A review on antiderivative and integral

March 2008

§1. Antiderivative (原函数)

Definition 1.1(antiderivative (原函数)) A function F is called an antiderivative of a function f on a given interval I if $F'(x) = f(x)$ for all x in the interval.

It is obvious that if F is an antiderivative of f then for any constant c $F + c$ is also an antiderivative of f. And, any antiderivative G of f must be a sum of F and a constant function, because if a function has derivative 0 on an **interval** then f must be a constant on the interval . Traditionally the following notation has been used to represent this fact:

$$\int f(x)dx = F(x) + c$$

where F denotes an antiderivative of f and c denotes constant function. The expression $\int f(x)dx$ is called the **indefinite integral** of f. Sometimes it is treated as a function, sometimes it is regarded as a set of all antiderivatives of f. The "arbitrariness" and "uncertainty" to use this notation is a historical result. Don't mind it!

How to find an antiderivative of a given function?

There is a technique called "*substitution*"(代入法).

Proposition 1(substitution(代入法)) If F is an antiderivative of f, then $f(g(x))g'(x)$ has antiderivative $F(g(x))$. Or,

$$\int f(g(x))g'(x)dx = F(g(x)) + c.$$

This is obvious. It is called "substitution" since it can be obtained by substituting $u = g(x)$ and $du = g'(x)dx$ into

$$\int f(u)du = F(u) + c.$$

Remark Substitution method is also called **changing variable method 变量替换法**.

Example 1 $\int (x^2+1)^{50} 2x\, dx$ **Example 2** $\int \sin(x+9)\, dx$

Example 3 $\int \cos 5x\, dx$ **Example 4** $\int \frac{dx}{(\frac{1}{3}x - 8)^5}$

Example 5 $\int \left(\frac{1}{x^2} + \frac{1}{\cos^2 \pi x}\right) dx$ **Example 6** $\int \sin^2 x \cos x\, dx$

Example 7 $\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$ **Example 8** $\int t^4\sqrt[3]{3-5t^5}\,dt$

Example 9 $\int x^2\sqrt{x-1}\,dx$ **Example 10** $\int \cos^3 x\,dx$

Example 11 Let $f_1(x)=\frac{1}{x}, x\neq 0,\ f_2(x)=\frac{1}{x},\ x>0,\ f_3(x)=\frac{1}{x}, x<0$. Find

$$\int f_k(x)\,dx, k=1,2,3.$$

Example 12 Evaluate $\int \frac{3x^2}{x^3+5}\,dx$.

Example 13 $\int e^{-x}\,dx \quad \int x^2e^{x^3}\,dx, \quad \int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx$

Proposition 2(integration by parts 分部积分法) If f has derivetive and g has an antiderivative of g, then

$$\int f(x)g(x)dx = f(x)G(x) - \int f'(x)G(x)\,dx.$$

Example 14 Evaluate $\int xe^x\,dx$.

Example 15 Evaluate $\int \ln x\,dx$.

Example 16 Evaluate $\int x^2e^{-x}\,dx$.

Example 17 Evaluate $\int e^x\cos x\,dx$.

Example 18

$$\begin{aligned}\int x^2\sqrt{x-1}\,dx &= \frac{2}{3}x^2(x-1)^{\frac{3}{2}} - \frac{4}{3}\int x(x-1)^{\frac{3}{2}}\,dx\\ &= \frac{2}{3}x^2(x-1)^{\frac{3}{2}} - \frac{8}{15}x(x-1)^{\frac{5}{2}} + \frac{8}{15}\int (x-1)^{\frac{5}{2}}\,dx\\ &= \frac{2}{3}x^2(x-1)^{\frac{3}{2}} - \frac{8}{15}x(x-1)^{\frac{5}{2}} + \frac{16}{105}(x-1)^{\frac{7}{2}} + c.\end{aligned}$$

Example 19 Evaluate $\int \sin^n x\,dx, \quad \int \cos^n x\,dx,\ n\in\mathbb{N}$.

