

Review on Doi-Onsager Model in Polymeric Fluids

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Abstract. We review the recent results on Doi-Onsager Equations in polymeric fluids, including the existence, uniqueness and properties of the solution to the evolution equations, all equilibrium solutions, stability of these equilibrium solutions, phase transitions from isotropic to nematic and hysteresis phenomenon. More importantly, it is shown the main approach to obtain these results. Comparably, we show the main difficulty to solve these problems with the Onsager potential, which is popularly used to investigate polymeric fluids.

1 Introduction

The Doi model for rod-like molecules has been very successful in describing the properties of liquid crystal polymers in a solvent [6, 7]. The basic object in the Doi model is the single rod-like molecule position-orientation distribution function. Interactions between molecules are modelled by a mean-field potential. Therefore, the Doi model can be regarded as a mean-field kinetic theory. Besides interaction with other rods, the rods are also interacting with the flow and are subject to Brownian forces. If the interaction strength is sufficiently strong, compared with the Brownian forces, or if the rod concentration is sufficiently high, then the system prefers to be in a nematic phase in which the rods tend to line up with each other. Otherwise the system is in an isotropic phase in which the orientation of the rods is

completely random. Such a phase transition problem was first described by Onsager in 1949 [23], using a variational approach. He used the free energy with the Onsager potential and an assumed form of the distribution function and was able to argue that in the limit of high concentration one has a transition from the isotropic uniform distribution to an ordered prolate distribution. So we call it by the Doi-Onsager model. Recently various phase transition diagrams to equilibrium states in rigid rod-like polymers have been observed in both experiments and numerical simulations, see e.g. [1, 8, 12, 13, 16, 20, 21]. Moreover, there are lots of papers published in which the Doi-Onsager model was concentrated to study theoretically [2, 3, 4, 5, 9, 10, 14, 15, 17, 18, 19, 24, 25]. In this paper, we will review the main results of the Doi-Onsager model, and which the approaches to obtain.

Now let us introduce the Doi-Onsager model, that is the Doi model in the absence of flow, in which case the model reduces to a non-local diffusion equation:

$$\frac{\partial f}{\partial t} = D_r \mathcal{R} \cdot (\mathcal{R}f + f\mathcal{R}U), \quad (1.1)$$

here $f(x, t)$ is the orientation distribution function, D_r is the rotational diffusivity, which, without loss of generality, will be set to 1; where $x \in \mathbb{S}^2$, \mathbb{S}^2 is the unit spherical surface and the operator is

$$\mathcal{R} = x \times \frac{\partial}{\partial x}. \quad (1.2)$$

U is the mean-field interaction potential with the form

$$U(x) \triangleq U(x, [f]) = \int_{|x'|=1} K(x, x') f(x') dx', \quad (1.3)$$

where $K(x, x')$ is a smooth, real valued, symmetric kernel. Different expression of this kernel leads to different models. Onsager considered the potential

$$K(x, x') = \alpha |x \times x'|, \quad (1.4)$$

where α is a parameter that measures the potential intensity (follows it is the same sense). What is more frequently to be used is the Maier-Saupe potential [6, 7, 22], defined by

$$K(x, x') = \alpha |x \times x'|^2. \quad (1.5)$$

There is also other potential forms, such as the dipolar potential [9, 10]

$$K(x, x') = -\alpha x \cdot x', \quad (1.6)$$

and the electric dipole-dipole potential [15]

$$K(x, x') = -K_B T (\gamma \mathbf{E} \cdot x + \alpha x \cdot x') - \frac{3K_B T}{2} |x \times x'|^2 - \frac{\alpha_0 K_B T}{2} |\mathbf{E} \times x'|^2, \quad (1.7)$$

where \mathbf{E} is an imposed electric field. γ is the strength of the permanent dipole on the molecule, α_0 is the difference of the polarizability parallel and perpendicular to the molecular direction known as the anisotropy, N is a dimensionless parameter describing the strength of the excluded volume potential and quadruple-quadruple interaction. When and $\gamma = \alpha_0 = 0$, it is not an imposed electric field in [14].

