to get for $j = 0, 1, \cdots, d+1,\ u \geq 0$,

$$\Big|\Big(\frac{d}{du}\Big)^j m_\ell(u)\Big| \leq C_d\Big(\frac{1}{u+1}\Big)^{\frac{d+1}{2}}, \tag{3.4.17}$$

which implies that there exists a number $\mu = \mu(\ell, d) > 0$, such that

$$|m_\ell(u)| \leq \frac{1}{2}, \tag{3.4.18}$$

whenever $u > \mu$. For the remainder of the proof, we always write μ to designate the number with this property.

Now fix a C^∞ function η, defined on $[0, \infty)$, with the properties that $\eta(x) = 0$ for $x > 2$, $\eta(x) = 1$ for $0 \leq x \leq 1$, and $0 \leq \eta(x) \leq 1$ for all $x \in [0, \infty)$. Let

$$\phi(u) = \Big(1 - \eta(\frac{u}{2\mu})\Big)\frac{(m_\ell(u))^3}{1 - m_\ell(u)}, \tag{3.4.19}$$

$$\psi(u) = \frac{u^{2\ell}\eta(\frac{u}{2\mu})}{A_\ell(u)}, \tag{3.4.20}$$

with $A_\ell(u)$, $m_\ell(u)$ as in Lemma 3.4.3.

For $t > 0$, we define three operators $V_t,\ \Phi_t,\ \Psi_t$ as follows:

$$\begin{aligned} \Big(V_t(f)\Big)^\wedge(\xi) &= \eta(t|\xi|)\hat{f}(\xi), \\ \Big(\Phi_t(f)\Big)^\wedge(\xi) &= \phi(t\xi)\hat{f}(\xi), \\ \Big(\Psi_t(f)\Big)^\wedge(\xi) &= \psi(t\xi)\hat{f}(\xi). \end{aligned} \tag{3.4.21}$$

By (3.4.17), (3.4.18) and (3.4.19), it is easy to verify that for $u \geq 0,\ j = 0, 1, \cdots, d+1$,

$$|\phi^{(j)}(u)| \leq C_d\Big(\frac{1}{u+1}\Big)^{\frac{3(d+1)}{2}}. \tag{3.4.22}$$

On the other hand, by (3.4.9) and an easy computation, we have

$$\frac{A_\ell(u)}{u^{2\ell}} \geq C_{d,\ell}\int_0^{\frac{2}{3}} (\sin\frac{uv}{2})^{2\ell}\, dv \geq C_{d,\ell} > 0, \quad \text{for } u \geq \frac{\pi}{2}, \tag{3.4.23}$$

$$\frac{A_\ell(u)}{u^{2\ell}} \geq C\frac{1}{u^{2\ell}}\int_0^1 (1 - v^2)^{\frac{d-1}{2}}(uv)^{2\ell}\, dv \geq C_{\ell,d} > 0, \quad \text{for } 0 < u < \frac{\pi}{2}. \tag{3.4.24}$$

Combining (3.4.20), (3.4.23)–(3.4.24) gives

$$\psi(v) \in C_c^\infty[0, \infty). \tag{3.4.25}$$

Therefore, according to Lemma 3.4.5, by (3.4.22), (3.4.25) and the fact $\eta \in C_c^\infty[0,\infty)$, we get from (3.4.21) that

$$\sup_{t>0}\|V_t(f)\|_p + \sup_{t>0}\|\Phi_t(f)\|_p + \sup_{t>0}\|\Psi_t(f)\|_p \le C_p\|f\|_p, \quad 1 \le p \le \infty. \tag{3.4.26}$$

Now the converse inequality (3.4.16) can be easily obtained from (3.4.26). In fact, it follows from the definition (3.4.1) that

$$K_{2\ell}(f,t) \le C_\ell\Big(\|f - V_{t'}f\|_p + t^{2\ell}\|\widetilde{\triangle}^\ell V_{t'}f\|_p\Big),$$

with $t' = \frac{t}{2\mu}$. Hence, it suffices to prove

$$\|f - V_{t'}f\|_p \le C\|f - B_{\ell,t}(f)\|_p, \tag{3.4.27}$$

and

$$t^{2\ell}\|\widetilde{\triangle}^\ell V_{t'}f\|_p \le C\|f - B_{\ell,t}(f)\|_p, \tag{3.4.28}$$

with $C > 0$ independent of f and t. Without loss of generality, we may assume $f \in C_c^\infty(\mathbb{R}^d)$.

Observe that

$$\begin{aligned}\Big(f - V_{t'}f\Big)^\wedge(\xi) =& \Big(1 - \eta(\frac{|\xi|t}{2\mu})\Big)\widehat{f}(\xi)\\ =& \Big(1 - \eta(\frac{|\xi|t}{2\mu})\Big)\Big(\frac{(m_\ell(t|\xi|))^3}{1 - m_\ell(t|\xi|)} + 1 + m_\ell(t|\xi|) + \big(m_\ell(t|\xi|)\big)^2\Big)\\ &\times (f - B_{\ell,t}f)^\wedge(\xi).\end{aligned}$$

We get

$$f - V_{t'}f = \Phi_t(f - B_{\ell,t}f) + (I - V_{t'})(I + B_{\ell,t} + B_{\ell,t}^2)(f - B_{\ell,t}f),$$

where I denotes the identity operator. This, together with (3.4.26) and the fact $\|B_{\ell,t}\|_{(p,p)} \le C_\ell$, gives (3.4.27).

Similarly, from the identities

$$\Big(t^{2\ell}\triangle^\ell V_{t'}f\Big)^\wedge(\xi) = \frac{t^{2\ell}|\xi|^{2\ell}\eta(\frac{|\xi|t}{2\mu})}{1 - m_\ell(t|\xi|)}(f - B_{\ell,t}f)^\wedge(\xi) = (\Psi_t(f - B_t(f))^\wedge(\xi),$$

we get

$$t^{2\ell}\widetilde{\triangle}^{\ell}V_{t'}f = \Psi_t(f - B_t(f)),$$

which, again by (3.4.26), implies (3.4.28). This completes the proof.

Chapter 4

Strong approximation by Fourier–Laplace series

4.1 Strong approximation by Cesàro means in $L^p(\mathbb{S}^{d-1})$ $(1 \le p \le \infty)$

Given a function $f \in L(\mathbb{S}^{d-1})$, recall that the Fourier–Laplace series of f is denoted by

$$\sigma(f)(x) := \sum_{k=0}^{\infty} Y_k(f)(x)$$

and the Cesàro means of order δ of $\sigma(f)$ are defined by

$$\sigma_N^\delta(f) := (A_N^\delta)^{-1} \sum_{k=0}^{N} A_{N-k}^\delta Y_k(f), \tag{4.1.1}$$

where

$$A_k^\delta = \frac{\Gamma(k+\delta+1)}{\Gamma(\delta+1)\Gamma(k+1)}, \ \delta > -1. \tag{4.1.2}$$

As stated in Chapter 0, the special value $\lambda := \frac{d-2}{2}$ of δ is known as the critical order of the Cesàro means σ_N^δ. In fact, for $\delta > \lambda$, $\sup\limits_N \|\sigma_N^\delta\|_{(p,p)} < \infty$ for all $1 \le p \le \infty$ while for $\delta \le \lambda$, this statement is no longer valid. It is well known that $\|\sigma_N^\lambda\|_{(1,1)} = \|\sigma_N^\lambda\|_{(\infty,\infty)} = O(\log N)$ as $N \to \infty$. (See [Wa-L].) Notice that $\lambda = 0$ when $d = 2$. We may regard σ_k^λ as a natural extension of the partial sum operator of the Fourier series in one variable.

To the best of our knowledge, so far very few results on the strong approximation were established on the sphere $\mathbb{S}^{d-1}$. The main goal in this section is to extend some well known results of Fourier series in one variable to the case of Fourier -Laplace series on the multi-dimensional sphere. Our idea is motivated by [Wa] and [BC].

4.1.1 Statements of the main results

Theorem 4.1.1. *Suppose f is a function defined on $\mathbb{S}^{d-1}$, $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.*

(i) If $2 \le p < \infty$ then for every $x \in \mathbb{S}^{d-1}$,

$$\Big(\sum_{k=0}^{\infty} |\sigma_k^\lambda(f)(x) - f(x)|^p\Big)^{\frac{1}{p}} \le C_p\Big(\int_0^\pi \Big|\frac{S_\theta(f)(x) - f(x)}{\theta}(\pi-\theta)^\lambda\Big|^q d\theta\Big)^{\frac{1}{q}}, \tag{4.1.3}$$

where C_p is a constant independent of f. Thus if

$$\int_0^\pi \Big|\frac{S_\theta(f)(x) - f(x)}{\theta}\Big(\pi-\theta\Big)^\lambda\Big|^q d\theta \in L^\infty(\mathbb{S}^{d-1}), \tag{4.1.4}$$

then

$$\sum_{k=0}^{\infty} |\sigma_k^\lambda(f)(x) - f(x)|^p \in L^\infty(\mathbb{S}^{d-1}). \tag{4.1.5}$$

(ii) If $1 \le p \le 2$, then for every $x \in \mathbb{S}^{d-1}$,

$$\Big(\int_0^\pi \Big|\frac{S_\theta(f)(x) - f(x)}{\theta}\Big(\pi-\theta\Big)^\lambda\Big|^q d\theta\Big)^{\frac{1}{q}} \le C_p\Big(\sum_{k=0}^{\infty} |\sigma_k^\lambda(f)(x) - f(x)|^p\Big)^{\frac{1}{p}}.$$

Thus if (4.1.5) holds, then (4.1.4) also holds.

Theorem 4.1.2. *Let $f \in L(\mathbb{S}^{d-1})$ and $1 < p < \infty$.*

(i) If $2 \le p < \infty$, then for every $x \in \mathbb{S}^{d-1}$

$$\sum_{k=0}^{\infty}\Big|\sigma_k^\lambda(f)(x) - f(x)\Big|^p \le C_p \int_0^\pi \frac{\Big|\big(S_\theta(f)(x) - f(x)\big)\big(\pi-\theta\big)^\lambda\Big|^p}{\theta^2} d\theta.$$

Thus if

$$\int_0^\pi \frac{\Big|\big(S_\theta(f)(x) - f(x)\big)\big(\pi-\theta\big)^\lambda\Big|^p}{\theta^2} d\theta \in L^\infty(\mathbb{S}^{d-1}), \tag{4.1.6}$$

then (4.1.5) is satisfied.