Example 20 $\int \cos(\ln x)\,dx$

§2. The definite integral (定积分)

Notation For convenience $F(b)-F(a)$ is written as $F\Big|_a^b$.

Example 1 $\int_0^{2\pi} \sin kx\,dx = \int_0^{2\pi} \cos jx\,dx = 0,\ \ k, j \in \mathbb{N}.$

Example 2 $\int_0^{\infty} \exp(-x)\,dx = 1.$ **Example 3** Evaluate $\int_{-1}^{2} |x|\,dx.$

Example 4 $\lim\limits_{a\to+\infty} \int_1^a \frac{1}{x}\,dx = +\infty.$ **Example 5** $\lim\limits_{a\to+\infty} \int_1^a \frac{1}{x^2}\,dx = 1.$

Example 6 Evaluate $\int_0^6 f(x)\,dx$ if $f(x) = \begin{cases} x^2, & x < 2, \\ 3x-2, & x \geq 2. \end{cases}$

Theorem Let $f \in L[a,b]$. Define $F(x) = \int_a^x f(t)\,dt,\ a \leq x \leq b$. If f is continuous at point $x \in [a,b]$ then $F'(x) = f(x)$.

§3. Evaluating definite integral by substitution and by parts

Example 1 $\int_0^2 x(x^2+1)^3\,dx$

Example 2 (a) $\int_0^{\frac{\pi}{8}} \sin^5 2x \cos 2x\,dx$ (b) $\int_2^5 (2x-5)(x-3)^9\,dx$

Example 3 Find the average value of the function $f(x) = \frac{\cos\frac{\pi}{x}}{x^2}$ over the interval $[1,3]$, i.e. the value $f_{\text{ave}} := \frac{1}{3-1}\int_1^3 f(x)\,dx.$

Solution $f_{\text{ave}} = \frac{1}{2}\int_1^3 \frac{\cos\frac{\pi}{x}}{x^2}\,dx \overset{u=\frac{\pi}{x}}{=\!=\!=\!=} \frac{1}{2}\int_{\pi}^{\frac{\pi}{3}} \frac{\cos u}{\pi^2}u^2\frac{-\pi}{u^2}\,du = \frac{1}{2\pi}\int_{\frac{\pi}{3}}^{\pi} \cos u\,du = -\frac{\sqrt{3}}{4\pi}.$

The formula of integration by parts

$$\int_a^b f(x)g'(x)\,dx = f(x)g(x)\Big|_a^b - \int_a^b f'(x)g(x)\,dx.$$

Example 4

$$\int_0^1 \tan^{-1} x\,dx = x\tan^{-1} x\Big|_0^1 - \int_0^1 \frac{x}{1+x^2}\,dx = \frac{\pi}{4} - \frac{1}{2}\ln(1+x^2)\Big|_0^1 = \frac{\pi}{4} - \ln\sqrt{2}.$$

Some reduction formulas (a) $\int \sec^n x\,dx$; (b) $\int \tan^n x\,dx$; (c) $\int x^n e^x\,dx$.

§4. Area and Volume

Area Formula If f and g are continuous functions on $[a, b]$ and if $f(x) \geq g(x)$ for all $x \in [a, b]$, then the area of the region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by the line $x = a$, and on the right by the line $x = b$ is (as a definition)

$$A = \int_a^b \big(f(x) - g(x)\big)\, dx.$$

Example 1 Find the area bounded above by $y = x + 6$, below by $y = x^2$, on the sides by the lines $x = 0$ and $x = 2$.

Solution $A = \displaystyle\int_0^2 (x + 6 - x^2)\, dx = (\frac{1}{2}x^2 + 6x - \frac{1}{3}x^3)\Big|_0^2 = \frac{34}{3}$. $\square$

Example 2 Find the area of the region that is enclosed between the curves $y = x^2$ and $y = x + 6$.