The equation (1.1) also is called the Smoluchowski equation, which can be rewritten as the more general form by P. Constsntin et al. in [2, 3, 4, 5] in local coordinates:

$$f_t = \frac{1}{\sqrt{g}} \partial_i (e^{-U} \sqrt{g} g^{ij} \partial_j (e^U f)), \quad \text{in } (M^n, g), \quad (1.8)$$

where it used the summation convention and (M^n, g) is a smooth compact connected Riemannian manifold without boundary. As what the authors said in [3]: The euqation (1.9) is a nonlinear Fokker-Planck equation (that is: it is a nonlinear equation, and it looks like a linear Fokker-Planck equation),

$$f_t = \Delta_g f + \text{div}_g(f \nabla U). \quad (1.9)$$

where

$$\Delta_g = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} \partial_j)$$

is the Laplace-Beltrami operator and

$$\text{div}_g(f \nabla U) = \frac{1}{\sqrt{g}} \partial_i (\sqrt{g} g^{ij} f \partial_j U).$$

Let $M^n = \mathbb{S}^{n-1}$, the $n - 1$ dimensional sphere (hereafter simple denoted by $(n - 1)$ D case). When $n = 2$, the unit circle has local coordinate $\phi \in [0, 2\pi]$ and one has $x_1(\phi) = \cos \phi, x_2(\phi) = \sin \phi, g^{11} = g = 1, \partial_1 = \partial_\phi$. When $n = 3$, the coordinates on the two dimensional unite sphere are (ϕ, θ) , and $x_1(\phi, \theta) = \sin \theta \cos \phi, x_2(\phi, \theta) = \sin \theta \sin \phi, x_3(\phi, \theta) = \cos \theta. g^{11} = 1, g^{22} = (\sin \theta)^{-2}, g^{ij} = 0, i \neq j, \partial_1 = \partial_\theta, \partial_2 = \partial_\phi$.

From the Doi equation (1.1) we see that $\int_{|x|=1} f(t, x) dx$ is conserved. Therefore (1.1) is usually solved together with an enforced normalization $\int_{|x|=1} f(t, x) dx = 1$. Now ones are concerned with the following questions for different kinds of potential:

- (1) existence and uniqueness of the nonnegative smooth solution to (1.1) with nonnegative initial data;
- (2) properties of the smooth solution;
- (3) all equilibrium solutions;
- (4) stability of all equilibrium solutions;
- (5) phase transition phenomena.

2 Existence

Firstly, P. Constantin et al. in [4], give the answer of (1). The proof can be done by successive approximations.

Theorem 2.1. *Let f_0 be a nonnegative, continuous function on \mathbb{S}^{n-1} . The solution of (1.1) with initial data $f(\cdot, 0) = f_0$ exists for all nonnegative time, are smooth, nonnegative and normalized*

$$\int_{\mathbb{S}^{n-1}} f(x, t) dx = \int_{\mathbb{S}^{n-1}} f_0(x) dx \quad (2.1)$$

the solutions are all real analytic for positive time.

Certainly, P. Constantin et al. in [4] also obtained some interesting estimates in certain norm (such as H^{-1}) for the bound to the smooth solution.

3 Steady states

It is more urgently wanted to know steady states of (1.1) than the existence. To our knowledge, there are many papers are concentrated to these problems [3, 4, 5, 8, 9, 10, 17, 18, 24, 25]. In the following we will review the main approach to obtain steady states for all kinds of potential forms.