(ii) If $1 < p \le 2$ then

$$\int_0^\pi \frac{\Big|\big(S_\theta(f)(x) - f(x)\big)\big(\pi-\theta\big)^\lambda\Big|^p}{\theta^2} d\theta \le C_p \sum_{k=0}^{\infty} |\sigma_k^\lambda(f)(x) - f(x)|^p.$$

Thus, if (4.1.5) is satisfied, then (4.1.6) is also satisfied.

Theorem 4.1.3. *Let $1 < p < \infty$ and $\delta > \lambda$. If $\min\{2, \frac{1}{\delta-\lambda}\} < p < \infty$, then for every $x \in \mathbb{S}^{d-1}$,*

$$\sum_{k=0}^{\infty} |\sigma_k^\delta(f)(x) - f(x)|^p \le C\Big(\frac{1}{\delta - \lambda - \frac{1}{p}}\Big)^p \int_0^\pi \frac{|S_\theta(f)(x) - f(x)|^p}{\theta^2}(\pi - \theta)^{\lambda p}\, d\theta. \qquad (4.1.7)$$

Corollary 4.1.4. *Suppose $x \in \mathbb{S}^{d-1}$ and f is a function defined on the sphere $\mathbb{S}^{d-1}$. Then*

$$\sum_{k=0}^{\infty} \big|\sigma_k^\lambda(f)(x) - f(x)\big|^2 < \infty$$

if and only if

$$\int_0^\pi \frac{\Big|S_\theta(f)(x) - f(x)\Big|^2 (\pi - \theta)^{d-2}}{\theta^2}\, d\theta < \infty.$$

We remark that in the special case $d = 2$, Theorems 4.1.1–4.1.3 were due to [Gi–Mo] while for the general case $d \ge 3$ only part of the above results were obtained in the Phd thesis [Zh].

Our last two theorems in this section are on the two side estimates of the quality $\sum_{k=0}^{\infty} \|\sigma_k^\lambda(f) - f\|_p^s$.

Theorem 4.1.5. *Suppose $1 < s < \infty$, $r > \frac{1}{s}$ and $f \in L^p(\mathbb{S}^{d-1})$ with $1 \le p \le \infty$.*

(i) If $1 < p < \infty$, then there exists a constant $C > 1$, independent of f, such that

$$C^{-1} \int_0^1 \frac{\omega_r(f,t)_p^s}{t^2}\, dt \le \sum_{k=0}^{\infty} \|\sigma_k^\lambda(f) - f\|_p^s \le C \int_0^1 \frac{\omega_r(f,t)_p^s}{t^2}\, dt. \qquad (4.1.8)$$

(ii) If $1 \le p \le \infty$ and $\delta > \lambda$, then (4.1.8) is satisfied with δ in place of λ.

The conditions on the index s in Theorem 4.1.5 are sharp in the following senses:

Theorem 4.1.6. *Suppose $1 \le p \le \infty$ and $f \in L^p(\mathbb{S}^{d-1})$.*

(i) If $0 < r \le \frac{1}{s}$, then

$$\int_0^1 \frac{\omega_r(f,t)_p^s}{t^2}\, dt < \infty$$

if and only if $f \equiv$ constant.

(ii) Suppose $\delta > 0$. Then

$$\sum_{k=0}^{\infty} \|\sigma_k^\delta(f) - f\|_p < \infty$$

if and only if $f \equiv$ constant.

For $1 \leq p \leq \infty$, $1 < s < \infty$ and $r > 0$, we define the space $H_p^{r,s}$ by

$$H_p^{r,s} := \Big\{ f \in L^p(\mathbb{S}^{d-1}) : \int_0^1 \frac{\omega_r(f,t)_p^s}{t^2}\,dt < \infty \Big\}.$$

We identify f with g in $H_p^{r,s}$ if $f - g \equiv constant$. It then follows from Theorem 4.1.6 that $H_p^{r,s}$ for $r \leq \frac{1}{s}$ has only one element zero and therefore is trivial. From Theorem 4.1.5, it follows that $H_p^{r,s}$ for $r > \frac{1}{s}$, $1 < p < \infty$ coincides with the function space

$$\Big\{ f \in L^p(\mathbb{S}^{d-1}) : \sum_{k=0}^{\infty} \|\sigma_k^\lambda(f) - f\|_p^s < \infty \Big\}, \quad 1 < p < \infty$$

and this statement for $p = 1$ and $p = \infty$ remains valid if we replace λ with a number $\delta > \lambda$. Therefore when $r > \frac{1}{s}$, the space $H_p^{r,s}$ is actually independent of r.

It should be pointed out that $H_p^{r,s}$ is never trivial whenever $r > \frac{1}{s}$. In fact, according to Theorem 3.1.4, one can easily deduce $W_p^r \subset H_p^{r,s}$ whenever $r > \frac{1}{s}$ and $\alpha > \frac{1}{s}$, where W_p^r is the function class defined as in Chapter 1, which clearly contains many nontrivial smooth functions.

This section is organized as follows. In Subsection 4.1.2, we introduce the concept of the equiconvergent operator from [Wa], describe some of its basic properties, and establish an equivalent relationship on the strong approximation between the Cesàro operator and the equiconvergent operator. Such an equivalence plays a basic role in the proofs of Theorems 4.1.1–4.1.3 in Subsections 4.1.1–4.1.3. Theorems 4.1.5 and 4.1.6 are proved in Subsection 4.1.5 by invoking some known properties of the modulus of continuity and applying some realization theorems in Chapter 3.

4.1.2 Equiconvergent operator and its equivalence to the Cesàro operator

The equiconvergent operator was introduced by Wang Kunyang [Wa] in the investigation of the pointwise convergence of the Cesàro means. We first give its definition and describe some of its basic properties.

We write

$$\sigma_N^\delta(\cos\theta) = \frac{1}{A_N^\delta}\sum_{k=0}^{N} A_{N-k}^\delta \frac{(k+\lambda)\Gamma(\lambda)}{2\pi^{\lambda+1}} P_k^\lambda(\cos\theta).$$

Then it follows from (4.1.1) and (0.0.6) that the Cesàro mean σ_N^δ can be expressed as

$$\sigma_N^\delta(f)(x) = \int_{\mathbb{S}^{d-1}} \sigma_N^\delta(xy) f(y)\,dy. \tag{4.1.9}$$

By formula (9.41.13) of Szegö's book [Sz] we can rewrite σ_N^δ as

$$\sigma_N^\delta(t) = \frac{1}{|\mathbb{S}^{d-1}|}\Big(a_N^\delta P_N^{(\frac{d-1}{2}+\delta,\frac{d-3}{2})}(t) + \sum_{v=1}^{\infty} b_v(N,d,\delta)\sigma_N^{\delta+v}(t)\Big), \tag{4.1.10}$$

where

$$a_N^\delta = \frac{1}{2^{d-2}\Gamma(\frac{d-1}{2})}\frac{\Gamma(N+\delta+d-1)\Gamma(2N+\delta+d)}{A_N^\delta\Gamma(N+\frac{d-1}{2})\Gamma(2N+2\delta+d)},$$

$$b_v(N,d,\lambda) = \frac{(-1)^{v+1}\delta(\delta-1)\cdots(\delta-v+1)\Gamma(N+\delta+v+1)\Gamma(2N+\delta+d)}{\Gamma(v+1)\Gamma(N+\delta+1)\Gamma(2N+\delta+d+v)}$$

It follows from [Wa] that

$$|b_v(N,d,\lambda)| \le C(d,\delta)v^{-d-1-\delta}. \tag{4.1.11}$$

Definition 4.1.1 ([Wa]). For any $f \in L(\mathbb{S}^{d-1})$, define

$$S_N^\delta(f)(x) := a_N^\delta \int_{\mathbb{S}^{d-1}} f(y) P_N^{(\frac{d-1}{2}+\delta,\frac{d-3}{2})}(xy)\,dy, \quad \delta > -1.$$

We call the following operator

$$E_N^\delta(f) = S_N^\delta(f)\Big(S_N^\delta(\mathbb{1})\Big)^{-1}$$

the equiconvergent operator of σ_N^δ, where $\mathbb{1}$ denotes the constant function of value 1.

One can easily verify

$$E_N^\delta(f)(x) := \gamma_N^\delta \int_{\mathbb{S}^{d-1}} f(y) P_N^{(\frac{d-1}{2}+\delta,\frac{d-3}{2})}(xy)\,d\sigma(y), \tag{4.1.12}$$

where

$$\gamma_N^\delta = \frac{\Gamma(\delta+1)\Gamma(N+1)\Gamma(N+d-1)}{(4\pi)^{\frac{d-1}{2}}\Gamma(N+\delta+1)\Gamma(N+\frac{d-1}{2})} \simeq N^{\frac{d-1}{2}-\delta}. \tag{4.1.13}$$

It was proved by Wang Kunyang [Wa] that for $f \in L(\mathbb{S}^{d-1})$, $x \in \mathbb{S}^{d-1}$ and $\delta > -1$,

$$\lim_{N\to\infty} \sigma_N^\delta(f)(x) = f(x)$$

if and only if

$$\lim_{N\to\infty} E_N^\delta(f)(x) = f(x).$$

In fact, more can be obtained from the proof of [Wa]: *For* $1 \le p \le \infty$, $\delta > -1$ *and* $f \in L^p(\mathbb{S}^{d-1})$,

$$\lim_{N\to\infty} \|\sigma_N^\delta(f) - f\|_p = 0$$

if and only if

$$\lim_{N\to\infty} \|E_N^\delta(f) - f\|_p = 0.$$

For detail of the above material, we refer the reader to [Wa] and [Wa-L, Chapter 4].

The main aim in this subsection is to prove the following equivalence theorem, which plays a basic role in our later proofs of the main results.

Theorem 4.1.7. *Suppose* $\delta > 0$, $1 < p < \infty$ *and* f *is a function defined on* $\mathbb{S}^{d-1}$. *Then there exists a constant* $C(d,\delta) > 0$ *such that for any* $x \in \mathbb{S}^{d-1}$,

$$\frac{1}{C(d,\delta)} \sum_{k=0}^\infty |E_k^\delta(f)(x)-f(x)|^p \le \sum_{k=0}^\infty |\sigma_k^\delta(f)(x)-f(x)|^p \le C(d,\delta)\Big(\frac{p}{p-1}\Big)^p \sum_{k=0}^\infty |E_k^\delta(f)(x)-f(x)|^p.$$

The proof of Theorem 4.1.7 is based on a series of lemmas. Its idea comes from [Wa].