Solution Let $f(x) = x+6-x^2$. Solving the equation $f(x) = 0$ we get two roots: $x_1 = -2$, $x_2 = 3$. We conclude that the region between the given curves is exactly between the lines $x = -2$ and $x = 3$. It is easy to verify that $f(x) > 0$ when $x \in [-2, 3]$. So we get the result $A = \displaystyle\int_{-2}^3 f(x)\, dx = (\frac{1}{2}x^2 + 6x - \frac{1}{3}x^3)\Big|_{-2}^3 = \frac{125}{6}$. $\square$

Example 3 Find the area of the region enclosed by $x = y^2$ and $y = x - 2$.

Solution Let $g(y) = y + 2 - y^2$. Solving the equation $g(y) = 0$ we get two roots: $y_1 = -1$, $y_2 = 2$. We conclude that the region enclosed by curves is exactly between the lines $y = -1$ and $y = 2$. It is easy to verify that $g(y) > 0$ when $y \in [-1, 2]$. So we get the result $A = \displaystyle\int_{-1}^2 g(y)\, dy = (\frac{1}{2}y^2 + 2y - \frac{1}{3}y^3)\Big|_{-1}^2 = \frac{10}{3} + \frac{7}{6} = \frac{9}{2}$. $\square$

Example 4 Let $s(t)$ be the position at time t of a particle moving on a coordinate line representing the time. We know that the instantaneous velocity $v(t)$ at time t is defined by $v(t) = s'(t)$. Hence by the Fundamental Theorem of Calculus we have

$$s(t_2) - s(t_1) = \int_{t_1}^{t_2} v(t)\, dt$$

which represents the change of the position of the particle over the time interval $[t_1, t_2]$.

Volume Formula Let S be a solid bounded by two parallel planes perpendicular to the $x-$axis at $x = a$ and $x = b$. If for each $x \in [a, b]$, the cross-sectional area of S perpendicular to the $x-$axis is $A(x)$, then the volume of the solid is (as a definition)

$$V = \int_a^b A(x)\, dx$$

provided $A \in R[a, b]$.

Example 5 Let f be continuous and nonnegative on $[a, b]$, and let R be the region that is bounded above by $y = f(x)$, below by the $x-$axis, and on the sides by the lines $x = a$ and $x = b$. Find the volume of the *solid of revolution* (旋转体) that is generated by revolving the region R about the $x-$axis.

Solution Since the cross section of the solid taken perpendicular to the $x-$axis at the point $x \in [a, b]$ is a circular desk of radius $f(x)$, its area is $A(x) = \pi(f(x))^2$. Thus by Formula 6.2.2 the volume of the solid is

$$V = \int_a^b \pi(f(x))^2 \, dx.$$

Example 6 Find the volume of the solid that is obtained when the region under the curve $y = \sqrt{x}$ over the interval $[1, 4]$ is revolved about the $x-$axis.

Solution $V = \displaystyle\int_1^4 \pi x \, dx = \frac{15\pi}{2}.$

Example 7 Find the volume of a ball of radius r.

Solution The ball can be regarded as the solid that is obtained when the region under the curve $y = \sqrt{r^2 - x^2}$ over the interval $[-r, r]$ is revolved about the $x-$axis. Then we get its volume

$$V = \int_{-r}^{r} \pi(r^2 - x^2) \, dx = (2 - \frac{2}{3})\pi r^3 = \frac{4}{3}\pi r^3.$$

Example 8 Let f and g be continuous and nonnegative on $[a, b]$, and suppose that $f(x) \geq g(x)$ for all $x \in [a, b]$. Let R be the region that is bounded above by $y = f(x)$, below by $y = g(x)$, and on the sides by the lines $x = a$ and $x = b$. Find the volume of the solid of revolution that is generated by revolving the region R about the $x-$axis.