This model has an energy functional

$$A(f) = \int_{|x|=1} \left[f(x) \ln f(x) + \frac{1}{2} f(x) U(x) \right] dx, \quad (3.1)$$

as its Lyapunov functional. That is a non-increasing function of time when evaluated on solutions. Indeed, taking the derivative of $A(f)$ when the time dependence comes from a smooth positive solution $f(x, t)$ of the Doi-Onsager equation, one obtains

$$\frac{d}{dt} A(f) = - \int_{\mathbb{S}^{n-1}} |\mathcal{R}(\ln f + U)|^2 f dx. \quad (3.2)$$

(3.1) can be also called a free energy, and the chemical potential μ

$$\mu = \frac{\delta A}{\delta f} = \ln f + U;$$

then (1.1) can be written as

$$f_t + \mathcal{R} \cdot (fv) = 0, \quad v = -\mathcal{R}\mu,$$

which is in the usual form of Fick's law. By definition, equilibrium solutions are steady states with constant chemical potential, $\mu = \text{constant}$. Indeed, it is equal to the steady states of the equation

$$\mathcal{R} \cdot (\mathcal{R}f + f\mathcal{R}U) = 0. \quad (3.3)$$

Follows we firstly show the method to study steady states. Then we illustrate this approach through some examples, which case were focused in different papers[3, 4, 5, 9, 10, 17, 18, 14, 15, 24]. The following steps are the main idea to obtain steady states.

Step 1: The solutions of (3.3) with normalization condition can be expressed in a Gibbs form:

$$f(x) = \frac{1}{Z} e^{-U}, \quad (3.4)$$

where Z is normalization. This is a nonlinear equation in f since U still depends on f . From the foregoing analysis we can see that the form (3.4) is important. By using it we can transform the problem to solve a partial differential equation (3.3) with the normalization $\int_{|x|=1} f(x) dx = 1$ into the problem to solve a nonlinear equation (3.4) without the partial differential operator.

Step 2: From the potential (1.3) with different kernel (1.4),(1.5), (1.6) and (1.7), we can see that U depends on finite modulus (e.g. (1.5), (1.6), (1.7)) or infinite modulus (e.g. (1.4)) if we transform the distribution f into the Fourier series for 2D case or spherical harmonics series for 3D case. But it is not all modulus to determine the steady states since the free energy (3.1) is not change for some modulus. This implies that each steady state is a kind of equilibrium solutions. Thus there will be at least a parameter in each kind equilibrium solution. So we only need to find a solution in each kind.

Step 3: By numerical simulation, we can find that there is axial symmetric property for all solutions of (3.4) for some potential. It will reduce the parameters to determine steady states. But the theoretical proof is the most difficult part to obtain all steady states. The authors in [18, 10, 24] are focused to prove this property by using most parts in their articles.

Step 4: From step 2 and 3, we can reduce the modulus parameters of U to determine the number of solutions to (3.4). Thus we will transform to solve the nonlinear equation (3.4) of the distribution function f into to solve some nonlinear equations of some modulus parameters. For some cases ones can completely analyze it[9, 10, 17, 18, 24]. But for some complex cases ones can obtain all steady states by numerical calculation[14, 15]. Concerning to the stability of these equilibrium solutions, it can be obtained by calculating the free energy (3.1). That is the minimum problem to the functional (3.1).

In the following we will give some examples to illustrate this idea in order to get steady states.

4 Examples

4.1 The Maier-Saupe potential in 2D

For the two-dimensional case, the Doi-Onsager equation (1.1) with (1.3) and (1.5) reduces to the following on a circle,

$$f_{\theta\theta} + (fU_{\theta})_{\theta} = 0, \quad \theta \in [0, 2\pi], \quad (4.1)$$

$$U(f) = \alpha \int_0^{2\pi} \sin^2(\theta - \theta') f(\theta') d\theta', \quad (4.2)$$

with the normalization

$$\int_0^{2\pi} f(\theta) d\theta = 1. \quad (4.3)$$

From (4.1)-(4.3), we can translate f into a Fourier form, writing

$$f(\theta) = \frac{1}{2\pi} + \sum_{k=1}^{\infty} [a_k \cos k\theta + b_k \sin k\theta]. \quad (4.4)$$