Lemma 4.1.8. *Suppose* $v,\ k \in \mathbb{N}$ *and* $\delta > 0$. *Then*

$$\sum_{j=k}^\infty \frac{A_{j-k}^{v-1}}{A_j^{\delta+v}} \le C(\delta) v^{\delta+1} k^{-\delta}, \tag{4.1.14}$$

where $C(\delta)$ *is a constant independent of* v *and* k.

Proof. We use the inequalities

$$C(b)^{-1}(a+1)^b \le \frac{\Gamma(a+b)}{\Gamma(a)} \le C(b)(a+1)^b, \tag{4.1.15}$$

where $a,\ b>0$ and $C(b)>0$ is independent of a. Then, by (4.1.2), we get

$$\sum_{j=k}^{\infty}\frac{A_{j-k}^{v-1}}{A_j^{\delta+v}}=C(\delta)\sum_{j=k}^{\infty}\frac{\Gamma(j-k+v)}{\Gamma(j-k+1)}\frac{\Gamma(j+1)}{\Gamma(j+\delta+v+1)}\frac{\Gamma(\delta+v+1)}{\Gamma(v)}$$
$$\leq C(\delta)v^{\delta+1}\sum_{j=k}^{\infty}\frac{j-k+1}{j+\delta+2}\frac{j-k+2}{j+\delta+3}\cdots\frac{j-k+v-1}{j+\delta+v}\Big(\frac{1}{j+1}\Big)^{\delta+1}.$$

Noticing that

$$\frac{j-k+1}{j+\delta+2}\frac{j-k+2}{j+\delta+3}\cdots\frac{j-k+v-1}{j+\delta+v}\leq 1,$$

we get (4.1.14) and complete the proof. □

Lemma 4.1.9. *Suppose* $1\leq p<\infty$, $\delta>0$ *and* $v\in\mathbb{N}$. *Then*

$$\sum_{k=0}^{\infty}|\sigma_k^{\delta+v}(f)-f|^p\leq C(\delta)v^{\delta+1}\sum_{k=0}^{\infty}|\sigma_k^{\delta+1}(f)-f|^p. \tag{4.1.16}$$

Proof. Without loss of generality, we may assume $v\geq 2$. We start from the representation

$$\sigma_k^{\delta+v}(f)=\frac{1}{A_k^{\delta+v}}\sum_{j=0}^{k}A_{k-j}^{v-2}A_j^{\delta+1}\sigma_j^{\delta+1}(f).$$

Noticing that

$$\sum_{j=0}^{k}A_{k-j}^{v-2}A_j^{\delta+1}=A_k^{\delta+v},$$

we get, by Hölder inequality,

$$\left|\sigma_k^{\delta+v}(f)-f\right|^p\leq\frac{1}{A_k^{\delta+v}}\sum_{j=0}^{k}A_{k-j}^{v-2}A_j^{\delta+1}\left|\sigma_j^{\delta+1}(f)-f\right|^p.$$

So,

$$\sum_{k=0}^{\infty}\left|\sigma_k^{\delta+v}(f)-f\right|^p\leq\sum_{j=0}^{\infty}A_j^{\delta+1}\left|\sigma_j^{\delta+1}(f)-f\right|^p\sum_{k=j}^{\infty}\frac{A_{k-j}^{v-2}}{A_k^{\delta+v}},$$

which, by Lemma 4.1.8, implies (4.1.16) and completes the proof.

□

The following lemma can be found in [Sz].

Lemma 4.1.10. *Suppose $\delta \geq 0$ is an integer. Then*

$$\sigma_N^\delta(x) = \frac{1}{A_N^\delta}\sum_{k=0}^{N} G_k(N,\delta) P_k^{(\frac{d-1}{2}+\delta, \frac{d-3}{2})}(x),$$

where

$$G_k(N,\delta) \leq \begin{cases} CN^{\delta-1}, & \text{if } k=0,\\ C\sum_{v=0}^{\delta-1}(N-k)^v k^{\frac{d-3}{2}-v}, & \text{if } 1\leq k\leq N-1,\\ CN^{\frac{d-1}{2}}, & \text{if } k=N. \end{cases}$$

Proof of Theorem 4.1.7. First, we prove the direct inequality

$$\sum_{k=0}^{\infty} |E_k^\delta(f) - f|^p \leq C(d,\delta)\sum_{k=0}^{\infty} |\sigma_k^\delta(f) - f|^p, \tag{4.1.17}$$

with $\delta > 0,\ 1 \leq p < \infty$.

By (4.1.9), (4.1.10) and (4.1.12), we obtain the following formula

$$\sigma_N^\delta(f)(x) = \beta_N^\delta E_N^\delta(f)(x) + \left|\mathbb{S}^{d-1}\right|^{-1} T_N^\delta(f)(x),$$

where

$$\beta_N^\delta = \frac{\Gamma(N+\delta+d-1)\Gamma(2N+\delta+d)}{\Gamma(N+d-1)\Gamma(2N+2\delta+d)} \tag{4.1.18}$$

and

$$T_N^\delta(f) := \sum_{v=1}^{\infty} b_v(N,d,\delta)\sigma_N^{\delta+v}(f). \tag{4.1.19}$$

Noticing that $E_N^\delta(\mathbb{1}) = \sigma^\delta(\mathbb{1}) = \mathbb{1}$, we obtain

$$E_N^\delta(f) - f = (\beta_N^\delta)^{-1}\left[\sigma_N^\delta(f) - f\right] + (\beta_N^\delta)^{-1}|\mathbb{S}^{d-1}|^{-1}\left[T_N^\delta(f) - T_N^\delta(\mathbb{1})f\right]. \tag{4.1.20}$$

Here $\mathbb{1}$ is the constant function of value 1.

(4.1.11) and Holder inequality give

$$\sum_{N=0}^{\infty} |T_N^\delta(f) - T_N^\delta(\mathbb{1})f|^p \leq C(d,\delta)\sum_{v=1}^{\infty}\sum_{N=0}^{\infty} |b_v(N,d,\delta)| \left|\sigma_N^{\delta+v}(f) - f\right|^p$$
$$\leq C(d,\delta)\sum_{v=1}^{\infty} v^{-(d+1+\delta)} \sum_{N=0}^{\infty}\left|\sigma_N^{\delta+v}(f) - f\right|^p,$$

which, by (4.1.16), is estimated by

$$C(d,\delta)\sum_{k=0}^{\infty}|\sigma_N^\delta(f)-f|^p. \tag{4.1.21}$$

On the other hand, it follows from (4.1.15) and (4.1.18) that

$$\beta_N^{-1}=O(1), \text{ as } N\to\infty. \tag{4.1.22}$$

Now combining (4.1.20), (4.1.21) and (4.1.22) yields (4.1.17).

Next, we prove the converse inequality.

Let $\delta_0=[\delta]+1$ and $\varepsilon=\delta_0-\delta$. We apply formula (9.4.3) of [Sz] to get

$$P_\mu^{(\frac{d-1}{2}+\delta_0,\frac{d-3}{2})}(t)=\sum_{v=0}^{\mu}d_v(\mu,d,\delta)P_v^{(\frac{d-1}{2}+\delta,\frac{d-3}{2})}(t), \tag{4.1.23}$$

where

$$d_v(\mu,d,\delta)=\frac{A_{\mu-v}^{\varepsilon-1}(2v+d+\delta)\Gamma(\mu+v+d+\delta+\varepsilon-1)\Gamma(v+d+\delta-1)\Gamma(\mu+\frac{d-1}{2})}{\Gamma(\mu+v+d+\delta)\Gamma(v+\frac{d-1}{2})\Gamma(\mu+d-1+\delta+\varepsilon)}.$$

Substituting (4.1.23) into (4.1.12) yields

$$E_\mu^{\delta_0}(f)-f=\sum_{v=0}^{\mu}\frac{\gamma_\mu^{\delta_0}}{\gamma_v^\delta}d_v(\mu,d,\delta)\Big(E_v^\delta(f)-f\Big),$$

with γ_μ^δ as in (4.1.13).

Noticing

$$|d_v(\mu,d,\delta)|\le C(d,\delta)\Big(\frac{v}{\mu}\Big)^{\frac{n+1}{2}+\delta}(\mu-v)^{\varepsilon-1},$$

we get from (4.1.13)

$$\Big|\frac{\gamma_\mu^{\delta_0}}{\gamma_v^\delta}d_v(\mu,n,\delta)\Big|\le C(d,\delta)(\mu-v)^{\varepsilon-1}\frac{v^{2\delta+1}}{\mu^{\varepsilon+2\delta+1}}.$$

Hence, by Hölder inequality,

$$\begin{aligned}\sum_{\mu=0}^{\infty}|E_\mu^{\delta_0}(f)-f|^p &\le C(d,\delta)\sum_{v=0}^{\infty}\sum_{\mu=v}^{\infty}(\mu-v)^{\varepsilon-1}\frac{v^{2\delta+1}}{\mu^{\varepsilon+2\delta+1}}\Big|E_v^\delta(f)-f\Big|^p\\ &\le C\sum_{v=0}^{\infty}\Big|E_v^\delta(f)-f\Big|^p. \end{aligned}\tag{4.1.24}$$

On the other hand, by Lemma 4.1.10 and (4.1.9), (4.1.12), one can easily obtain

$$\sigma_L^{\delta_0}(f)(x) - f(x) = \frac{1}{A_L^{\delta_0}} \sum_{\ell=0}^{L} H_\ell(L, \delta_0)(E_\ell^{\delta_0}(f)(x) - f(x)),$$

where

$$H_\ell(L, \delta_0) \le \begin{cases} CL^{\delta_0 - 1}, & \ell = 0, \\ C \sum_{\rho=0}^{\delta_0 - 1} (L-\ell)^\rho \ell^{-1-\rho+\delta_0}, & 1 \le \ell \le L-1, \\ CL^{\delta_0}, & \ell = L. \end{cases}$$

Therefore, by an elementary computation, we get

$$|\sigma_L^{\delta_0}(f)(x) - f(x)| \le C \frac{1}{L+1} \sum_{\ell=0}^{L-1} |E_\ell^{\delta_0}(f)(x) - f(x)| + |E_L^{\delta_0}(f) - f|.$$

Now applying the following Hardy–Landau inequality:

$$\sum_{k=1}^{m} \Big| \frac{1}{k} \sum_{j=1}^{k} a_j \Big|^p \le \Big(\frac{p}{p-1} \Big)^p \sum_{k=1}^{m} a_k^p, \ a_k \ge 0, \ p > 1,$$

we get from (4.1.24)

$$\sum_{k=0}^{\infty} |\sigma_k^{\delta_0}(f) - f|^p \le C(d, \delta) \Big(\frac{p}{p-1} \Big)^p \sum_{k=0}^{\infty} |E_k^{\delta}(f) - f|^p. \tag{4.1.25}$$

By (4.1.19), (4.1.16) and (4.1.11), we conclude

$$\sum_{k=0}^{\infty} |T_k^{\delta}(f) - f|^p \le C(d, \delta) \sum_{k=0}^{\infty} |\sigma_k^{\delta+1}(f) - f|^p. \tag{4.1.26}$$

Noticing $\delta + 1 \ge \delta_0$, by Hölder inequality, one can easily verify that the right-hand side of (4.1.26) is dominated by a multiple of the left -hand side of (4.1.25). This, together with (4.1.20), yields the converse inequality and completes the proof. □

4.1.3 Proofs of Theorems 4.1.1 and 4.1.2

To prove Theorems 4.1.1 and 4.1.2, we need some known results on orthogonal series.