Solution $V = \displaystyle\int_a^b \pi\Big((f(x))^2 - (g(x))^2\Big) \, dx.$

Because the cross section are washer (垫圈) shaped, the application of this formula is called *method of washers*.

Example 9 Find the volume of the solid generated when the region between the graphs of the functions $f(x) = \frac{1}{2} + x^2$ and $g(x) = x$ over the interval $[0, 2]$ is revolved about the $x-$axis.

Solution $V = \displaystyle\int_0^2 \pi\Big((\frac{1}{2} + x^2)^2 - x^2\Big) \, dx = \int_0^2 \pi\Big((\frac{1}{4} + x^4)\Big) \, dx = \frac{69\pi}{10}.$

Example 10 Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 2$, and $x = 0$ is revolved about the $y-$axis.

Solution It is obvious that the the solid is obtained when the region under the curve $x = y^2$ over the interval $[0, 2]$ is revolved about the $y-$axis. Hence $V = \displaystyle\int_0^2 \pi y^4 \, dy = \frac{32\pi}{5}.$

Example 11 Let f be continuous and nonnegative on $[a, b]$ $a \geq 0$, and let R be the region that is bounded above by $y = f(x)$, below by the $x-$axis, and on the sides by the lines $x = a$ and $x = b$. Find the volume of the solid of revolution (旋转体) S that is generated by revolving the region R about the $y-$axis.

Analysis If $f(x) = h$ a constant, then S is a cylindrical shell which volume is obviously the difference

$$\pi b^2 h - \pi a^2 h = 2\pi \frac{a+b}{2}(b-a)h. \tag{1}$$

In general case we make a partition $T_n = \{x_j = a + j\frac{b-a}{n} : j = 0, 1, \cdots, n\}$, let R_j be the region that is bounded above by $y = f(x)$, below by the $x-$axis, and on the sides by the lines x_{j-1} and x_j, $j = 1, \cdots, n$. Then we denote by S_j the solid of revolution that is generated by revolving the region R_j about the $y-$axis. If n is big enough then on each $[x_{j-1}, x_j]$ $(j = 1, \cdots, n)$, f may be regarded as a constant $f(\frac{x_{j-1}+x_j}{2})$ approximately. Then the volume of S_j has an approximate value $V_j \approx 2\pi \dfrac{x_{j-1}+x_j}{2}(x_j - x_{j-1})f(\dfrac{x_{j-1}+x_j}{2})$ by the formula (1). Hence the volume V of S has the following approximate value.

$$V = \sum_{j=1}^{n} S_j \approx 2\pi \frac{x_{j-1}+x_j}{2}(x_j - x_{j-1})f(\frac{x_{j-1}+x_j}{2}).$$

We denote by

$$\Lambda_n = \{\lambda_j = \frac{x_{j-1}+x_j}{2} = a + (j - \frac{1}{2})\frac{b-a}{n} : j = 1, \cdots, n\}$$

the special "choice" with respect to T_n, and write $g(x) = 2\pi x f(x)$. Then we see

$$V \approx \sum_{j=1}^{n} g(\lambda_j)\frac{b-a}{n} = S(g, T_n, \Lambda_n) = S_n(g)$$

which is just a special Riemann sum of g (see the Definition 5.5.1 and the followed Theorem 1). We may image that when $n \to +\infty$ the limit of $S_n(g)$ is the exact value of V. Then we get (as a definition) the formula (see (6.3.2), P.413)

$$V = \int_a^b g(x)\,dx = \int_a^b 2\pi x f(x)\,dx. \tag{2}$$

Example 12 Find the volume of the torus (环状体) that results when the region enclosed by the circle of radius r with center at $(h, 0), h > r$, is revolved about the $y-$axis.