Insert this form (4.4) into (4.2), we can find that

$$U \text{ only depends on the second modulus } a_2 \text{ and } b_2. \quad (4.5)$$

Later we will see that this is the key observation. On the other hand, from (4.2), we can get

$$\begin{aligned} U(f) &= \frac{\alpha}{2} \int_0^{2\pi} [1 - \cos 2(\theta - \theta')] f(\theta') d\theta' \\ &= \frac{\alpha}{2} \left[1 - \int_0^{2\pi} [\cos 2\theta \cos 2\theta' + \sin 2\theta \sin 2\theta'] f(\theta') d\theta' \right] \\ &:= \frac{\alpha}{2} [1 - a_2 \cos 2\theta - b_2 \sin 2\theta], \end{aligned} \quad (4.6)$$

where $a_2 := \langle \cos 2\theta' \rangle = \int_0^{2\pi} \cos 2\theta' f(\theta') d\theta'$, $b_2 := \langle \sin 2\theta' \rangle = \int_0^{2\pi} \sin 2\theta' f(\theta') d\theta'$. Moreover, From the equation (4.1), we can see that $f(\theta + \theta_0)$ is also a solution if $f(\theta)$ is a solution. This property implies that there is a free parameter in the solution(step 2). With the Gibbs form (3.4), we can claim that the only one parameter is left to determine the number of the solutions in the two second modulo parameters a_2, b_2 . This can be easily observed when we change the expression of (4.6) into the form

$$U(f) = \frac{\alpha}{2} + \eta \cos 2(\theta - \theta_0), \quad (4.7)$$

where $\eta = \sqrt{a_2^2 + b_2^2}$, and $\tan \theta_0 = b_2/a_2$. This is also the general solution of the equation

$$U_{\theta\theta} + 4U = 2\alpha. \quad (4.8)$$

(4.8) is a linear decoupled equation satisfied by the Maier-Saupe potential (4.2), which was pointed out in [18]. Combination the expression (4.7) related to the second modulo parameters a_2, b_2 and the Gibbs' form (3.4) yields

$$f(\theta) = \frac{e^{-\eta \cos 2(\theta - \theta_0)}}{\int_0^{2\pi} e^{-\eta \cos 2(\theta - \theta_0)} d\theta}, \quad (4.9)$$

where θ_0 is arbitrary and η^* satisfies:

$$\frac{\alpha}{2} + \eta \cos 2(\theta - \theta_0) = \alpha \int_0^{2\pi} \sin^2(\theta - \theta') f(\theta') d\theta'. \quad (4.10)$$

Inserting (4.9) into (4.10), we obtain a simple relation

$$\frac{\int_0^{2\pi} \cos 2\theta e^{-\eta \cos 2\theta} d\theta}{\int_0^{2\pi} e^{-\eta \cos 2\theta} d\theta} + \frac{2\eta}{\alpha} = 0. \quad (4.11)$$

Then, one only need to determine the number of zeros of η related the intensity parameter α in (4.11)(step 4). Through the detail argument in [18], it can be found that there is a critical value of $\alpha = 4$, when $\alpha \leq 4$, the only equilibrium solution is the constant $f = 1/2\pi$; and when $\alpha > 4$, then besides the constant solution, all other equilibrium solutions can be given by (4.9). Moreover, the stability of these solutions becomes to discuss minimum points to the functional (3.1) depending on the parameter η . The authors in [9, 25] gave the detail discussion.

Concerning this question, Constantin et al. [3] in 2004 established a bound on the number of stationary solutions for the 2D case, and at the same time, gave a sharp estimate on the region of stability for the isotropic solution. In 2005, we used the Fourier series (4.4) and continuous fraction analysis in [17] gave these results. Moreover, in [18], which is focus on the 3D case, we also obtained the same result by using the decouple equation (4.8) in order to highlight this observation. Almost at the same time these results were also claimed in [5, 9] by using the similar idea.