Let $\{\phi_k\}_{k=0}^{\infty}$ be a complete orthonormal system satisfying

$$\sup_k \|\phi_k\|_\infty \le C < \infty$$

with respect to the Lebesgue measure on $(0,\pi)$. For $f \in L(0,\pi)$, write

$$a_k(f) = \int_0^\pi f(t)\phi_k(t)\,dt.$$

The following lemma can be easily obtained by applying the Riesz -Thorin interpolation theorem.

Lemma 4.1.11. *Suppose $1 \le p \le \infty$, $\frac{1}{p}+\frac{1}{q}=1$ and the notations are the same as the above.*

(i) If $2 \le p \le \infty$ and $f \in L^q$, then

$$\Big(\sum_{k=0}^{\infty} |a_k(f)|^p\Big)^{\frac{1}{p}} \le C\|f\|_q,$$

with $C>0$ independent of f.

(ii) If $1 \le p \le 2$ and $\{a_k(f)\}_{k=0}^{\infty} \in \ell^p$, then

$$\|f\|_q \le \Big(\sum_{k=0}^{\infty} |a_k(f)|^p\Big)^{\frac{1}{p}}.$$

The following two lemmas were due to [St-W2].

Lemma 4.1.12. *Suppose $f \in L^p(0,\pi)$, $1<p<\infty$ and the notations are as the above.*

(i) If $1<p\le 2$, then

$$\Big(\sum_{k=0}^{\infty} |a_k(f)|^p(k+1)^{p-2}\Big)^{\frac{1}{p}} \le C\|f\|_p.$$

(ii) If $2 \le p < \infty$, then

$$\|f\|_p \le C\Big(\sum_{k=0}^{\infty} |a_k(f)|^p(k+1)^{p-2}\Big)^{\frac{1}{p}}.$$

Lemma 4.1.13. *Let $f(x)$ be a function on $(0,\pi)$ with Fourier coefficients $a_k(f)$ with respect to $\{\phi_k\}$. Then*

$$\Big(\sum_{k=1}^{\infty} |a_k(f)|^q(k+1)^{-\beta q}\Big)^{\frac{1}{q}} \le C\Big(\int_0^1 |f(x)|^p|x|^{\alpha p}\,dx\Big)^{\frac{1}{p}},$$

whenever $0 \le \alpha < \frac{1}{p'}$, $q \ge p$ and $\beta = \frac{1}{q}+\frac{1}{p}-1+\alpha \ge 0$.

Proof of Theorem 4.1.1. We get the idea from [Gi-Mo]. Invoking (4.1.12) with $\delta = \lambda := \frac{d-2}{2}$, noting $E_k^\lambda(\mathbb{1}) = \mathbb{1}$, we have

$$E_k^\lambda(f)(x) - f(x) = \gamma_k^\lambda \int_0^\pi \Big(S_\theta(f)(x) - f(x)\Big) P_k^{(d-\frac{3}{2},\frac{d-3}{2})}(\cos\theta)\sin^{d-2}\theta\, d\theta, \tag{4.1.27}$$

with γ_k^λ as in (4.1.13). Let

$$\phi_k(\theta) = \Big(\frac{(2k+\frac{3}{2}d-2)\Gamma(k+1)\Gamma(k+\frac{3}{2}d-2)}{\Gamma(k+d-\frac{1}{2})\Gamma(k+\frac{d-1}{2})}\Big)^{\frac{1}{2}} P_k^{(d-\frac{3}{2},\frac{d-3}{2})}(\cos\theta)\sin^{d-1}\frac{\theta}{2}\cos^{\frac{d-2}{2}}\frac{\theta}{2}.$$

Then it follows from (0.0.21) that $\{\phi_k\}_{k=0}^\infty$ forms an orthonormal basis of $L^2[0,\pi]$. By a straightforward computation, (4.1.27) can be rewritten as

$$E_k^\lambda(f)(x) - f(x) = \beta_k \int_0^\pi F_x(\theta)\phi_k(\theta)\, d\theta =: \beta_k a_k(F_x), \tag{4.1.28}$$

where

$$F_x(\theta) = \frac{S_\theta(f)(x) - f(x)}{\sin\frac{\theta}{2}}\cos^\lambda\frac{\theta}{2}$$

and

$$\beta_k = \Big(\frac{(2k+\frac{3}{2}d-2)\Gamma(k+\frac{3}{2}d-2)\Gamma(k+\frac{d-1}{2})}{\Gamma(k+d-\frac{1}{2})\Gamma(k+1)}\Big)^{\frac{1}{2}} \frac{(4\pi)^{\frac{d-1}{2}}\Gamma(k+\frac{d}{2})}{\Gamma(\frac{d}{2})\Gamma(k+d-1)}.$$

It is easy to verify

$$0 < c_d \le \beta_k \le d_d.$$

(i) Applying Lemma 4.1.11 (i) to the orthonormal system $\{\phi_k(\theta)\}_{k=0}^\infty$ above and the function $F_x(\theta)$, we get for $2 \le p \le \infty$,

$$\Big(\sum_{k=0}^\infty |E_k^\lambda(f)(x) - f(x)|^p\Big)^{\frac{1}{p}} \le C\Big(\int_0^\pi |F_x(\theta)|^q\, d\theta\Big)^{\frac{1}{q}},$$

which, together with Theorem 4.1.7 and the fact

$$|F_x(\theta)| \sim \Big|\frac{S_\theta(f)(x) - f(x)}{\theta}(\pi-\theta)^\lambda\Big|, \quad \theta \in (0,\pi), \tag{4.1.29}$$

implies (4.1.3) and completes the proof of (i).

(ii) Invoking Lemma 4.1.11 (ii), we get for $1 \le p \le 2$,

$$\Big(\int_0^\pi |F_x(\theta)|^q\, d\theta\Big)^{\frac{1}{q}} \le C\Big(\sum_{k=0}^\infty |E_k^\lambda(f)(x) - f(x)|^p\Big)^{\frac{1}{p}},$$

which, again by Theorem 4.1.7 and the fact (4.1.29), implies the desired result of (ii). This completes the proof. □

Theorem 4.1.2 can be proved by using Theorem 4.1.7 and Lemmas 4.1.12 and 4.1.13. Since the proof runs along the same lines as that of [Gi-Mo], we omit the detail.

4.1.4 Proof of Theorem 4.1.3

The following lemma can be found in [Be-Sh, P124].

Lemma 4.1.14. *Let ψ be a nonnegative measurable function on $(0,\infty)$ and suppose $\alpha \in (0,\infty)$ and $1 \le p < \infty$. Then*

$$\Big(\int_0^\infty \Big(\frac{1}{t^\alpha}\int_0^t \psi(s)\,ds\Big)^p \frac{dt}{t}\Big)^{\frac{1}{p}} \le \frac{1}{\alpha}\Big(\int_0^\infty (t^{1-\alpha}\psi(t))^p \frac{dt}{t}\Big)^{\frac{1}{p}} \tag{4.1.30}$$

and

$$\Big(\int_0^\infty \Big(t^\alpha \int_t^\infty \psi(s)\frac{ds}{s}\Big)^p \frac{dt}{t}\Big)^{\frac{1}{p}} \le \frac{1}{\alpha}\Big(\int_0^\infty (t^{\alpha}\psi(t))^p \frac{dt}{t}\Big)^{\frac{1}{p}}. \tag{4.1.31}$$

Proof of Theorem 4.1.13. (4.1.7) for $2 \le p < \infty$ is a consequence of part (i) of Theorem 4.1.2. For the case $p < 2$, we start from the representation

$$E_k^\delta(f)(x) - f(x) = \gamma_k^\delta \int_0^\pi [S_\theta(f)(x) - f(x)] P_k^{(\frac{d-1}{2}+\delta,\frac{d-3}{2})}(\cos\theta)\sin^{d-2}\theta\,d\theta. \tag{4.1.32}$$

We break the integral in the right-hand side of (4.1.32) into three parts:

$$\gamma_k^\delta \int_0^{k^{-1}} + \gamma_k^\delta \int_{k^{-1}}^{\frac{\pi}{2}} + \gamma_k^\delta \int_{\frac{\pi}{2}}^{\pi} := I_{k,1} + I_{k,2} + I_{k,3}.$$

Now write $\phi_x(\theta) = S_\theta(f)(x) - f(x)$. By Hölder inequality and elementary calculations, we get

$$\begin{aligned}\sum_{k=0}^\infty |I_{k,1}|^p &\le C\sum_{k=0}^\infty \Big((k+1)\int_0^{(k+1)^{-1}} |\phi_x(\theta)|\,d\theta\Big)^p \\ &\le C\int_0^\infty \Big|s^{-1-\frac{1}{p}}\int_0^s |\phi_x(\theta)|\chi_{[0,\frac{\pi}{2}]}(\theta)\,d\theta\Big|^p \frac{ds}{s},\end{aligned}$$

which, by (4.1.30), is estimated by

$$C\int_0^{\frac{\pi}{2}} \frac{|\phi_x(\theta)|^p}{\theta^2}\,d\theta.$$

Similarly, for $p > \frac{1}{\delta-\lambda}$,

$$\begin{aligned}\sum_{k=0}^{\infty} |I_{k,2}|^p &\le C \sum_{k=0}^{\infty} \Big((k+1)^{\lambda-\delta} \int_{(k+1)^{-1}}^{\frac{\pi}{2}} |\phi_x(\theta)| \theta^{\frac{d}{2}-2-\delta}\, d\theta \Big)^p \\ &\le C \int_0^{\infty} \Big| s^{\delta-\frac{d-2}{2}-\frac{1}{p}} \int_s^{\infty} \frac{\phi_x(\theta)\theta^{\frac{d}{2}-\delta-1}\chi_{[0,\frac{\pi}{2}]}(\theta)}{\theta}\, d\theta \Big|^p \frac{ds}{s}, \\ &\le C \Big(\frac{1}{\delta-\lambda-\frac{1}{p}} \Big)^p \int_0^{\frac{\pi}{2}} \frac{|\phi_x(\theta)|^p}{\theta^2}\, d\theta,\end{aligned}$$

where the last inequality follows from (4.1.31).