Solution By formula (2) we get the volume

$$\begin{aligned} V &= 2\int_{h-r}^{h+r} 2\pi x\sqrt{r^2 - (x-h)^2}\,dx \\ &\overset{u=x-h}{=\!=\!=} 4\pi \int_{-r}^{r} (u+h)\sqrt{r^2 - u^2}\,du \\ &= 8\pi h \int_0^r \sqrt{r^2 - u^2}\,dx \\ &\overset{u=r\sin t}{=\!=\!=\!=} 8\pi h \int_0^{\frac{\pi}{2}} r^2\cos^2 t\,dt = 4\pi h r^2 \int_0^{\frac{\pi}{2}} (1 + \cos 2t)\,dt = 2\pi^2 h r^2. \end{aligned}$$

Example 13 Use cylindrical shells to find the volume V of the solid generated when the region enclosed between $y = \sqrt{x}$, $x = 1, x = 4$, and the $x-$axis is revolved about the $y-$axis.

Solution $V = \displaystyle\int_1^4 2\pi x\sqrt{x}\,dx = 2\pi \int_1^4 x^{\frac{3}{2}}\,dx = \frac{124\pi}{5}.$

Example 14 Use cylindrical shells to find the volume V of the solid generated when the region R in the first quadrant enclosed between $y = x$ and $y = x^2$ is revolved about the $y-$axis.

Solution Let $f(x) = x$, $g(x) = x^2$. The region R is on the sides by the lines $x = 0$ and $x = 1$. Hence

$$V = \int_0^1 2\pi x f(x)\,dx - \int_0^1 2\pi x g(x)\,dx = \int_0^1 2\pi x(f(x) - g(x))\,dx = \frac{\pi}{6}.$$

Example 15 Use cylindrical shells to find the volume V of the solid generated when the region R under $y = x^2$ over the interval $[0, 2]$ is revolved about the $x-$axis.

Solution The line $y = x^2, x \in [0, 2]$ is the same $x = \sqrt{y}, y \in [0, 4]$. The region R has right side $x = 2$. So, $V = \displaystyle\int_0^4 2\pi y(2 - \sqrt{y})\,dy = \frac{32\pi}{5}$.

§5. Length of a plane curve, Area of a surface of revolution(旋转面) and Applications in Physics

After a suitable explanation we have the following definition.

Definition 5.1(Length of a plane curve) If $y = f(x)$ is a smooth curve on $[a, b]$ (i.e. f has continuous derivative on $[a, b]$), then the arc length L of this curve over $[a, b]$ is defined as

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\,dx. \tag{1}$$

Example 1 Find the arc length L of the curve $y = x^{\frac{3}{2}}$ from $(1, 1)$ to $(2, 2\sqrt{2})$.

Solution

$$L = \int_1^2 \sqrt{1 + \left(\frac{3}{2}x^{\frac{1}{2}}\right)^2}\,dx = \int_1^2 \sqrt{1 + \frac{9}{4}x}\,dx \overset{u=1+\frac{9}{4}x}{=\!=\!=} \frac{4}{9}\int_{\frac{13}{4}}^{\frac{11}{2}} u^{\frac{1}{2}}\,du = \frac{22\sqrt{22} - 13\sqrt{13}}{27}. \qquad \square$$

If there is function $x = \varphi(t), t \in [\alpha, \beta], \alpha < \beta$ such that φ' is continuous and positive, and $a = \varphi(\alpha), b = \varphi(b)$ then we may use substitution $x = \varphi(t)$ in (1) and get

$$L = \int_\alpha^\beta \sqrt{1 + (f'(\varphi(t)))^2}\varphi'(t)\,dt = \int_\alpha^\beta \sqrt{(\varphi'(t))^2 + (f'(\varphi(t))\varphi'(t))^2}\,dt.$$

If we write $x = x(t)$ instead of $x = \varphi(t)$ and $y = y(t)$ instead of $y = f(\varphi(t))$ then we are motivated to the following