4.2 The Maier-Saupe potential in 3D

This section is devoted to the analysis of the three-dimensional case. Consider 3D Doi-Onsager model with the Maier-Saupe potential of the form:

$$\mathcal{R} \cdot \mathcal{R}f + \mathcal{R} \cdot (f\mathcal{R}U) = 0, \quad x \in \mathbb{S}^2, \quad (4.12)$$

$$U = \alpha \int_{|x'|=1} |x \times x'|^2 f(x') dx', \quad (4.13)$$

with the normalization

$$\int_{|x|=1} f(x) dx = 1. \quad (4.14)$$

Firstly, we can see from (4.13) that the Maier-Saupe potential also only depends on the second modulus similar to the 2D case. This can be easily shown when we change the expression of (4.13) into

$$\begin{aligned} U &= \alpha \int_{|x'|=1} (1 - (x \cdot x')^2) f(x', t) dx' \\ &= \alpha \left\{ 1 - x \otimes x : \int_{|x'|=1} x' \otimes x' f(x', t) dx' \right\}. \end{aligned} \quad (4.15)$$

The here second order tensor $B := \int_{|x'|=1} x' \otimes x' f(x', t) dx'$ only depends on the second modulus when we denote the orientation distribution function by the spherical harmonics series

$$f = \frac{1}{4\pi} + \sum_{l=1}^{\infty} \sum_{m=-l}^l b_{lm} Y_l^m, \quad (4.16)$$

where Y_l^m are spherical harmonics. Since the spherical harmonics elements are orthogonal each other, so

$$U = \alpha \left\{ \frac{2}{3} - \frac{8\pi}{15} (b_{22} Y_2^{-2} + b_{2-2} Y_2^2 + b_{20} Y_2^0 - b_{21} Y_2^{-1} - b_{2-1} Y_2^1) \right\}. \quad (4.17)$$

Now one knows that U only depends on the second modulus coefficients $b_{22}, b_{2-2}, b_{21}, b_{2-1}, b_{20}$ of f . Additionally, we know that f is real, so is also U . Therefore, there are **five** freedom variables in these complex coefficients $b_{22}, b_{2-2}, b_{21}, b_{2-1}, b_{20}$ because $\bar{b}_{22} = b_{2-2}, \bar{b}_{21} = b_{2-1}$ and b_{20} is real, where \bar{b}_{22} denotes the complex conjugation of b_{22} . We can also see this point in [18] from the decouple equation of U in (4.13) satisfying

$$\mathcal{R} \cdot \mathcal{R}U + 6U = 4\alpha. \quad (4.18)$$

The dimension of the space of its general solutions is five. In other word, U only depend on the limit coefficients of f in the spherical harmonics series. It is why ones can completely solve the equation (4.12) with the Maier-Saupe potential (4.13)[2, 18, 10]. In virtue to the Gibbs' form (3.4) and the foregoing analysis, we know that there are only five freedom variables in the distribution function. Furthermore, one can observe that the potential U has the following important properties in [18]: $U(x)$ is a solution of (4.13) if and only if it satisfies

$$U(x) = \alpha \int_{|x'|=1} |x \times x'|^2 e^{-U(x')} dx' \left[\int_{|x'|=1} e^{-U(x')} dx' \right]^{-1}. \quad (4.19)$$

Moreover, if $U(x)$ is a solution to (4.19), $U(Tx)$ is also a solution, where T is an arbitrary rotation operator in \mathbb{R}^3 (step 2). Since there are three freedom variables in an arbitrary rotation operator in \mathbb{R}^3 , this properties implies that it is the rest two freedom to determine the distribution function. In order to more clearly see this point, we can choose a rotation operator T such that

$$U = \alpha \left(1 - \sum_{i=1}^3 q_i x_i^2 \right), \quad (4.20)$$

where $q_1 + q_2 + q_3 = 1$ ref. [18]. In fact, where for $i = 1, 2, 3$,

$$q_i = \int_{|x'|=1} x_i^2 e^{-U(x')} dx' / \int_{|x'|=1} e^{-U(x')} dx'. \quad (4.21)$$

Then it will be two independent variables among $q_i (i = 1, 2, 3)$ to determine U , hence to determine f .