Finally, since $p > \frac{1}{\delta-\lambda}$, the following estimate is obvious:

$$\sum_{k=0}^{\infty} |I_{k,3}|^p \le C \int_{\frac{\pi}{2}}^{\pi} |\phi_x(\theta)|^p (\pi-\theta)^{\lambda p}\, d\theta.$$

Putting these together, we complete the proof. □

4.1.5 Proofs of Theorems 4.1.5 and 4.1.6

Proof of Theorem 4.1.5. (i) We first prove (4.1.8) for $r = 1$. It follows from Theorem 3.1.11 and Corollary 3.2.16 that

$$\|\sigma_k^{\lambda}(f) - f\|_p \approx \omega_1(f, \frac{1}{k})_p. \tag{4.1.33}$$

From the monotonicity of the function $\omega_1(f,\cdot)$, we have

$$\int_0^1 \frac{\omega_1(f,t)_p^s}{t^2} \approx \sum_{k=1}^{\infty} \omega_1(f, \frac{1}{k})_p^s \quad 1 < p < \infty. \tag{4.1.34}$$

Now combining (4.1.33) and (4.1.34) yields (4.1.8) for $r = 1$.

Next, we prove (4.1.8) for general $r > 0$. From the first case, it will suffice to prove that for $\alpha > \beta > \frac{1}{s}$,

$$\int_0^1 \frac{\omega_\alpha(f,t)_p^s}{t^2}\, dt \approx \int_0^1 \frac{\omega_\beta(f,t)_p^s}{t^2}\, dt.$$

The inequality

$$\int_0^1 \frac{\omega_\alpha(f,t)_p^s}{t^2}\, dt \le C \int_0^1 \frac{\omega_\beta(f,t)_p^s}{t^2}\, dt$$

is obvious, since (see [Rus1])

$$\omega_\alpha(f,t)_p \le C\omega_\beta(f,t)_p$$

whenever $\alpha > \beta > 0$.

Now we consider the converse inequality

$$\int_0^1 \frac{\omega_\beta(f,t)_p^s}{t^2}\,dt \le C\int_0^1 \frac{\omega_\alpha(f,t)_p^s}{t^2}\,dt. \tag{4.1.35}$$

We use the following result

$$\omega_\beta(f,t)_p \le Ct^\beta \int_t^1 \frac{\omega_\alpha(f,u)_p}{u^{\beta+1}}\,du, \tag{4.1.36}$$

where $\alpha > \beta,\ t > 0$. The inequality (4.1.36) can be derived by following the standard method in [Di1].

Now by (4.1.36),

$$\begin{aligned}\int_0^1 \frac{\omega_\beta(f,t)_p^s}{t^2}\,dt &\le C\int_0^1 t^{\beta s-2}\Big(\int_t^1 \frac{\omega_\alpha(f,u)_p}{u^{\beta+1}}\,du\Big)^s dt\\ &\le C\int_0^\infty \Big(t^{\beta-\frac{1}{s}}\int_t^\infty \frac{\omega_\alpha(f,u)_p\chi_{[0,1]}(u)}{u^\beta}\,\frac{du}{u}\Big)^s\,\frac{dt}{t},\end{aligned}$$

which, by Hardy inequality (4.1.31), implies (4.1.35).

The proof of (ii) is almost identical to that of (i). (The only difference is that (4.1.33) is valid for all $1 \le p \le \infty$ in this case.) □

Proof of Theorem 4.1.6. (i) Notice that

$$\frac{\omega_r(f,t)_p^s}{t} \le \int_t^{2t} \frac{\omega_r(f,\theta)_p^s}{\theta^2}\,d\theta = o(1), \quad \text{as } t\to 0+.$$

We get

$$\omega_r(f,t)_p = o(t^{\frac{1}{s}}) = o(t^r), \quad \text{as } t\to 0+.$$

But it follows from [Wa-L, P192, Corollary 4.6.4] that $\omega_r(f,t)_p = o(t^r)$, as $t\to 0+$ if and only if $f \equiv constant$. This finishes the proof of (i).

(ii) Notice that for $\delta > 0$ and $1 \le p \le \infty$,

$$\sum_{j=0}^{\infty} \|\sigma_j^{\delta+1}(f) - f\|_p \le C_\delta \sum_{j=0}^{\infty} \|\sigma_j^{\delta}(f) - f\|_p.$$

Without loss of generality, we may assume $\delta > \lambda$. From the proof of Theorem 4.1.1 (i), it follows that (4.1.8) is valid with $r = s = 1$ and with λ replaced by δ. Thus, from (i), we conclude $f \equiv$ *constant*. This completes the proof. □

4.2 Strong approximation by the Cesàro means with critical index in the Hardy spaces $H^p(\mathbb{S}^{d-1})$ $(0 < p \le 1)$

Let $H^p \equiv H^p(\mathbb{S}^{d-1})$ $(0 < p \le 1)$ denote the real Hardy space on $\mathbb{S}^{d-1}$. Given a distribution f on $\mathbb{S}^{d-1}$, assume $\sigma_k^\delta(f)$ denotes its Cesàro mean of order $\delta > -1$. (We give the precise definitions of $H^p(\mathbb{S}^{d-1})$ and $\sigma_k^\delta(f)$ in Section 4.2.1 below.) The following statements on the uniform boundedness of the Cesàro means in $H^p(\mathbb{S}^{d-1})$ were proved in [CTW].

(i) For $0 < p \le 1$ and $\delta > \delta(p) := \frac{d-1}{p} - \frac{d}{2}$,

$$\sum_k \|\sigma_k^\delta\|_{(H^p,H^p)} < \infty.$$

(ii) The estimate in (i) is sharp in the sense that

$$\sup_k \|\sigma_k^\delta\|_{(H^p,H^p)} = \infty$$

whenever $\delta \le \delta(p)$. In fact, more can be stated: For $\delta < \delta(p)$,

$$\sup_k \|\sigma_k^\delta\|_{w(H^p,L^p)} = \infty,$$

while for $\delta = \delta(p)$,

$$\|\sigma_k^\delta\|_{(H^p,L^p)} \ge C(\log k)^{\frac{1}{p}}.$$

(iii) For $0 < p < 1$ and the critical index $\delta = \delta(p)$,

$$\sup_k \|\sigma_k^\delta\|_{w(H^p,L^p)} < \infty.$$

For $p = 1$ and $\delta = \lambda$, the above weak type estimate fails.

The main goal in this section, as suggested in the title, is to consider the strong approximation by the Cesàro means with critical index. Before stating the main result, we have to introduce some necessary notations. Given a distribution f on $\mathbb{S}^{d-1}$, let $f^{(r)}$ $(r > 0)$ denote the r-th order derivative of f. (See Section 4.2.1 for precise definition.) For $r > 0$ and $0 < p \le 1$, let $\widetilde{W}_p^r$ denote the function class

$$\widetilde{W}_p^r := \Big\{ f \in H^p(\mathbb{S}^{d-1}) : \; f^{(r)} \in H^p(\mathbb{S}^{d-1}) \Big\}.$$

Define the K-functional $K_r(f,t)_{H^p}$ by

$$K_r(f,t)_{H^p} := \inf\Big\{ \|f - g\|_{H^p} + t^r \|g^{(r)}\|_{H^p} : \; g \in \widetilde{W}_p^r \Big\}.$$

For $0 < p \le 1$ and $f \in H^p$, let $E_j(f, H^p)$ $(j = 0, 1, \cdots)$ be the best approximation of f, by spherical harmonics of degree less than or equal to j, in the space $H^p(\mathbb{S}^{d-1})$.

The main result in this section can be stated as follows.

Theorem 4.2.1. *Let $0 < p < 1$ and $\delta = \delta(p) := \frac{d-1}{p} - \frac{d}{2}$. Then for $f \in H^p(\mathbb{S}^{d-1})$,*

$$\sum_{j=1}^{N} \frac{1}{j} \|\sigma_j^\delta(f) - f\|_{H^p}^p \approx \sum_{j=1}^{N} \frac{1}{j} E_j^p(f, H^p).$$

As a consequence, we have

Corollary 4.2.2. *Suppose $0 < p < 1$ and $\delta = \delta(p) := \frac{d-1}{p} - \frac{d}{2}$. Then for $f \in H^p(\mathbb{S}^{d-1})$,*

$$\frac{1}{\log N} \sum_{k=1}^{N} \frac{\|\sigma_k^\delta(f) - f\|_{H^p}^p}{k} \le C K_1^p\Big(f, \Big(\frac{1}{\log N}\Big)^{\frac{1}{p}}\Big)_{H^p}.$$

In the case of Riesz means for the periodic functions, Theorem 4.2.1 was due to [Be2] while Corollary 4.2.2 was due to [C-J-Lu]. Our proof here relies heavily on a new characterization of the Hardy space $H^p(\mathbb{S}^{d-1})$, which is in terms of the maximal Cesàro operator. Our approach is different from those of [Be2] and [C-J-Lu]. We remark that it appears to be difficult to obtain Theorem 4.2.1 and Corollary 4.2.2 simply by following the technique developed in [Be2] and [C-J-Lu]. (This can be seen from the arguments in [CTW] and [Chen].)

Similar results for the generalized Riesz means $R_k^{\delta,\alpha}(f)$ can be stated as follows. (We give the precise definition of $R_k^{\delta,\alpha}(f)$ in Section 4.2.1 below.)