Length formula for parametric curves If no segment of the curve represented by the parametric equation

$$x = x(t),\ \ y = y(t)\ (\alpha \le t \le \beta)$$

is traced more than once as t increases from α to β, and if $x'(t)$ and $y'(t)$ are continuous functions for $\alpha \le t \le \beta$, then the arc length of the curve is given by

$$L = \int_\alpha^\beta \sqrt{(x'(t)^2 + (y'(t))^2}\,dt. \tag{2}$$

Example 2 Use (2) to find the circumference (周长) L of a circle of radius r from the parametric equation

$$x = r\cos t,\ \ y = r\sin t, (0 \le t \le 2\pi).$$

Solution

$$L = \int_0^{2\pi} \sqrt{(-r\sin t)^2 + (r\cos t)^2}\,dt = \int_0^{2\pi} r\,dt = 2\pi r.$$

Example 3 Find the arc length of $y = \sin x$ from $x = 0$ to $x = \pi$.

Solution The length is $L = \int_0^{\pi} \sqrt{1+\cos^2 x}\,dx$. This integral cannot be evaluated in terms of elementary functions. Using Maple :

```
> int((1+(cos(x))^2)^(1/2),x=0..Pi);
                        2 sqrt(2) EllipticE(1/2 sqrt(2))

> evalf(%);
                                  3.820197788
```

We get $L \approx 3.820197788$. ☐

Definition 5.2(Area of a surface of revolution (旋转面面积)) If f is a smooth, nonnegative function on $[a, b]$ (i.e. f has continuous derivative and $f \ge 0$ on $[a, b]$), then the area S of the surface of revolution that is generated by revolving the curve $y = f(x)$ over $[a, b]$ about the $x-$axis is defined as

$$S = \int_a^b 2\pi f(x)\sqrt{1+(f'(x))^2}\,dx. \tag{3}$$

Example 4 Find the area of the surface that is generated by revolving the portion of the curve $y = x^3$ between $x = 0$ and $x = 1$ about the $x-$axis.

Solution $S = \int_0^1 2\pi x^3\sqrt{1+(3x^2)^2}\,dx \xlongequal{u=1+9x^4} \int_1^{10} \frac{2\pi}{36}u^{\frac{1}{2}}\,du = \frac{\pi}{27}(10^{\frac{2}{3}} - 1).$

Example 5 Find the area of the surface that is generated by revolving the portion of the curve $y = x^2$ between $x = 1$ and $x = 2$ about the $y-$axis.

Solution The curve is also expressed as $x = \sqrt{y},\ 1 \le y \le 4$. Hence the area required is

$$S = \int_1^4 2\pi\sqrt{y}\sqrt{1+(\frac{1}{2\sqrt{y}})^2}\,dy = \pi\int_1^4 \sqrt{1+4y}\,dy \xlongequal{u=1+4y} \frac{\pi}{4}\int_5^1 7\sqrt{u}\,du = \frac{\pi}{6}(17^{\frac{3}{2}} - 5^{\frac{3}{2}}).$$

Definition 5.3(Work performed by the force) (力做的功)) Suppose that an object moves in the positive direction along a coordinate line over the interval $[a, b]$ while subjected to a variable force $F(x)$ that is applied in the direction of motion. Then we define the **work(功)** W performed by the force on the object to be $W = \int_a^b F(x)\,dx$.

Example 6 A spring exerts a force of 5 N when stretched 1 m beyond its natural length.

(a) Find the spring constant (弹性系数) k.

(b) How much work is required to stretch the spring 1.8 m beyond its natural length?

Solution (a) Substituting $x = 1$(m) and $F(1) = 5$(N) into the formula $F(x) = kx$ we get $k = 5$(N/m).

(b) $W = \int_0^{1.8} kx\,dx = \frac{5}{2}(1.8)^2$(Nm)=8.1(J)(焦耳).

Example 7 A conical water tank of radius 10 meters and height 30 meters is filled with water to a depth of 15 meters. How much work is required to pump all of the water out through a hole in the top of the tank?