Additionally, by numerical calculation we find the solutions to (4.12)-(4.14) are axial symmetry. Indeed, this can be proved through showing that one of the following three cases is true: $q_1 = q_2$ or $q_2 = q_3$ or $q_3 = q_1$ when we use the expression (4.20) [18, 10, 24](step 3). It is difficult to prove this property which shows the axial symmetry of the distribution functions. Now there are three proofs which the readers can refer to [10, 18, 24]. Now it is the rest one freedom to determine the solutions. We can more clearly see the foregoing analysis result by constructing the general solutions satisfying (4.12), (4.18) and (4.14). In [18] we gave that all explicit stationary distributions are

$$f = k e^{-U}, \quad (4.22)$$

where

$$U = \frac{2\alpha}{3} - \eta \left(|x \times y|^2 - \frac{2}{3} \right), \quad (4.23)$$

$\eta \in \mathbb{R}$ is a parameter. $y \in \mathbb{S}^2$ is the symmetric axial of f . k is normalization. Then the parameter η in (4.23) is to determine the solutions. That is, η satisfies

$$\frac{2\alpha}{3} - \eta \left(|x \times y|^2 - \frac{2}{3} \right) = \alpha \int_{|x'|=1} |x \times x'|^2 k e^{-U(x')} dx'. \quad (4.24)$$

To solve this nonlinear equation of the parameter η depending on the intensity α , one can obtain the complete picture of the equilibrium solutions(step 4). We gave all the detail in 2005 in [18]. By rigorous analysis, we found that the number of stationary solutions of the Doi-Onsager equation (4.12) with (4.13), (4.14) hinges on whether the intensity α crosses two critical values: $\alpha^* \approx 6.731393$ and 7.5, where

$$\alpha^* = \min_{\eta} \frac{\int_0^1 e^{-\eta z^2} dz}{\int_0^1 (z^2 - z^4) e^{-\eta z^2} dz}. \quad (4.25)$$

And there are different kind solutions-isotropic(Figure 1), prolate(Figure 3) and oblate(Figure 2) solutions when α is in different internals. Moreover there exist multiply solutions when $\alpha > \alpha^*$. With the expression (4.22) depending on the parameter η , then further to study the stability of these solutions can be transformed into to solve the minimum of the nonlinear function (3.1) of the parameter η [10, 2, 25]. By the way, we point out that Liu [19]

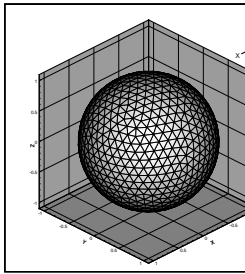


Figure 1: $\eta = 0$

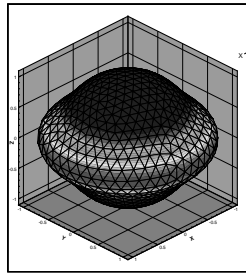


Figure 2: $\eta = 10$

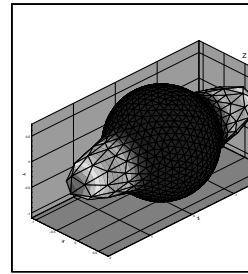


Figure 3: $\eta = -10$

study the dynamic steady states for different intensity α .