Theorem 4.2.3. *Let* $\alpha > 0,\ 0 < p < 1$ *and* $\delta = \delta(p) := \frac{d-1}{p} - \frac{d}{2}$. *Then for* $f \in H^p(\mathbb{S}^{d-1})$,

$$\sum_{j=1}^{N} \frac{1}{j} \|R_j^{\delta,\alpha}(f) - f\|_{H^p}^p \approx \sum_{j=1}^{N} \frac{1}{j} E_j^p(f, H^p).$$

Corollary 4.2.4. *Let* $\alpha > 0,\ 0 < p < 1$ *and* $\delta = \delta(p) := \frac{d-1}{p} - \frac{d}{2}$. *Then for* $f \in H^p(\mathbb{S}^{d-1})$,

$$\frac{1}{\log N} \sum_{k=1}^{N} \frac{\|R_k^{\delta,\alpha}(f) - f\|_{H^p}^p}{k} \le C K_\alpha^p\Big(f, \Big(\frac{1}{\log N}\Big)^{\frac{1}{p}}\Big)_{H^p}.$$

Since the proofs of Theorem 4.2.3 and Corollary 4.2.4 run along the same lines as those of Theorem 4.2.1 and Corollary 4.2.2, (in fact, simpler), in the remainder of this section, we only prove Theorem 4.2.1 and Corollary 4.2.2.

The organization of this section is as follows. Section 4.2.1 contains some basic definitions and results on the real Hardy spaces $H^p(\mathbb{S}^{d-1})$. A new characterization of $H^p(\mathbb{S}^{d-1})$ is given in terms of the maximal Cesàro operator in Section 4.2.2. In Section 4.2.3, we briefly sketch some of the basic approximation results in $H^p(\mathbb{S}^{d-1})$. Theorem 4.2.1 and Corollary 4.2.2 are then proved in Section 4.2.4 by invoking the characterization theorem of Section 4.2.2 and the approximation results obtained in Section 4.2.3.

4.2.1 Hardy spaces on $\mathbb{S}^{d-1}$

Most material described in this subsection can be found in [CTW] and [Col].

Let $\mathcal{S} \equiv \mathcal{S}(\mathbb{S}^{d-1})$ denote the set of indefinitely differentiable functions on $\mathbb{S}^{d-1}$ endowed with the usual test function topology and let $\mathcal{S}' \equiv \mathcal{S}'(\mathbb{S}^{d-1})$ be the dual of $\mathcal{S}$. $\mathcal{S}$ is called the space of test functions and $\mathcal{S}'$ the space of distributions. $\Big($ One may think of a function on $\mathbb{S}^{d-1}$ as a function defined on annulus about $\mathbb{S}^{d-1}$ by extending the function to be constant along rays through the origin. This allows us to associate with

$$\gamma = (\gamma_1, \cdots, \gamma_d), \quad D^\gamma = \Big(\frac{\partial}{\partial x_1}\Big)^{\gamma_1} \cdots \Big(\frac{\partial}{\partial x_d}\Big)^{\gamma_d}, \quad |\gamma| = \gamma_1 + \cdots + \gamma_d$$

a differential operator of order $|\gamma|$ by differentiating in $\mathbb{R}^d$ and restricting to $\mathbb{S}^{d-1}$. The topology on $\mathcal{S}$ is that induced by the seminorms

$$N_m(\varphi) = \sum_{|\gamma|=m} \|D^\gamma \varphi\|_\infty, \quad m = 0, 1, 2, \cdots .\Big)$$

The pairing of $f \in \mathcal{S}'$ and $\varphi \in \mathcal{S}$ is given by $< f, \varphi >$. If f is an integrable function on $\mathbb{S}^{d-1}$, we set $< f, \varphi >= \int_{\mathbb{S}^{d-1}} f(u)\varphi(u)\, d\sigma(u)$.

For $x \in \mathbb{S}^{d-1}$ and $z \in B_d := \Big\{(z_1, \cdots, z_d) \in \mathbb{R}^d : \; z_1^2 + \cdots + z_d^2 \le 1\Big\}$, let

$$P_z(x) = c_d \frac{1 - |z|^2}{|z - x|^d}.$$

P_z belongs to $\mathcal{S}$ and is called the Poisson Kernel. c_d is chosen so that $\displaystyle\int_{\mathbb{S}^{d-1}} P_z(x)\, d\sigma(x) = 1$ for all $z \in B_d$. For $f \in \mathcal{S}'$, we write

$$f(z) =< f, P_z > .$$

$f(z)$ is called the Poisson integral of f.

For a distribution f we define the *radial maximal function,* $P^+f(x)$,

$$P^+f(x) = \sup_{0 \le r < 1} |f(rx)|, \quad x \in \mathbb{S}^{d-1}.$$

Definition 4.2.1 ([Col]). The Hardy space $H^p(\mathbb{S}^{d-1})$ is the linear space of distributions f with $\|P^+f\|_p < \infty$. We set $\|f\|_{H^p} = \|P^+f\|_p$.

Definition 4.2.2 ([Col]). Let m be a non-negative integer and $x \in \mathbb{S}^{d-1}$. We say that $\varphi \in K_m$ if $\varphi \in \mathcal{S}$ and

i) $\operatorname{supp} \varphi \subset B(x, h)$ for some $h > 0$ and

i) $\sup\Big\{|h|^{d-1+|\gamma|} |D^\gamma \varphi(z)| : \; z \in B_d, \; |\gamma| \le m\Big\} \le 1.$

Definition 4.2.3 ([Col]). For $f \in \mathcal{S}'$, *the grand maximal function* f^* of f is the function

$$f^*(x) = \sup\Big\{ |<f, \varphi>| : \quad \varphi \in K_m(x) \Big\}, \quad x \in \mathbb{S}^{d-1}.$$

Theorem 4.2.5 ([Col]). *If* $m > (d-1)/p,\ 0 < p < \infty$, *then*

$$A\|P^+ f\|_p \le \|f^*\|_p \le B\|P^+ f\|_p,$$

where $0 < A < B < \infty$ *are constants that depend on* p, m *and* d.

It is well known that if $p > 1$, $\|P^+ f\|_p$ is equivalent to $\|f\|_p$. Thus, $H^p(\mathbb{S}^{d-1})$ coincides with $L^p(\mathbb{S}^{d-1})$ if $p > 1$. For the remainder of this section we assume $0 < p \le 1$.

The conclusion of Theorem 4.2.5 is often described as the "grand maximal function" characterization of Hardy spaces. We now turn to the " atomic characterization."

Definition 4.2.4. A regular p-atom, $0 < p \le 1$, centered at $x \in \mathbb{S}^{d-1}$, is a function $a \in L^\infty(\mathbb{S}^{d-1})$ satisfying:

i) supp $a \subset B(x, s)$ for some $s > 0$,

ii) $\|a\|_\infty \le s^{-\frac{d-1}{p}}$, and

iii) $\displaystyle\int_{\mathbb{S}^{d-1}} a(u) Y(u)\, d\sigma(u) = 0$, for every spherical harmonic of degree less than or equal to $[(d-1)(\frac{1}{p} - 1)]$.

An exceptional atom is a function $a \in L^\infty(\mathbb{S}^{d-1})$ with $\|a\|_\infty \le 1$.

Theorem 4.2.6 ([Col]). *Let* $0 < p \le 1$. *If* $\{a_j\}_{j=0}^\infty$ *is a sequence of exceptional or regular* p*-atoms, and* $\{c_j\}_{j=0}^\infty$ *is a sequence of complex numbers with*

$$\Big(\sum_{j=0}^\infty |c_j|^p\Big)^{\frac{1}{p}} < \infty,$$

then $\displaystyle\sum_{j=0}^\infty c_j a_j$ *converges in* H^p *and*

$$\|\sum_j c_j a_j\|_{H^p} \le A\Big(\sum_j |c_j|^p\Big)^{\frac{1}{p}},$$

where $A > 0$, *depends on* p *and* d.

Conversely, if $f \in H^p(\mathbb{S}^{d-1})$ there exists a sequence $\{a_j\}$ of complex numbers such that

$$f = \sum_j c_j a_j \quad and \quad \Big(\sum_j |c_j|^p\Big)^{\frac{1}{p}} \le B\|f\|_{H^p},$$

where B depends on p and d.

For $f \in \mathcal{S}'(\mathbb{S}^{d-1})$ we associate its expansion in spherical harmonics, $f \sim \sum_{k=0}^{\infty} Y_k(f)$, where $Y_k(f)(x) =< f, Z_x^{(k)} >$, with $Z_x^{(k)}$ the zonal harmonic of degree k with pole x. For $\delta > -1$ and $\alpha > 0$, the *Cesàro mean* σ_k^δ and the *Riesz mean* $R_k^{\delta,\alpha}$ are defined by

$$\sigma_k^\delta(f)(x) := \sum_{j=0}^{k} \frac{A_{k-j}^\delta}{A_k^\delta} Y_j(f)(x)$$

and

$$R_k^{\delta,\alpha}(f)(x) := \sum_{j=0}^{k} \Big(1 - \Big(\frac{j}{k+1}\Big)^\alpha\Big)^\delta Y_j(f)(x)$$

respectively. Given $f \in \mathcal{S}'$ and a number $r > 0$, we say $g := f^{(r)}$ is the r*-th order derivative of* f if $g \in \mathcal{S}'$ and for any nonnegative integer k

$$Y_k(g) = (-k(k+d-2))^{\frac{r}{2}} Y_k(f).$$

4.2.2 A new characterization of the Hardy space $H^p(\mathbb{S}^{d-1})$

The main goal in this subsection is to establish a "maximal Cesàro operator" characterization of $H^p(\mathbb{S}^{d-1})$, which will play a basic role in our later proof of Theorem 4.2.1.

We first introduce some necessary notations. Let $\{\mu_k\}$ be a sequence of complex numbers. Given a positive integer ℓ, we define $\triangle^\ell \mu_k$ by

$$\triangle \mu_k = \mu_k - \mu_{k+1}, \quad \triangle^{i+1}\mu_k = \triangle(\triangle^i \mu_k), \quad i = 1, \cdots, \ell - 1,$$

and define $\overleftarrow{\triangle}^\ell \mu_k$ by

$$\overleftarrow{\mu_k} = (-1)^\ell \triangle^\ell \mu_k.$$

Given $f \in \mathcal{S}'$, its maximal Cesàro mean $\sigma_*^\delta(f)$ of order δ is

$$\sigma_*^\delta(f) = \sup_k |\sigma_k^\delta(f)|.$$

The main result in this subsection can be stated as follows.

Theorem 4.2.7. *Suppose* $0 < p \le 1$, $\delta > \delta(p) := \frac{d-1}{p} - \frac{d}{2}$ *and* f *is a distribution on* $\mathbb{S}^{d-1}$. *Then* $f \in H^p(\mathbb{S}^{d-1})$ *if and only if* $\sigma_*^\delta(f) \in L^p(\mathbb{S}^{d-1})$. *Furthermore*

$$\|f\|_{H^p(\mathbb{S}^{d-1})} \approx \|\sigma_*^\delta(f)\|_{L^p(\mathbb{S}^{d-1})}.$$

To the best of our knowledge, Theorem 4.2.7 is new even in the case $d = 2$ (the periodic case), for which only the boundedness of σ_*^δ from $H^p(\mathrm{T})$ to $L^p(\mathrm{T})$ is known. (See [Mor].)