Solution The weight density of water is 1 gram per cubical centimeter that is 10^3 Kg/m^3 (kilogram per cubical meter). Then as is shown in Figure 6.6.5 the work

$$W = 10^3 \int_{15}^{30} \frac{\pi}{9} x^3\,dx = 2 \cdot 3^3 \cdot 5^8 \pi(\text{kg}\cdot\text{m}).$$

Work-energy relationship Assume that an object moves in the positive direction along a coordinate line over the interval $[a, b]$ while subjected to a force $F(x)$ that is applied in the direction of the motion. Let $x(t), v(t) = x'(t)$, and $v'(t)$ denote the respective position, velocity, and acceleration of the object at time t. It follows from Newton's Second Law of Motion that $F(x(t)) = mv'(t)$ where m is the mass of the object. Assume that $x(t_0) = a, x(t_1) = b$ with $v(t_0) = v_i$ and $v(t_1) = v_f$ the initial (初) and final (终) velocities of the object, respectively. Then

$$\begin{aligned} W &= \int_a^b F(x)\,dx = \int_{x(t_0)}^{x(t_1)} F(x)\,dx = \int_{t_0}^{t_1} F(x(t))x'(t)\,dt \\ &= \int_{t_0}^{t_1} mv'(t)v(t)\,dt = \int_{v_i}^{v_f} mv\,dv = \frac{1}{2}mv^2\Big|_{v_i}^{v_f} \\ &= \frac{1}{2}v_f^2 - \frac{1}{2}v_i^2. \end{aligned}$$

We define the "energy of motion" or **kinetic energy** of the object by

$$K = \frac{1}{2}mv^2.$$

Then we have the **work-energy relationship**

$$W = K_f - K_i$$

with K_i and K_f denoting the initial and final kinetic energy of the object.

Example 8 A space probe (探测器) of mass $m = 5.00 \times 10^4$ kg travels in deep space subjected only to the force of its engine. Starting at a time when the speed of the probe is $v = 1.10 \times 10^4$m/s, the engine is fired continuously over a distance of 2.50×10^6m with a constant force of 4.00×10^5N in the direction of motion. What is the final speed of the probe?

Solution We have

$$
\begin{aligned}
K_f = W + K_i &= \int_0^{2.50\times 10^6\mathrm{m}} F(x)\,dx + \frac{1}{2}mv_i^2 \\
&= (4.00\times 10^5\mathrm{m})\times(2.50\times 10^6\mathrm{N}) + \frac{1}{2}(5.00\times 10^4\mathrm{kg})(1.10\times 10^4\mathrm{m/s})^2 \\
&= 4.025\times 10^12\mathrm{J}
\end{aligned}
$$

Then we get

$$v_f = \sqrt{2\frac{K_f}{m}} = \sqrt{\frac{2(4.025\times 10^12)}{5.00\times 10^4}}\mathrm{m/s} \approx 1.27\times 10^4\mathrm{m/s}.$$

Definition 5.4(Fluid force on a vertical surface (液体在竖直面上的压力) Suppose that a flat surface is immersed (浸入)vertically in a fluid of weight density (比重) ρ and that the submerged (淹没的) portion of the surface extends from $x = a$ to $x = b$ along an $x-$axis whose positive direction is down (Figure 6.7.4a). For $a \le x \le b$, suppose that $w(x)$ is the width of the surface and that $h(x)$ is the depth of the point x. Then we define the **fluid force** on the surface (that is perpendicular to the surface) to be

$$F = \int_a^b \rho h(x)w(x)\,dx.$$

Example 9 The face of a dam is a vertical rectangle of height 100 m and width 200 m. Find the total fluid force exerted on the face when the water surface is level with the top of the dam.

Solution $\int_0^{100} 200x\,dx = 10^6\mathrm{T}(吨).$