From the above analysis, we now know that the key observation to solve completely the Doi-Onsager model in the sphere is that the potential U depends on the limit modulus

similar to the 2D case. Moreover, one can transform the problem to solve PDE (4.12)-(4.14) into to solve a nonlinear equation (4.24) of the parameter η without any differential operators. Constantin et al. [2, 4] also pointed out this idea. They inserted (4.20) into (4.21) to obtain a nonlinear systems of q_1, q_2, q_3 . Then they gave the bound on the number of equilibrium solutions by analyzing the nonlinear equations of two independent parameters. Meanwhile it was analyzed the behavior of steady states of when α very small and very large. Thus they obtained the stable isotropic and prolate solutions [3]. In [2], Constantin reviewed the results in [3, 4]. Meanwhile he studied the general potential smooth, real valued, symmetric kernel in (1.2) and obtained that the prolate nematic state concentrated at one configuration direction is generically high intensity limit. Certainly, he also studied the dynamic behavior of (1.1) in [2]. Lui et al. [18] combined the equations (4.21) and (4.18). Then we showed the solutions of (4.12)-(4.14) are shown to be necessarily a set of axially symmetric functions, and a complete classification of parameters for phase transitions to these stationary solutions is obtained. It was shown that the number of stationary solutions hinges on whether the potential intensity crosses two critical values $\alpha_1 \approx 6.731393$ and $\alpha_2 = 7.5$. Furthermore, we present explicit formulas for all stationary solutions. Almost at the same time, Fatkullin et al. [10] gave the different proof of the axial symmetry in virtue of Bessel functions and showed the critical points of α by numerical calculation. Stability of steady states were discussed in [10] without the detail. Later Zhou et al. [24] gave the more transparent proof of the axial symmetry. They constructed a function by using the parameters q_1, q_2, q_3 then deduce a contradiction if q_1, q_2, q_3 are distinct besides $q_i \geq 0 (i = 1, 2, 3)$ and $q_1 + q_2 + q_3 = 1$. Meanwhile they also computed again the critical points of α .

Before we end this section we illustrate these equilibrium situation for the sake of clarity as [11] and show phase transition, hysteresis phenomenon. Figure 4 shows the scalar order parameter s reported versus the nondimensional potential intensity α . The scalar order parameter represents a scalar measure of the degree of the order of the sample. Here

$$s = \frac{3}{2}\Lambda, \quad (4.26)$$

where Λ is the eigenvalue with the largest absolute value of the traceless second-rank order

tensor \mathbf{S} given by

$$\mathbf{S} = \langle x \otimes x \rangle - \frac{1}{3}\mathbf{I} = \int_{|x|=1} x \otimes x f(x) dx - \frac{1}{3}\mathbf{I}, \quad (4.27)$$

with \mathbf{I} as the second-rank unit tensor. By using the explicit formula (4.22) of the equilibrium solutions, we obtain

$$s = -\frac{\eta}{\alpha}.$$

The scalar order parameter s is zero when the system is isotropic, the distribution function

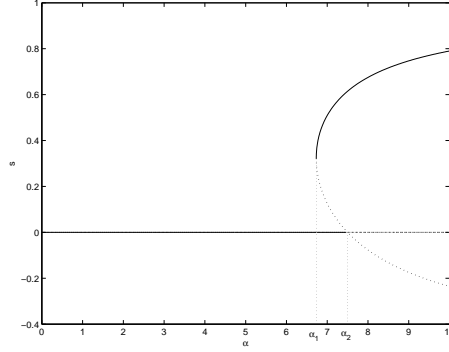


Figure 4: s vs α : the dashed line is the unstable solution, the solid line is the stable and metastable solution, where $\alpha_1 = \alpha^* = 6.731393$ and $\alpha_2 = 7.5$

like Figure 1. Conversely, its value is nonzero if some degree of orientation is present. It is positive in the case of prolate distributions (e.g. Figure 3), and negative for oblate ones (e.g. Figure 2). In Figure 4, solid lines represent stable and meta-stable stationary solutions and the dashed lines are unstable stationary solutions. From this figure we can see the hysteresis phenomenon. That is, the stable stationary state is from the isotropic phase to the prolate phase when the potential intensity α crosses the critical point 7.5 from small to large, and from the prolate phase to the isotropic phase when α crosses the other critical point 6.731393 from large to small.