Proof. Suppose $\sigma_*^\delta(f) \in L^p(\mathbb{S}^{d-1})$. Then for a.e. $x \in \mathbb{S}^{d-1}$, $\sigma_*^\delta(f)(x) < \infty$. Observing

$$\sigma_*^{[\delta]+1}(f)(x) \le \sigma_*^\delta(f)(x),$$

we get

$$\Big|Y_k(f)(x)\Big| \le C_\delta k^{\delta+1}\sigma_*^\delta(f)(x), \quad a.e. \;\; x \in \mathbb{S}^{d-1},$$

since

$$Y_k(f)(x) = \overleftarrow{\triangle}^{[\delta]+2}\Big[A_k^{[\delta]+1}\sigma_k^{[\delta]+1}(f)(x)\Big].$$

Thus for every $r \in (0,1)$,

$$\sum_{k=0}^{\infty} r^k \Big|Y_k(f)(x)\Big| < \infty, \quad \text{a.e.} \;\; x \in \mathbb{S}^{d-1}.$$

Taking account into

$$(1-r)^{-1-\delta} = \sum_{k=0}^{\infty} A_k^\delta r^k, \tag{4.2.1}$$

we get for a.e. $x \in \mathbb{S}^{d-1}$,

$$\begin{aligned}(1-r)^{-1-\delta}\sum_{k=0}^{\infty} r^k Y_k(f)(x) &= \Big(\sum_{k=0}^{\infty} A_k^\delta r^k\Big)\Big(\sum_{k=0}^{\infty} r^k Y_k(f)(x)\Big)\\ &= \sum_{k=0}^{\infty} A_k^\delta r^k \sigma_k^\delta(f)(x).\end{aligned}$$

Noticing

$$P_r(f)(x) = \sum_{k=0}^{\infty} r^k Y_k(f)(x),$$

we obtain that for a.e. $x \in \mathbb{S}^{d-1}$

$$P_r(f)(x) = (1-r)^{1+\delta}\sum_{k=0}^{\infty} A_k^\delta r^k \sigma_k^\delta(f)(x),$$

which, together with (4.2.1), implies

$$P^+(f)(x) \le \sigma_*^\delta(f)(x), \quad \text{a.e. } x \in \mathbb{S}^{d-1}.$$

So according to Definition 4.2.1, we conclude that $f \in H^p(\mathbb{S}^{d-1})$ and

$$\|f\|_{H^p} = \|P^+(f)\|_p \le \|\sigma_*^\delta(f)\|_p.$$

The proof of the converse part of the theorem was essentially contained in [CTW]. In fact, with a slight modification of the proof of Lemma 4.2 of [CTW], we have

$$\sigma_*^\delta(a)(x) \le \begin{cases} Cr^{-\frac{d-1}{p}+\frac{d}{2}+\delta}|x-y|^{-(\frac{d}{2}+\delta)}, & 0<|x-y|\le \frac{\pi}{2}, \\ Cr^{-\frac{d-1}{p}+\frac{d}{2}+\delta}|x+y|^{-(\frac{d}{2}+\delta)}, & 0<|x+y|\le \frac{\pi}{2}, \\ Cr^{-\frac{d-1}{p}}, & x \in \mathbb{S}^{d-1}, \end{cases}$$

where a is a regular p-atom, supported in $B(y,r)$ and with the property $\|a\|_\infty \le r^{-\frac{d-1}{p}}$. Taking into account

$$\|\sigma_*^\delta\|_{(\infty,\infty)} < \infty,$$

we get by standard method that

$$\|\sigma_*^\delta(f)\|_p \le C\|f\|_{H^p}.$$

This concludes the proof. □

As a consequence, we have the following multiplier theorem.

Corollary 4.2.8. *Let $\{\mu_k\}_{k=0}^\infty$ be a sequence of complex numbers, $0<p\le 1$, $\delta(p) := \frac{d-1}{p} - \frac{d}{2}$ and $\ell = [\delta(p)]+1$. Suppose the following conditions are satisfied:*

$$(i) \qquad \sup_k |\mu_k| \le M < \infty,$$

$$(ii) \qquad \sum_{k=0}^\infty \left|\triangle^{\ell+1}\mu_k\right|(k+1)^\ell \le M.$$

Then

$$\Big\|\sum_{k=0}^\infty \mu_k Y_k(f)\Big\|_{H^p} \le CM\|f\|_{H^p},$$

where $C>0$ is independent of M, $\{\mu_k\}$ and f.

Proof. Let

$$T(f) := \sum_{k=0}^{\infty} \mu_k Y_k(f).$$

Then by Theorem 4.2.7, it suffices to prove

$$\sigma_*^{\ell+2}(Tf) \le CM\sigma_*^{\ell}(f). \tag{4.2.2}$$

Applying Abelian transform $\ell+1$ times gives

$$\sigma_N^{\ell+2}(Tf) = \sum_{k=0}^{N} \triangle^{\ell+1}\Big(\frac{A_{N-k}^{\ell+2}}{A_N^{\ell+2}}\mu_k\Big) A_k^{\ell}\sigma_k^{\ell}(f)(x), \tag{4.2.3}$$

where $A_j^{\ell+2} = 0$ for $j < 0$.

On the other hand, according to conditions (i) and (ii), one can easily verify that for all $v = 0, 1, \cdots, \ell$,

$$\sum_{k=0}^{\infty} |\triangle^{v+1}\mu_k| k^v \le CM.$$

Thus

$$\begin{aligned}\sum_{k=0}^{N}\Big|\triangle^{\ell+1}\Big(\frac{A_{N-k}^{\ell+2}}{A_N^{\ell+2}}\mu_k\Big)\Big| A_k^{\ell} \le & C\sum_{v=0}^{\ell+1}\sum_{k=0}^{N}\Big|\triangle^{\ell+1-v}\Big(\frac{A_{N-k-v}^{\ell+2}}{A_N^{\ell+2}}\triangle^{v}\mu_k\Big)\Big|(k+1)^{\ell} \\ \le & CM, \end{aligned} \tag{4.2.4}$$

where we define $\triangle^0\mu_k = \mu_k$.

Now combining (4.2.3) with (4.2.4), we get (4.2.2) and complete the proof. □

4.2.3 Basic approximation results in the spaces $H^p(\mathbb{S}^{d-1})$ $(0 < p \le 1)$

Lemma 4.2.9. ***(Bernstein inequality.)*** *Let $r > 0$. Then for every spherical harmonic T_N of degree less than or equal to N,*

$$\|T_N^{(r)}\|_{H^p} \le CN^r \|T_N\|_{H^p},$$

where $C > 0$ is independent of N and T_N.

Lemma 4.2.10. *Suppose $r > 0$ and η is a C^∞ function with the properties that $0 \le \eta \le 1$ for $x \in \mathbb{R}$, $\eta(x) = 1$ for $0 \le |x| \le 1$ and $\eta(x) = 0$ for $|x| > 2$. For $N > 0$, let η_N be an operator defined by*

$$\eta_N(f) := \sum_{k=0}^{2N} \eta(\frac{k}{N}) Y_k(f).$$

Then for $f \in H^p(\mathbb{S}^{d-1})$

$$\|f - \eta_N f\|_{H^p} + \frac{1}{N^r} \left\| (\eta_N f)^{(r)} \right\|_{H^p} \approx K_r(f, \frac{1}{N})_{H^p}.$$

Lemma 4.2.9 can be obtained by applying Corollary 4.2.8, while Lemma 4.2.10 is a consequence of Lemma 4.2.9 and Corollary 4.2.8. The proofs of both lemmas follow the standard method. (See the proofs of Theorems 3.2 and 5.1 in [Di1].) We omit the detail.

Lemma 4.2.11. *Suppose $0 < p \le 1$ and $r > 0$. Then for $f \in H^p(\mathbb{S}^{d-1})$,*

$$K_r^p(f, \frac{1}{N})_{H^p} \le C N^{-rp} \sum_{k=1}^{N} k^{rp-1} E_k^p(f, H^p).$$

Lemma 4.2.11 can be obtained by invoking Lemma 4.2.10 and the routine method. (See the proof of Theorem 6.4 of [Di1].)

Lemma 4.2.12. *Suppose $0 < p \le 1$ and $\delta > \delta(p) := \frac{d-1}{p} - \frac{d}{2}$. Then for $f \in H^p$,*

$$\|\sigma_N^\delta(f) - f\|_{H^p} \approx K_1(f, \frac{1}{N})_{H^p}.$$

In the special case $d = 2$, Lemma 4.2.12 for the Riesz means was due to [Be2], while for the general case $d \ge 3$, its proof runs along the same lines as that of Theorem 3.1.4. We omit the detail here.

4.2.4 Proofs of Theorem 4.2.1 and Corollary 4.2.2

To prove Theorem 4.2.1, we need a series of lemmas.

Lemma 4.2.13 ([Chen]). *Suppose $0 < p < 1$, $\delta = \delta(p) := \frac{d-1}{p} - \frac{d}{2}$ and $f \in H^p(\mathbb{S}^{d-1})$. Then*

$$\frac{1}{\log N} \sum_{k=1}^{N} \frac{\|\sigma_k^\delta(f)\|_{L^p}^p}{k} \le C \|f\|_{H^p}^p. \tag{4.2.5}$$

We remark that Lemma 4.2.13 was essentially contained in [CTW] though not explicitly stated there. In fact, it is a direct consequence of the following estimates of $\sigma_L^\delta(a)(x)$, which were obtained in the proof of Lemma 4.2 of [CTW]:

$$\left|\sigma_L^\delta(a)(x)\right| \le \begin{cases} C(rL)^{s-(d-1)(\frac{1}{p}-1)}|x-y|^{-\frac{d-1}{p}}, & 0<|x-y|\le \frac{\pi}{2}, \\ C(rL)^{s-(d-1)(\frac{1}{p}-1)}|x+y|^{-\frac{d-1}{p}}, & 0<|x+y|\le \frac{\pi}{2}, \\ Cr^{-\frac{d-1}{p}}, & x\in \mathbb{S}^{d-1}, \end{cases}$$

where a is a regular p-atom, supported in $B(y,r)$, $\|a\|_\infty \le r^{-\frac{1}{p}}$ and $s=[(d-1)(\frac{1}{p}-1)]+1$ or 0.

The key part of the proof of Theorem 4.2.1 is to show that Lemma 4.2.13, with the L^p-norms in the left hand side of (4.2.5) replaced by H^p -norms, remains valid. More precisely, we prove

Lemma 4.2.14. *Let* $0<p<1$, $\delta(p) := \frac{d-1}{p}-\frac{d}{2}$ *and* $f\in H^p$. *Then*

$$\frac{1}{\log N}\sum_{k=1}^{N}\frac{\|\sigma_k^\delta(f)\|_{H^p}^p}{k} \le C_p\|f\|_{H^p}^p.$$

In the case of the Riesz means of the multiple periodic functions, Lemma 4.2.14 was duo to [J-Liu-Lu]. The proof in [J-Liu-Lu] relies on the atomic decomposition theorem, while our proof here relies on the "maximal Cesàro operator" characterization theorem (Theorem 4.2.7). We point out that it appears to be difficult to obtain Lemma 4.2.14 simply by following the technique developed in [J-Liu-Lu]. (This can be seen from the proof of the theorem in [Chen, p154] and that of Theorem 4.10 of [CTW].)

For the remainder of this section, we define $A_j^\delta = 0$ for $j<0$ and regard A_j^δ as a function of all $j\in\mathbb{Z}$.

Proof. According to Theorem 4.2.7 and Lemma 4.2.13, it suffices to prove

$$\sigma_*^{\ell+4}(\sigma_L^\delta(f))(x) \le C\sigma_*^\ell(f)(x) + C\left|\sigma_L^\delta(f)(x)\right|, \tag{4.2.6}$$

where $\ell=[\delta]+1$. To this end, we have to estimate the quality $\sigma_N^{\ell+4}(\sigma_L^\delta(f))(x)$. We consider three cases.

Case 1. $0 \le N \le L$.

Let $\mu_k = \frac{A_{N-k}^{\ell+4} A_{L-k}^{\delta}}{A_N^{\ell+4} A_L^{\delta}}$ for all $k \in \mathbb{Z}_+$. It is easy to verify that

$$\left|\triangle^{\ell+1}\mu_k\right| \le C \begin{cases} \frac{1}{N^{\ell+1}}, & \text{if } 0 \le N \le \frac{L}{2}, \\ \frac{(L-k+1)^{\delta+3}}{L^{\delta+\ell+4}}, & \text{if } \frac{L}{2} \le N \le L. \end{cases}$$

So, invoking the Abelian transform $\ell + 1$ times yields

$$|\sigma_N^{\ell+4}\sigma_L^{\delta}(f)(x)| \le C\sigma_*^{\ell}(f)(x). \tag{4.2.7}$$

Case 2. $N \ge L + 1$ *and* $\delta > 0$ *is not an integer.*

We write

$$\sigma_N^{\ell+4}\sigma_L^{\delta}(f)(x) = \sum_{k=0}^{L} a_k b_k Y_k(f)(x) + O(1)\sigma_L^{\delta}(f)(x), \tag{4.2.8}$$

where

$$a_k = \begin{cases} \frac{A_{N-k}^{\ell+4}}{A_N^{\ell+4}} - \frac{A_{N-L}^{\ell+4}}{A_N^{\ell+4}}, & 0 \le k \le N, \\ 0, & k \ge N+1, \end{cases}$$

and

$$b_k = \begin{cases} \frac{A_{L-k}^{\delta}}{A_L^{\delta}}, & 0 \le k \le L, \\ 0 & \text{otherwise.} \end{cases}$$

The following inequalities can be verified for non-integer δ by straightforward computation.

$$|a_k| \le C\frac{L-k+1}{N}, \tag{4.2.9}$$

$$|\triangle^i a_k| \le C\left(\frac{1}{N}\right)^i, \quad i = 1, \cdots, \ell, \tag{4.2.10}$$

$$|\triangle^i b_k| \le C\frac{(L-k+1)^{\delta-1}}{(L+1)^{\delta}}, \quad i = 0, 1, \cdots. \tag{4.2.11}$$

Invoking (4.2.9)– (4.2.11), we conclude that for a non-integer δ,

$$\sum_{k=0}^{L}\left|\triangle^{\ell+1}(a_k b_k)\right|(k+1)^{\ell} \le C, \tag{4.2.12}$$

which, by applying the Abelian transform $\ell+1$ times to the first sum of the right hand side of (4.2.8), implies

$$\left|\sigma_N^{\ell+4}(\sigma_L^\delta(f))(x)\right| \le C\sigma_*^\ell(f)(x) + C|\sigma_L^\delta(f)(x)| \tag{4.2.13}$$

Case 3. $N \ge L+1$, *δ is an integer.*

We start with the identity (4.2.8). Observing $\ell = \delta + 1$, we can easily verify that

$$\triangle^\ell a_k = \begin{cases} O(\frac{1}{L^\delta}), & L-\ell \le k \le L, \\ 0, & \text{otherwise.} \end{cases} \tag{4.2.14}$$

$$\triangle^{\ell+1} b_k = \begin{cases} O(\frac{1}{L^\delta}), & L-\ell \le k \le L, \\ 0, & \text{otherwise.} \end{cases} \tag{4.2.15}$$

Now invoking (4.2.9)— (4.2.11) and (4.2.14)— (4.2.15) gives (4.2.12), which again implies (4.2.13).

Putting the above three cases together, we obtain (4.2.6) and conclude the proof. □

Proof of Theorem 4.2.1. We get the idea from [Be2]. The lower estimate is evident. For the proof of the upper estimate we take $\alpha > \delta(p)$. Then

$$\begin{aligned} \sum_{j=1}^N \frac{1}{j}\|f-\sigma_j^\delta(f)\|_{H^p}^p &= \sum_{j=1}^{\log\log N} \sum_{m=2^{2^j}+1}^{2^{2^{j+1}}} \frac{1}{m}\|\sigma_m^\delta(f)-f\|_{H^p}^p \\ &\le \sum_{j=1}^{\log\log N} \sum_{m=2^{2^j}+1}^{2^{2^{j+1}}} \frac{1}{m}\left\|(\sigma_m^\delta(f)-f)-\sigma_m^\alpha\left(\sigma_m^\delta(f)-f\right)\right\|_{H^p}^p \\ &\quad + \sum_{j=1}^{\log\log N} \sum_{m=2^{2^j}+1}^{2^{2^{j+1}}} \frac{1}{m}\left\|\sigma_m^\alpha\left(\sigma_m^\delta(f)-f\right)\right\|_{H^p}^p. \end{aligned} \tag{4.2.16}$$

With a slight modification of the proof of Lemma 3.1.6, we have

$$\|\sigma_m^\alpha(\sigma_m^\delta(f)-f)\|_{H^p}^p \le CK_1^p(f,\frac{1}{m})_{H^p}. \tag{4.2.17}$$

Therefore, the second sum in (4.2.16) can be estimated by

$$C\sum_{j=1}^{N}\frac{1}{j}K_1^p(f,\frac{1}{j})_{H^p},$$

which, by the monotonicity of the K-functional $K_1(f,\cdot)_{H^p}$, is bounded by

$$C\sum_{j=1}^{\log N}K_1^p(f,\frac{1}{2^j})_{H^p}.$$

According to Lemma 4.2.12, we have that for $2^{2^j}+1\le m\le 2^{2^{j+1}}$ and every $g\in H^p$,

$$\Big\|g-\sigma_m^\alpha(g)\Big\|_{H^p}\le C\Big\|g-\sigma_{2^{2^j}}^\alpha(g)\Big\|_{H^p}. \tag{4.2.18}$$

Now applying (4.2.18), with $g=\sigma_m^\delta(f)-f$, to the first sum in (4.2.16), we have

$$\begin{aligned}\Big\|(\sigma_m^\delta(f)-f)-\sigma_m^\alpha\Big(\sigma_m^\delta(f)-f\Big)\Big\|_{H^p}^p &\le C\Big\|(\sigma_m^\delta(f)-f)-\sigma_{2^{2^j}}^\alpha\Big(\sigma_m^\delta(f)-f\Big)\Big\|_{H^p}^p\\ &\le C\|\sigma_m^\delta(f-\sigma_{2^{2^j}}^\alpha(f))\|_{H^p}^p+C\|f-\sigma_{2^{2^j}}^\alpha(f)\|_{H^p}^p.\end{aligned}$$

Applying Lemma 4.2.14 to the sum

$$\sum_{m=2^{2^j}}^{2^{2^{j+1}}}\frac{1}{m}\|\sigma_m^\delta(f-\sigma_{2^{2^j}}^\alpha(f))\|_{H^p}^p$$

and using Lemma 4.2.12, we obtain, by the monotonicity of the K-functional,

$$\sum_{j=1}^{N}\frac{1}{j}\|\sigma_j^\delta(f)-f\|_{H^p}^p\le C\sum_{j=1}^{\log N}K_1^p(f,\frac{1}{2^j})_{H^p}.$$

Lemma 4.2.11 gives

$$\begin{aligned}\sum_{j=1}^{\log N}K_1^p(f,\frac{1}{2^j})_{H^p}&\le C\sum_{j=1}^{\log N}2^{-jp}\sum_{k=1}^{2^j}k^{p-1}E_k^p(f,H^p)\\ &\le C\sum_{k=1}^{N}\frac{E_k^p(f,H^p)}{k},\end{aligned}$$

which is the required upper estimate. This concludes the proof. □

Proof of Corollary 4.2.2. By Lemma 4.2.12, we have the following Jackson type inequality:

$$E_k(f,H^p)\le CK_r(f,\frac{1}{k})_{H^p},\quad r>0. \tag{4.2.19}$$

Invoking (4.2.19) and by the monotonicity of the K-functional, we conclude

$$\begin{aligned}\sum_{j=1}^{N} \frac{1}{j} E_j^p(f, H^p) \leq & C \sum_{j=1}^{N} \frac{1}{j} K_1^p(f, \frac{1}{j})_{H^p} \\ & \approx \int_1^{N+1} \frac{K_1^p(f, \frac{1}{t})_{H^p}}{t}\, dt. \end{aligned} \tag{4.2.20}$$

We decompose the integral into two parts $\int_1^A + \int_A^{N+1}$, with $A > 0$ a constant to be decided later. Then a straightforward computation shows that (4.2.20) is estimated by

$$CK_1^p(f, \frac{1}{A})_{H^p} \log N + CK_1^p(f, \frac{1}{A})_{H^p} A^p.$$

Taking $A = (\log N)^{\frac{1}{p}}$ and invoking Theorem 4.2.1 give the corollary. This concludes the proof. □

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