4.3 The dipole-dipole potential

The authors in [14] were focused on the potential (1.7) with the parameter $\gamma = \alpha_0 = 0$. Firstly one can rewrite U in this case as

$$U = \frac{3NK_B T}{2} - (\alpha K_B T \langle x' \rangle) \cdot x - \frac{3NK_B T}{2} \langle x' x' \rangle : xx. \quad (4.28)$$

From this form we can see that U only depends on the first and second modulus. So it can be solved through step 1-4. Ji et al. in [14] found that the equilibrium solutions are also invariant with respect to the $SO(3)$ group(step 2 in Sec. 3). Then it was proved that the first moment vector either vanishes or must parallel to one of the eigenvectors tensor in Theorem 1 [14](step 3 in Sec. 3). Thus they can transform to solve (3.4) to solve the nonlinear equations of the three parameters. All equilibrium solutions and stability were found and critical points of phase transition by numerical calculation(step 4 in Sec. 3). The details can be found in [14]. It was studied in [15] for the potential (1.7) with the parameters γ, α_0 by using similar approach. But the steady states and phase transition phenomena more complicated than the case $\gamma = \alpha_0 = 0$.

4.4 The Onsager potential

From the analysis in Sec. 4.1, ones know that (4.5) is the real cause to completely solve the equation (4.1) with the Maier-Saupe potential (4.2) for 2D case. From Sec 4.1 and 4.3, we can predict that (4.1)-(4.3) can be solved when the potential U only depends on the limit modulus. However, for the Onsager potential (1.4), it is $|\sin(\theta - \theta')|$ in 2D case, we can find U depends on all modulus a_i and b_i ($i = 1, 2, \dots$). In the same thing, it can be obtained that the Onsager potential depends on all modulus of f in the spherical harmonics series for 3D case. This is the main difficult to the Onsager potential. From the other viewpoint, we can verify that the Onsager potential satisfies

$$U_{\theta\theta\theta} + U_{\theta}U_{\theta\theta} + UU_{\theta} + U_{\theta} = 0 \quad (4.29)$$

with the period π , $U(\theta) = U(\theta + \pi)$. Comparing with (4.8), we can see that (4.29) is a nonlinear equation. From our knowledge it is difficult to obtain the explicit expression of the solutions. Therefore, there is few theoretical result concerned with the Onsager potential (1.4).

However, we believe that the equilibrium solutions are axial symmetry for the Onsager potential. But it is open that how to rigorously prove it and how to reduce the parameters in order to obtain all steady states. We hope to complete it by step 2-4 in Sec. 3.

5 Conclusion Remark

Here we recall the recent results about the Doi-Onsager model. One can see that it is popularly absorbed its steady states, which show the state of the rod-like particles in polymeric fluid and phase transition phenomena. It is our hope to completely solve it. But up to now we conclude that Doi-Onsager model can be solved for the finite moment potential, such as the Maier-Saupe potential (1.5), the dipole potential (1.6), the electric dipole-dipole potential (1.7). The approaches to study these finite moment potentials are similar. Firstly it is to transform the state equation, which is a partial differential equation, into a nonlinear equation without any partial differential operator. Then thank to the properties of the equilibrium solutions, such as $SO(3)$ group invariant, the moment parameters in potential can be reduced. If the steady states is axial symmetry, the parameters to determine the number of the equilibrium solutions can be reduced again. Thus one can obtain all equilibrium solutions by studying nonlinear equations of these parameters through theoretical and numerical calculation. In these cases, stability of the steady states is a minimum problem of a functional. For the onsager potential, now it is few result since it depends on infinite moments although there are possibly similar properties to the Maier-Saupe case.

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